SOME NEW ALGEBRAIC METHODS IN TOPOLOGY

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1. **Introduction.** The purpose of this address is solely expository. It is intended to give the mathematician who is not an expert in algebraic topology a picture of some of the newer algebraic techniques and machinery which have recently become common in that subject. The expert is warned that this exposition contains no methods or results which have not already been published.

We shall concentrate attention on the spectral sequence, a topic initiated by Leray. By its use, one can investigate the homology structure of a fibre space in terms of the base space and fibre. This method has been applied with considerable success to the study of the topological structure of Lie groups and homogeneous manifolds. Other important applications have been made in the subject of differential geometry in the large.

2. **Graded groups.** In algebraic topology, one associates with each topological space certain algebraic structures, such as groups, rings, vector spaces, modules, etc. In this address, we shall for the sake of simplicity restrict our attention mainly to certain abelian groups which are associated with topological spaces. Almost everything we shall say could be equally well applied to the case of vector spaces over a given field, or more generally, to modules over a given commutative ring. And with a little additional effort, we could consider the various rings that are associated with a space.

Usually it turns out that one associates with a topological space X not a single abelian group, but a whole sequence of abelian groups. The most important examples are the following:

The n-dimensional homology group of X with coefficients in an arbitrary abelian group G, denoted by $H_n(X, G)$ $(n=0, 1, 2, \cdots)$.

The n-dimensional cohomology group of X with coefficients in an arbitrary abelian group G, denoted by $H^n(X, G)$ $(n=0, 1, 2, \cdots)$.

The n-dimensional homotopy group of X, denoted by $\pi_n(X)$ $(n=1, 2, 3, \cdots)$. These groups were introduced by Hurewicz in 1935. $\pi_1(X)$ is the ordinary fundamental group, which need not be abelian. However, for $n>1$, $\pi_n(X)$ is abelian.

These groups are the very basis of algebraic topology. Many im-

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portant topological properties of a space X are reflected in certain algebraic properties of these groups. Many questions of a purely topological nature about spaces can be reduced to questions of a purely algebraic nature about the homology, cohomology, or homotopy groups of X. For these, and various other reasons, it is important to be able to determine as much as possible about their structure, and to determine any relations that may exist between the groups associated with different spaces under various conditions.

When one has to consider sequences of abelian groups, as in the examples just mentioned, it is often more convenient to collect all the groups together into one bigger group, and then consider that one bigger group instead. One does this by forming the (weak) direct sum of all the groups of the given sequence. The larger group which results from this process is now generally called a *graded group*; it is the direct sum of a certain indexed sequence of subgroups. Thus as examples we have the graded groups

$$H(X, G) = \sum_{n=0}^{\infty} H_n(X, G), \qquad H^*(X, G) = \sum_{n=0}^{\infty} H^n(X, G),$$

associated with any pair consisting of a topological space X and an abelian group G.[1]

3. **Groups with a differential operator.** One naturally asks at this point how the homology, cohomology, and homotopy groups of a space are defined. To go into this question in detail would take us too much time. However, there is a standard algebraic procedure by which the homology and cohomology groups of a space are derived from what are called a group of chains and a group of cochains respectively. It is the purpose of this section to explain this procedure.

Let A be an abelian group. An endomorphism, $d: A \rightarrow A$, of A is called a *differential operator* if $d^2 = 0$, i.e., for any $x \in A$, $d[d(x)] = 0$. A pair (A, d) consisting of an abelian group A and a differential operator d on A is called a *differential group*. Given a differential group (A, d), we shall denote by $\mathcal{Z}(A)$ the kernel of d, and by $\mathcal{B}(A)$ the image,[2] $d(A)$. The condition $d^2 = 0$ implies that $\mathcal{B}(A) \subset \mathcal{Z}(A)$, and hence one may form the factor group $\mathcal{Z}(A)/\mathcal{B}(A)$. This factor group will be denoted by $\mathcal{H}(A)$, and called the *derived group*.

[1] It is customary to extend the above definitions by defining $H_n(X, G)$ and $H^n(X, G)$ for n negative to be the trivial group consisting of the zero element alone.

[2] In topological applications, $\mathcal{Z}(A)$ is generally called the group of cycles or cocycles. Similarly, $\mathcal{B}(A)$ is called the group of boundaries, or coboundaries, or bounding cycles.

