BOOK REVIEWS

Logic for mathematicians. By J. B. Rosser, New York, McGraw-Hill, 1953. 14+540 pp. $10.00.

This book is undoubtedly a major addition to the literature of mathematical logic. Yet it is hardly the kind of addition which one would expect from its title, its preface, and the reputation of its author. The preface states that the book is intended as a textbook for mature mathematicians; that, as such, it aims to be relatively complete; and that matters of considerable logical importance which are not interesting for the mathematician are purposely omitted. The book turns out to be a detailed exposition, modeled on the *Principia Mathematica*, of a particular logistic system.

For the purposes of criticism the book will be divided into two parts. The first part, comprising the first eight chapters, develops what may be called the basic logic, i.e. logic through the restricted predicate calculus, including descriptions and equality. The second part deals with what may be called the higher logistic, which in this case means abstract set theory, including the more elementary portions of cardinal and ordinal arithmetic. It will be convenient to discuss what seems to be the main aspect of the second part first, and then to take up the first part along with the certain auxiliary aspects of the second.

The system developed in the second part is Quine's *New foundations*, which will be called NF. This is the system proposed by Quine at the winter meeting of 1936 and published in the American Mathematical Monthly in 1937. It followed a series of papers in which Quine, who was at first an advocate of the theory of types, made a study of axiomatic set theory. He showed in these that the axiom of subsets (Aussonderung), which guarantees the existence of the subset of those elements of a given set which have a certain property, is the essential axiom scheme of the theory. In NF he proposed that the restriction to subsets of a given set can be abandoned, provided that the property is required to be stratified, i.e. such that its variables (as they appear in that particular formula) can be classified so that the restrictions of the theory of types are satisfied. In the resulting system there is a single universal class, rather than a hierarchy of universal classes in the various types; also each class has a unique complement; and cardinal numbers exist as sets without typical ambiguity.

The consistency of the system was at first in doubt (cf. Zentralblatt für Mathematik 16: 193). In 1939 Rosser made a serious attempt to prove it inconsistent, and later Lyndon and Rosser did prove the inconsistency of the stronger system of Quine's book of 1940; but NF withstood all these onslaughts. On the other hand certain anomalies soon became apparent in it. As Quine himself showed in 1937, Cantor's paradox is only avoided because a class α is not always cardinally similar to the set of its unit subsets (which Rosser calls USC (α)). There are various anomalies connected with mathematical induction; thus in the present book one needs a special axiom to the effect that the first n natural numbers form a set with the cardinal number n. The restriction to stratified properties is peculiarly annoying; not only is the criterion tedious to verify (much more so than the restriction to subsets of a given set) in case defined terms are present, but it is apt to fail in the most unexpected places. The most striking anomaly, however, is that shown recently by Specker (Proceedings of the National Academy for September, 1953; the presentation is based on the present book). Specker showed that the axiom of choice is incompatible with NF, in fact the cardinals are not well ordered. Informally, Specker's conclusion can be made plausible as follows. For any cardinal n, let $T(n)$ be the cardinal of USC(α) when that of α is n, and let ∞ be the cardinal of the universal class; then the cardinals

$$\infty, T(\infty), T(T(\infty)), \cdots$$

form a decreasing sequence with no first member. In addition to the ordinary sets and cardinals which ascend from below, there are thus extraordinary sets which descend from ∞. Only the ordinary sets are of any interest in mathematics. It seems clear that, in order to develop a mathematics which remotely resembles what we are used to, a restriction to ordinary sets will have to be introduced at some point. Such a restriction will be similar to those in more orthodox set theory; and in that case the restriction to stratified properties is an unnecessary and bothersome complication.

Notwithstanding all this—indeed because of it—the system NF is of great logical interest. Its promulgation was a major help to deepening our understanding. In presenting the most elaborate treatment of this system which has yet appeared, Rosser has rendered a real service to logic. But NF is a strange choice for a logic for mathematicians. Even before Specker's discovery enough was known of the system's anomalous and dangerous character to have warned Rosser of the risks he was taking. These risks, moreover, could have been largely avoided by using a more modern, postulational point of view. Much of the theory of the book is independent of the particular logistic system chosen, and one would expect a cautious author to point out this independence. A large part of the theory of relations, for example, could be deduced (using basic logic) from certain postulates which would be valid in NF for homogeneous relations and in orthodox set theory for relations in a restricted universe. Such an approach would be more suitable for mathematicians than an unqualified commitment to a system of such questionable character. In choosing the latter course, the reviewer thinks that Rosser has done the exact opposite of what he claims to have done in his preface.

Let us now turn to the criticism of the first part of the book. No attempt will be made here to outline the contents of the various chapters. Rather, certain general features will be discussed. These include: first, features which seem to be advantages of the book as a textbook; second, contributions of the book to technical mathematical logic; and third, criticism of certain faults.