For any topological space X and coefficient group G one may define in various ways two differential groups: the group of chains, $(C(X, G), \partial)$, and the group of cochains, $(C^*(X, G), \delta)$. The differential operators $\partial: C(X, G) \to C(X, G)$ and $\delta: C^*(X, G) \to C^*(X, G)$ are called the boundary and coboundary operators respectively. Although one can define the groups of chains and cochains in many different ways, it is a fundamental theorem (or rather, set of theorems) of algebraic topology that the derived groups are always the same (up to an isomorphism) for "reasonably nice" spaces. These derived groups are by definition the homology and cohomology groups, $H(X, G)$ and $H^*(X, G)$ respectively.

The graded structure on $H(X, G)$ and $H^*(X, G)$ came from the fact that $C(X, G)$ and $C^*(X, G)$ have a naturally defined graded structure:

$$\begin{aligned} C(X, G) &= \sum_n C_n(X, G), \\ C^*(X, G) &= \sum_n C^n(X, G). \end{aligned} \qquad \text{(direct sum)}$$

Here $C_n(X, G)$ is called the group of n-dimensional chains, and $C^n(X, G)$ is called the group of n-dimensional cochains. With respect to these graded structures, the differential operators ∂ and δ always turn out to be homogeneous of degrees -1 and $+1$ respectively, i.e.,

$$\partial[C_p(X)] \subset C_{p-1}(X), \qquad \delta[C^p(X)] \subset C^{p+1}(X).$$

From this it follows immediately that the subgroups of cycles and bounding cycles also split up into direct sums of sequences of subgroups. Therefore the factor group, cycles modulo bounding cycles, also has a graded structure.

This process of forming the derived group, $\mathcal{H}(A)$, from a given differential group, (A, d), is very common in algebraic topology. Moreover, it is also becoming common in some other branches of mathematics.

4. **Exact triangles of groups.** Another frequently occurring algebraic notion in algebraic topology is that of an "exact triangle" of groups and homomorphisms. Suppose we have three abelian groups, A, B, and C, and homomorphisms $f: A \to B$, $g: B \to C$, $h: C \to A$. These groups and homomorphisms may be conveniently exhibited by a triangular diagram, as follows:

$$\begin{array}{ccc} A & \longrightarrow & B \\ & \nwarrow \quad \swarrow & \\ & C & \end{array}$$

This triangle is said to be *exact* in case the image of each homomorphism is precisely the kernel of the following homomorphism, i.e.,

$$f(A) = g^{-1}(0), \qquad g(B) = h^{-1}(0), \qquad h(C) = f^{-1}(0).$$

The notion of an exact triangle has the advantage that it considerably shortens the statement of many results, making them seem more natural and easier to remember.

A purely algebraic situation of frequent occurrence which gives rise to an exact triangle is the following (we shall need to make use of this example shortly). Let (A, d) be a differential group, and let B be a subgroup of A which is "stable" under d, i.e., $d(B) \subset B$. Then the restriction of d to the subgroup B defines a differential operator on B; also, d maps cosets of A (modulo B) into cosets, and defines an endomorphism of the factor group A/B which is obviously a differential operator also. Hence we can form the derived groups, $\mathcal{H}(A)$, $\mathcal{H}(B)$, and $\mathcal{H}(A/B)$. The inclusion homomorphism of B into A induces a homomorphism i: $\mathcal{H}(B) \to \mathcal{H}(A)$ of the derived groups in a natural way. Similarly, the natural homomorphism $A \to A/B$ which assigns to each element $x \in A$ its coset, $x + B$, induces a homomorphism j: $\mathcal{H}(A) \to \mathcal{H}(A/B)$. Finally, by using the differential operator d, it is possible to define[3] a homomorphism d': $\mathcal{H}(A/B) \to \mathcal{H}(B)$ in a natural way. These three homomorphisms, i, j, and d', fit together to form a triangle,

$$\begin{array}{ccc} \mathcal{H}(B) & \longrightarrow & \mathcal{H}(A) \\ & \nwarrow \quad \swarrow & \\ & \mathcal{H}(A/B) & \end{array}$$

which may be easily proved to be exact.

5. Spectral sequences of differential groups. The algebraic apparatus we have described so far has been more or less standard in this subject for many years now. We shall now take up an idea of much more recent vintage, which has been of central importance in much of the rapid progress that algebraic topology has made in the last five years.

DEFINITION. An infinite sequence of differential groups, (A_n, d_n), $n = 1, 2, 3, \cdots$, is called a *spectral sequence*[4] in case each group in

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[3] If $u \in \mathcal{H}(A/B)$, then $d'(u)$ is defined as follows: Choose a cycle z belonging to the homology class u. Then z is a coset, $z = y + B$, where $y \in A$. The fact that z is a cycle implies that $d(y) \in B$. Then $d'(u)$ is defined to be the homology class of $d(y)$. This definition can be shown to be independent of the choices of z and y.

[4] The notion of a spectral sequence is due mainly to J. Leray. For references to early work on the subject, see [2].

the sequence is the derived group of its predecessor, i.e.,

$$A_{n+1} = \mathcal{H}(A_n).$$

Associated with any spectral sequence is a certain *limit group*. Roughly speaking, this limit group is defined as follows. Let $\mathcal{Z}(A_n)$ denote the kernel of d_n, as before, and let

$$\kappa_n : \mathcal{Z}(A_n) \to A_{n+1}$$

be the natural homomorphism which is defined by assigning to each element of the subgroup $\mathcal{Z}(A_n)$ its coset modulo $\mathcal{B}(A_n)$. Then κ_n is a homomorphism of a subgroup of A_n onto A_{n+1}. Consider the sequence of groups A_n and homomorphisms κ_n, $n=1, 2, \cdots$. This sequence of groups and homomorphisms almost satisfies the standard definition of a "direct sequence" of groups; it fails only because the homomorphisms κ_n are not defined over all of A_n. By a slight modification of the usual definition of the limit group of a direct sequence of groups, one defines the limit group[5] of the sequence $\{A_n, \kappa_n\}$.

6. **An example.** We shall now illustrate by means of an example how some of the main results of algebraic topology can be best stated in terms of spectral sequences.

One of the most important concepts of present day topology is that of a *fibre space*. Not only does this concept occur in many different situations in topology itself, but it also is of central importance in some other branches of mathematics. Fibre spaces arise everywhere in differential geometry in the consideration of tangent spaces and tensor spaces over differentiable manifolds; and it is a natural concept in considering a coset space of a Lie group. The notion of a fibre space is a generalization of the familiar idea of the Cartesian product of two spaces. In fact, Pontrjagin has called them "skew products."

Various different definitions are now current for the term "fibre space." For our purposes it will suffice if we say that a fibre space is locally a product space. To be precise, a fibre space is a quadruple (B, p, X, F), where B, X, and F are topological spaces, called respectively the "total space," the "base space," and the "fibre," and p is a continuous map of B onto X, called the "projection." These four things are required to satisfy the following condition: For any point $x \in X$ there exists a neighborhood V of x and a homeomorphism ϕ of $V \times F$ onto $p^{-1}(V)$ such that

$$p[\phi(v, y)] = v$$

[5] For the benefit of the interested reader, we give the precise details in Appendix I.

for any points $v \in V$ and $y \in F$.

Given any two topological spaces X and F, one can always construct the following trivial fibre space having X for base space and F for fibre: choose B to be the product $X \times F$, and let p be the projection of $X \times F$ onto X. The main interest attaches to those fibre spaces which are *not* product spaces "in the large." An example of a nontrivial fibre space is obtained by letting B be a Möbius strip, X be the circumference of a circle, and F a segment of the real line. The definition of the projection p: $B \rightarrow X$ is left to the reader. Another example is obtained by taking B to be a covering space of X, and p: $B \rightarrow X$ the usual projection of the covering space onto the base space. In this case the fibre is a discrete space. In any case, one can usually construct many different (i.e., non-isomorphic) fibre spaces with a given base space and fibre.[6]

It is an important problem for many different reasons to determine what relations exist between the homology (or cohomology) groups of the total space, base space, and fibre in a fibre space. Various partial results along this line were obtained as long ago as the 1930's. However, the first nearly complete answer to this problem was obtained by J. Leray[7] about 1947. His final results for cohomology groups may be stated as follows. Let (B, p, X, F) be a fibre space, and G an abelian group. Then there exists a spectral sequence (E_n, d_n), $n = 2, 3, \cdots$, and a decreasing sequence of subgroups of $H^*(B, G)$,

$$H^*(B, G) = D^0 \supset D^1 \supset D^2 \supset \cdots \supset D^p \supset \cdots$$

such that the following two facts are true:

(a) The first term of the spectral sequence, E_2, is naturally isomorphic[8] to $H^*(X, H^*(F, G))$.