Any one acquainted with Rosser's previous work will know that at his best he is a master of lucid exposition. This book contains many instances of his skill. A notable example is the theorem on the completeness of propositional algebra. Here a clear presentation of the proof by Kalmar's method is made even more perspicuous by a preliminary consideration of a special case. Another instance is the explanation of the dot notation on p. 19 ff. In other cases, where the clarity is not so noticeable, the discussion is stimulating. Nearly every important logical idea is discussed from the standpoint of its significance first, and these discussions are usually noteworthy on one or both of these counts. Some further examples are the discussion of variables and quantifiers at the beginning of Chapter VI; of what Rosser calls Rule C in Chapter VI, §7; of classes and the Russell paradox in Chapter IX, §1; and of functions in Chapter X, §5.

The book also contains a large number of exercises. Furthermore, considerable space is devoted to the analysis of concepts and proofs from ordinary mathematics. These illustrations are drawn from such fields as elementary geometry, advanced calculus, basic topology (Hausdorff spaces), etc. Although the proofs analyzed are among the simpler ones from the point of view of the mathematician, yet they serve well as illustrations. There is also some attention to developing techniques which would be useful for such analysis. The reviewer regards this contact with ordinary mathematics as one of the strong points of the book.

We turn now to the technical contributions in the book. The reviewer has selected three for special comment.

The first of these is the formulation of propositional algebra. The author presents an axiomatic formulation in which all the connectives are defined in terms of conjunction and negation. There are three axiom schemes. These differ from the four scheme formulation of the reviewer (*Leçons de logique algébrique*, Paris, 1952, p. 114), in that the commutative law is eliminated by introducing a twist in one of the other schemes—a similar trick was used by Nicod (1917) and also works for the *Principia Mathematica*. The only other formulation, so far as the reviewer knows, which uses these primitives is one of Sobocinski (1939). (However, these primitives were used, in combination with others, by Lewis and other writers on strict implication, and they were used by Peano.)

The second point is the elaboration of certain rules of the restricted predicate calculus having to do with quantification over restricted ranges on the one hand, and with rules for introducing quantifiers on the other. The first of these is related to work of A. Schmidt (1938) and Wang (1952); the second is equivalent to the corresponding rules of Gentzen (see below); but the author's treatment has features of interest.

The third point is the axiomatic theory of descriptions. The notion $(\iota x)F(x)$ is taken here as primitive and is taken so that it is always an object. The idea is extended to variables over a restricted range in a peculiar way. Let A be an object specified in advance but belonging to the range; then when $(\iota x)F(x)$ (over the restricted range) does not exist in the ordinary sense it is defined to be A. This does not always agree with our intuitions. Thus, if the range consists of human beings living in 1953, let A be Winston Churchill; then

The King of France is bald

turns out to be true. On the whole, however, the theory of descriptions is an improvement over that of Russell.

Let us now turn to some entries on the other side of the ledger.

In the first place there is serious confusion about the basic nature of formal reasoning. In several places Rosser states explicitly that he adheres to the view that in such reasoning one is talking about meaningless marks. That requires, among other things, the use of quotation marks. He explains the usual use of these on page 50 and claims (on p. 51) to have used them according to that rule up to that point. Yet on page 12 we read, "Consider two statements '*P*' and '*Q*' of symbolic logic which are translations of English sentences '*A*' and '*B*'." But "*P*," "*Q*," "*A*," and "*B*" are capital letters; by no stretch of the imagination can they be either statements or sentences.

On the other hand on p. 83 we read that x (rather than "x") is the twenty-fourth letter of the alphabet. These are not isolated instances; there is a general confusion between use and mention throughout. There is no use of Quine's quasi-quotation marks or other devices which are indispensable for correct thinking along syntactical lines. In fact the evidence is that Rosser is not actually thinking in that way; he is thinking not of symbols but of their meanings. If he were to abandon the fiction that he is talking about symbols and say instead that he is using symbols according to prescribed rules without regard to their meaning, then most of what he says after p. 51 (where he allows himself to be careless) would be correct, whereas the earlier part where he has "scrupulously tried to be correct" is a muddle in any event. This confusion also obscures considerably the discussion of variables and functions. In these cases it is literally not clear just what the author means.

Even apart from this criticism the discussion of the nature of logic leaves much to be desired. In his amusing introductory chapter he fails (so the reviewer thinks, of course) to bring out with sufficient clarity that the aim of logic is objective analysis; that it strives to replace subjective feelings of correctness by objective criteria; that without it there is no objective meaning to be attached to mathematical rigor or truth. The lack of this emphasis gives this introduction an air of pedantry. Later on he does not characterize with sufficient definiteness what the reviewer would call epimethods, including the nature of constructiveness, and the relation of the constructive epitheorems to the fundamental requirements of objectivity. The reviewer finds the discussion of three logics on p. 79 a bit confusing. To him these three "logics" seem so little alike that to call them all by the same name is hardly more than a three-way pun.