(b) Let E_∞ denote limit group of this spectral sequence. Then E_∞ is naturally isomorphic to the weak direct sum, $\sum_p D^p/D^{p+1}$.

At first sight, this result does not look as if it would be very helpful in deducing precise relations between the cohomology groups of B, X, and F. However, it is much more useful than would at first appear due to the following additional facts:

(c) As mentioned above, cohomology groups of topological spaces

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[6] A closely related concept is that of a fibre bundle. A fibre bundle is a fibre space which has certain additional elements of structure attached to it.

[7] This result of Leray's was first announced in various notes in C. R. Acad. Sci. Paris, cf. the references given in [2] and [3].

[8] To be strictly correct, E_2 is isomorphic to the cohomology group of X with the local coefficient system $H^*(F, G)$. However the formulation of the text is correct in most cases in which spectral sequences are actually applied to fibre spaces (e.g., the case in which X is simply connected).

are graded groups. Therefore $E_2=H^*(X, H^*(F, G))$ is "bi-graded," i.e., is the direct sum of the *doubly* indexed family of subgroups, $H^p(X, H^q(F, G))$. We shall denote $H^p(X, H^q(F, G))$ by $E_2^{p,q}$.

(d) In most cases that occur in practice, $H^*(X, H^*(F, G))$ can be expressed in terms of $H^*(X, G)$ and $H^*(F, G)$ by means of tensor products and other purely algebraic operations, provided G is chosen suitably.

(e) The group E_3 inherits a bi-graded structure from the bi-graded structure on E_2. That is, E_3 is the direct sum of a doubly indexed family of subgroups, $E_3^{p,q}$, where $E_3^{p,q}$ is the image of a subgroup of $E_2^{p,q}$ under the homomorphism κ_2 described in §5 above. One now sees inductively that all the groups E_n are bi-graded, and the homomorphism κ_n is compatible with the bi-graded structures on E_n and E_{n+1} (i.e., κ_n maps $E_n^{p,q}$ into $E_{n+1}^{p,q}$). This implies that the limit group, E_∞, is also bi-graded.

(f) For any pair of integers p and q, there exists an integer N (depending on p and q) such that the groups $E_n^{p,q}$ are isomorphic for all $n>N$. This implies that the component $E_\infty^{p,q}$ of E_∞ is determined in N steps. Naturally, this fact that any given component of E_∞ can be determined by a finite process is of essential importance.

Practically all the known results about the cohomology groups of a fibre space can be derived from this fundamental theorem of Leray. And this is in spite of the fact that the spectral sequence says nothing about the various group extensions involved if one tries to determine $H^*(B, G)$ from E_∞. There is also the further difficulty that, in various applications, the problem of computing the successive E_r's and d_r's is not effectively solved.

An analogous result about the homology groups of a fibre space is due to J. P. Serre [5].

7. **Exact couples.** The question naturally arises, how are spectral sequences of a fibre space defined? Here again it would take us too long to go into details, but there is a certain algebraic mechanism which gives rise to a spectral sequence which can be easily described. This algebraic mechanism seems to be applicable to many problems of topology.

DEFINITION. An *exact couple*[9] is an exact triangle in which two of the groups are the same; to be precise, an exact couple consists of two abelian groups, A and C, and three homomorphisms, $f\colon A\to A$, $g\colon A\to C$, and $h\colon C\to A$. These homomorphisms are required to satisfy the following "exactness" conditions:

[9] Exact couples were introduced by the author; cf. [4].

$$f(A) = g^{-1}(0), \; g(A) = h^{-1}(0), \text{ and } h(C) = f^{-1}(0).$$

These three conditions can be easily kept in mind if one makes the following triangular diagram,

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ & {}_{h}\nwarrow \quad \swarrow_{g} & \\ & C & \end{array}$$

and observes that the kernel of each homomorphism is required to be the image of the preceding homomorphism. We shall denote such an exact couple by the notation $\langle A, C; f, g, h\rangle$, or more briefly $\langle A, C\rangle$.

There is an important operation which assigns to an exact couple $\langle A, C; f, g, h\rangle$ another exact couple $\langle A', C'; f', g', h'\rangle$, called the *derived* exact couple. This derived exact couple is defined as follows.

Define an endomorphism d: $C\rightarrow C$ by $d = g \circ h$. Then $d^2 = d \circ d = g \circ h \circ g \circ h = 0$, since $h \circ g = 0$ by exactness. Therefore d is a differential operator on C. Let $C' = \mathcal{H}(C)$, the derived group of the differential group (C, d). Let $A' = f(A)$, which is a subgroup of A. Define $f' = f|A'$, the restriction of f to the subgroup A'. The homomorphism h': $C'\rightarrow A'$ is induced by h; it is readily verified that $h[\mathcal{Z}(C)]\subset A'$, and $h[\mathcal{B}(C)] = \{0\}$, hence h induces a homomorphism of the factor group $C' = \mathcal{Z}(C)/\mathcal{B}(C)$ into A'. The definition of g': $A'\rightarrow C'$ is more complicated. Let $a\in A'$; choose an element $b\in A$ such that $f(b) = a$. Then $g(b)\in \mathcal{Z}(C)$, and $g'(a)$ is defined to be the coset of $g(b)$ modulo $\mathcal{B}(C)$. It is easily verified that this definition is independent of the choice made of the element $b\in A$, and that g' is actually a homomorphism.

Of course, it is necessary to verify that the homomorphisms f', g', and h' satisfy the exactness condition of an exact couple. This verification is purely mechanical.

It is clear that this process of derivation can be applied to the derived exact couple $\langle A', C'; f', g', h'\rangle$ to obtain another exact couple $\langle A'', C''; f'', g'', h''\rangle$, called the *second derived exact couple*, and so on. In general, we shall denote the nth derived couple by $\langle A^{(n)}, C^{(n)}; f^{(n)}, g^{(n)}, h^{(n)}\rangle$, and $d^{(n)} = g^{(n)} \circ h^{(n)}$: $C^{(n)}\rightarrow C^{(n)}$ will denote the differential operator on $C^{(n)}$. Then the sequence of differential groups, $(C^{(n)}, d^{(n)})$, is readily seen to be a spectral sequence. It will be reerred to as *the spectral sequence associated with the exact couple* $\langle A, C\rangle$.

We can now indicate how the spectral sequence of a fibre space is defined. Let (B, p, X, F) be a fibre space as described above, and let $C^*(B)$ denote the group of cochains of B (we are omitting the coefficient group G from our notation). The fibre space structure on B defines in a rather natural way a nested sequence of subgroups[10] of the group $C^*(B)$:

$$C^*(B) = A^{-1} \supset A^0 \supset A^1 \supset A^2 \supset \cdots \supset A^p \supset \cdots .$$

Each of these subgroups is stable under the coboundary operator, δ:

$$\delta(A^p) \subset A^p.$$

Hence we can form the derived groups, and there is associated with each adjacent pair of groups in this sequence an exact triangle, as described above:

$$\begin{array}{ccc} \mathcal{H}(A^{p+1}) & \xrightarrow{\ i_p\ } & \mathcal{H}(A^p) \\ & {\scriptstyle \delta_p'} \nwarrow \quad \swarrow {\scriptstyle j_p} & \\ & \mathcal{H}(A^p/A^{p+1}) & \end{array}$$

Now define graded groups A and C as follows:

$$A = \sum_n \mathcal{H}(A^n), \qquad C = \sum_n \mathcal{H}(A^n/A^{n+1}).$$

Then the homomorphisms i_n, j_n, and δ_n' define homomorphisms $i: A \to A$, $j: A \to C$, and $\delta': C \to A$ respectively, and the necessary exactness conditions are fulfilled, so that $\langle A, C; i, j, \delta' \rangle$ is an exact couple. The spectral sequence associated with this exact couple is the desired spectral sequence of the fibre space (B, p, X, F).

It should be mentioned here that Leray originally obtained the spectral sequence of a fibre space directly from the nested sequence of subgroups, $A^0 \supset A^1 \supset \cdots$, without the use of the intermediate notion of an exact couple. However, it is probably easier to keep in mind the definition of an exact couple and its derived exact couple than it is to remember Leray's method. In addition, the method of exact couples applies to some situations to which Leray's method does not apply (see the examples below).