Another point is that certain standard terms are used in senses quite different from those generally accepted. For example the terms "dual" (Chapter IV) and "indeterminate" (Chapter VI) are used in nonstandard senses. In regard to "substitution theorem," there is confusion between that sort of substitution (German "Ersetzung") and that indicated (p. 209) by "$\{\text{Sub in } S{:}A \text{ for } x_1\}$" (German "Einsetzung"). The distinction between these two can be maintained by using "replacement" for the former (see Carnap, *Logical syntax of language*, 1937, p. 36). The usage of Carnap deserves to be more generally followed than it is. Finally the reviewer finds it confusing to talk about certain statements in proofs by Rule C as "steps" in the proof; these statements are really suppositions which are eliminated later. This circumstance is made much clearer by the Gentzen formulation of the rule.

The author's discussion of functions has already been commented on. It would be well for every mathematician to read this section. There is much truth in it. Because of the lack of a suitable notation for functions at least one eminent mathematician has published an erroneous result. Nevertheless, the passage should be read with a critical eye. Before making sweeping changes the more basic confusion between use and mention must first be cleared up. As to whether the notation "$f(x)$" denotes a function or a function value, that is a question of whether the variable is bound or free; in such a statement as

$$\frac{d}{dx}f(x) = g(x)$$

all instances of "x" are bound, either explicitly or by the context, and therefore the reviewer would say that the statement has to do with functions. A notation for explicitly indicating the binding of variables would of course be useful in advanced work. (It would be useful too in Chapter VI; in fact the reviewer wonders why the λ-notation was not introduced before considering quantifiers.) Since the λ-notation conflicts with other uses of "λ" one of the other notations such as "$[x]A$" (reviewer), "$x \rightarrow A$" (H. T. Davis), "${}_xA$" (suggested verbally by Quine), "$[A]_x$" (reviewer) might be used.

Again, the author's opinion as to what sort of logic is suitable for mathematicians is certainly very different from the reviewer's. The particular sort of axiomatic propositional algebra used seems to the reviewer a technicality; the system of Hilbert and Bernays is more natural. Likewise, the long list of technical developments of set theory, developed in sickening detail, has an interest which seems rather special. On the other hand, the reviewer thinks the following topics deserve greater emphasis: the Gentzen rules, which are far closer to intuitive reasoning than Rosser's Rule C, for example; the intuitionistic and minimal logics; the relations to other algebraic systems, which can be developed to a certain extent as independent systems without basing them on logic; the theorems of Gödel; independent recursive or combinatory number theory, etc.

In conclusion, the reviewer can see both merits and faults in this work of Rosser. It is the most complete treatment of Quine's *New foundations* which now exists; and it makes several minor contributions to technical logic. It has several virtues from the expository viewpoint. As a logic for mathematicians, however, it is something of a disappointment. It makes an unfortunate choice of logistic system and goes overboard in commitment to it. Furthermore, it seems to the reviewer a bit old-fashioned; contrary to the statement in the preface it considers several topics whose interest seems technical and special; while other topics of modern logic, whose mathematical interest seems far greater, are ignored.

H. B. CURRY

Theory and applications of distance geometry. By L. M. Blumenthal. Oxford University Press, 1953. 12+348 pp. $10.00.

The first systematic development of geometric aspects of metric spaces is due to Karl Menger, who published a number of results in his *Untersuchungen über allgemeine Metrik* in 1928. Since then several authors have made contributions to this new domain. In 1938 Blumenthal gave a survey of the material available at that time in his book *Distance geometries*, one of the University of Missouri Studies. Considerable progress has been made since 1938 and with this new book the author aims to give a detailed introduction to the subject. About two-thirds of the text is devoted to imbedding and characterization problems. This shows clearly that the writer's own taste has played a large part in selecting the topics.

In the first chapter the author introduces all the principal notions which play a part in the theory of abstract metric spaces. Some have a topological character. It should be remarked that the definition given for compact spaces (p. 29) is different from the usual one, which may lead to some confusion. Chapter II is devoted to the notions of betweenness, convexity and metric segments. It contains Aronszajn's proof of one of Menger's theorems concerning segments: Each two points of a complete and convex space are joined by a metric segment. Chapter III gives in 30 pages a somewhat scanty treatment of metric curve theory. It contains several possible definitions of length and curvature and a survey of the results obtained so far, but for the proofs of many results the reader is referred to the original papers. A fuller treatment is given of the characterization problem. A subclass of metric spaces is characterized metrically whenever necessary and sufficient conditions (in terms of the metric) are obtained which have to be satisfied in order that a metric space be congruent with a member of this subclass. If the subclass consists of the set of subspaces of a metric space the problem is called the imbedding problem. An extensive treatment of this problem for the Euclidean spaces is given in Chapter IV. Most of these results are due to Menger but the author has succeeded in simplifying several of the proofs. If the subclass consists of one space only the problem is to characterize this space metrically among the metric spaces. Several characterizations are given for Euclidean spaces and Hilbert spaces. In a wide class of