8. **Some examples of other applications of spectral sequences and exact couples.** For our first example, let X be a connected topological space, and let $\tilde{X}$ be a covering space of X. Let Π denote the fundamental group of X, and $\tilde{\Pi}$ the subgroup of Π to which $\tilde{X}$ corresponds. We shall assume that $\tilde{X}$ is a *regular* covering; this means that $\tilde{\Pi}$ is a normal subgroup of Π. Also, this condition implies that the quotient

[10] For the benefit of the reader who has some familiarity with the concepts of singular cohomology theory, we indicate one way of defining this nested sequence of subgroups in Appendix II.

group $\Pi/\tilde{\Pi}$ operates on $\tilde{X}$ as a group of homeomorphisms, none of which have any fixed points (with the exception of the identity element, of course). One may consider that X is obtained from $\tilde{X}$ by identifying points which correspond under the operations of $\Pi/\tilde{\Pi}$.

It is a problem of long standing in topology to determine what relations must exist between the homology (or cohomology) groups of X and $\tilde{X}$. Any such relations will clearly have also to involve something about the groups Π and $\tilde{\Pi}$; for, it is known that given any subgroup $\tilde{\Pi}$ of Π, one can construct a covering $\tilde{X}$ of X which corresponds to the subgroup $\tilde{\Pi}$.

In 1948, H. Cartan[11] proved the following result: There exists a spectral sequence (E_n, d_n), $n=2, 3$, and a nested sequence of subgroups of $H^*(X)$,

$$H^*(X) = D^0 \supset D^1 \supset D^2 \supset \cdots \supset D^p \supset \cdots,$$

such that the following two facts are true:

(a) The first term of the spectral sequence, E_2, depends only on the factor group, $\Pi/\tilde{\Pi}$, the cohomology group $H^*(\tilde{X})$, and the manner in which $\Pi/\tilde{\Pi}$ operates on $H^*(\tilde{X})$. (Note: since $\Pi/\tilde{\Pi}$ operates on $\tilde{X}$, it also operates on $H^*(\tilde{X})$.) To be precise, E_2 is isomorphic to *the cohomology group of the group* $\Pi/\tilde{\Pi}$ *with coefficients* $H^*(\tilde{X})$,

$$E_2 = H^*(\Pi/\tilde{\Pi}, H^*(\tilde{X})).$$

(b) Let E_∞ denote the limit group of this spectral sequence. Then E_∞ is naturally isomorphic to the direct sum $\sum_p D^p/D^{p+1}$.

The remarks (c), (d), (e), and (f) that were made in §6 about the spectral sequence of a fibre bundle apply verbatim to this spectral sequence also.

In the statement of this result, there occurs a concept which we have not mentioned before: the cohomology group of an arbitrary abstract group Π (which need not be abelian) with coefficients in an abelian group G, denoted by $H^*(\Pi, G)$. This concept (which is purely algebraic) had its origin in some purely topological results of Hurewicz, but since then it was proven of use in algebraic number theory.

It should also be pointed out that although a covering space is a special case of a fibre space, this spectral sequence of H. Cartan is *not* the same as that obtained by applying the spectral sequence of Leray to the case of a covering space. The spectral sequence of Leray is of little interest in the case of a covering space.

Our second example refers to the homotopy groups of a space.

[11] H. Cartan announced this result without proof in [1]; he has not published a complete proof as yet, except in some mimeographed notes of limited circulation.

The homotopy groups of a space X, denoted by $\pi_p(X)$ $(p \geqq 1)$, were mentioned in passing at the beginning of this address. They are topological invariants of X, as are the homology and cohomology groups of X. In some problems they give more complete information about a space, or give deeper insight into its topology. Unfortunately it is usually much more difficult to determine the homotopy groups of X than it is to determine the homology or cohomology groups. Any theorem which contributes to a better understanding of the structure of the homotopy groups of spaces is usually rather significant. Spectral sequences come into this picture as follows: Given any topological space X, there exists a spectral sequence[12] (E_n, d_n), $n=2, 3, \cdots$, such that the limit group E_∞ bears the same relation to homotopy groups of X that the limit group of the spectral sequence of a fibre space bears to the cohomology groups of the total space (see §6). The first term, E_2, is not completely known. It is a bigraded group, and some of its homogeneous components are ordinary homology groups of X. Much of our meager knowledge about the homotopy groups of a general space can be derived from the existence of this spectral sequence. Leray's method for defining spectral sequences does not apply to this example.

It should be mentioned that an analogous result holds for the Borsuk-Spanier cohomotopy groups of a finite-dimensional space X. There exists a spectral sequence,[13] (E_n, d_n) such that the limit group, E_∞, is closely related to the cohomotopy groups of X, and the initial term, E_2, is closely related to the cohomology groups of X.

Our final example is of a purely algebraic nature. As mentioned above, if Π is an arbitrary group, then the cohomology group $H^*(\Pi, G)$ (with coefficients in the abelian group G) is defined. Suppose Π' is a normal subgroup of Π. Then the question arises, What relations exist between the three cohomology groups, $H^*(\Pi, G)$, $H^*(\Pi', G)$, and $H^*(\Pi/\Pi', G)$? This question is answered by a theorem[14] of Serre and Hochschild: There exists a spectral sequence (E_n, d_n) such that $E_2 = H^*(\Pi/\Pi', H^*(\Pi'))$, and the limit group E_∞ is related to $H^*(\Pi, G)$ in a way we have described before (cf. the discussion of the spectral sequence of a fibre bundle above).

9. **Conclusion.** The main point of this paper may be stated as follows: The algebraic notion of a spectral sequence is essential for the statement of many basic theorems of algebraic topology. These

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[12] This spectral sequence was introduced by the author in [**4**, part II].

[13] This spectral sequence was introduced by the author in [**4**, part III].

[14] This theorem is stated and proved in [**6**].

spectral sequences often arise as the associated spectral sequences of certain exact couples.

10. Appendix I: The precise definition of the limit groups of a spectral sequence. Let (A_n, d_n), $n=1, 2, \cdots$, be a spectral sequence, and let κ_n: $Z(A_n) \to A_{n+1}$ be as defined in §5. Define a homomorphism κ_n^p of a certain subgroup of A_n onto A_{n+p} by composition of the homomorphisms $\kappa_n, \kappa_{n+1}, \cdots, \kappa_{n+p-1}$:

$$\kappa_n^p = \kappa_{n+p-1} \circ \cdots \circ \kappa_{n+1} \circ \kappa_n.$$

Let $\overline{A}_n$ denote the subgroup of A_n consisting of those elements $a \in A_n$ such that $\kappa_n^p(a)$ is defined for all values of p. Define $\bar{\kappa}_n$: $\overline{A}_n \to \overline{A}_{n+1}$ to be the restriction of κ_n to the subgroup $\overline{A}_n$. Then the sequence of groups $\{\overline{A}_n\}$ and homomorphisms $\{\bar{\kappa}_n\}$ is a direct sequence of groups in the usual sense, and its limit group is defined to be the limit group of the given spectral sequence.

11. Appendix II: The definition of the nested sequence of subgroups of the group of cochains of a fibre space. Let (B, p, X, F) be a fibre space, as defined in §6, such that the base space X and the fibre F are finite polyhedra. Choose a simplicial decomposition of X which is fine enough to satisfy the following condition: For any simplex $\sigma \subset X$, there exists a homeomorphism ϕ of $\sigma \times F$ onto $p^{-1}(\sigma)$ such that

$$p[\phi(x, y)] = x$$

for any points $x \in \sigma$ and $y \in F$. It is readily seen that such simplicial decompositions must exist. Let X^n denote the n-dimensional skeleton of X with respect to this simplicial decomposition (i.e., the union of all simplexes of dimension $\leqq n$), and let $B^n = p^{-1}(X^n)$ for $n = 0, 1, 2, \cdots$.

Let $C^*(B, G)$ denote the group of singular cochains of B with coefficients in G. An element $f \in C^*(B, G)$ is a function which assigns to a singular simplex T in B an element $f(T) \in G$. Define A^n to be the subgroup of $C^*(B, G)$ consisting of those cochains which vanish on all singular simplexes contained in the subspace B^n of B. Then

$$C^*(B) \supset A^0 \supset A^1 \supset A^2 \supset \cdots \supset A^n \supset \cdots$$

and

$$\delta(A^n) \subset A^n$$

as required. For full details of this method, see [**4**, part V].

This method does not apply in case X is not a polyhedron. The method to be used in the general case, using "cubical" singular cochains, has been described by J. P. Serre [5].

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