THE NOVEMBER MEETING IN LOS ANGELES

The four hundred eighty-seventh meeting of the American Mathematical Society was held at the University of Southern California, Los Angeles, California, on Saturday, November 29, 1952. Attendance was approximately 100, including the following 81 members of the Society:

M. G. Arsove, E. W. Barankin, E. F. Beckenbach, M. M. Beenken, Clifford Bell, R. E. Bellman, L. D. Berkovitz, H. F. Bohnenblust, J. V. Breakwell, L. H. Chin, E. A. Coddington, L. M. Coffin, J. M. Danskin, G. B. Dantzig, P. H. Daus, A. C. Davis, A. H. Diamond, R. P. Dilworth, Milton Drandell, Roy Dubisch, L. K. Durst, H. A. Dye, Arthur Erdélyi, Harley Flanders, G. E. Forsythe, I. L. Glicksberg, J. W. Green, L. D. Gregory, C. J. Halberg, Jr., L. A. Henkin, M. R. Hestenes, P. G. Hodge, Jr., P. G. Hoel, D. G. Humm, Rufus Isaacs, P. B. Johnson, Jan Kalicki, L. D. Kovach, D. H. Lehmer, R. B. Leipnik, H. A. Linstone, M. M. Loève, Charles Loewner, N. M. Martin, H. A. Osborn, T. K. Pan, Emanuel Parzen, R. S. Phillips, D. H. Potts, W. T. Puckett, Jr., R. M. Redheffer, Edgar Reich, R. E. Roberson, J. B. Robinson, Raphael M. Robinson, G. F. Rose, Peter Scherk, Raymond Sedney, Bernard Sherman, Seymour Sherman, I. M. Singer, James C. Smith, Ernst Snapper, R. H. Sorgenfrey, D. V. Steed, Robert Steinberg, E. G. Straus, A. C. Sugar, J. D. Swift, Alfred Tarski, A. E. Taylor, F. B. Thompson, R. A. Wagner, L. F. Walton, W. R. Wasow, M. A. Weber, P. A. White, G. T. Whyburn, B. R. Wicker, F. L. Wolfe, P. S. Wolfe.

The two sections at 10:00 A.M. for contributed papers were presided over by Dr. G. E. Forsythe and Professor Ernst Snapper. At 2:00 P.M., Professor M. M. Loève of the University of California, Berkeley, delivered the invited address, *Ergodic theorems in probability theory.* Professor Loève was introduced by Professor P. G. Hoel. At 3:15 there was a section for late papers, at which Professor D. V. Steed presided.

After the meetings, those attending were the guests of the Mathematics Department of the University of Southern California at tea at the Town and Gown.

Abstracts of papers presented at the meeting follow. Those preceded by the letter "*t*" were presented by title. Paper 114 was presented by Professor Tarski, paper 116 by Professor Kalicki, and paper 137 by Professor Scherk. Mr. Mack was introduced by Professor G. C. Evans, Mr. Scott by Professor Kalicki, and Mr. Adams by Professor J. C. C. McKinsey.

Algebra and Theory of Numbers

110*t*. Anne C. Davis: *Square roots of some denumerable order types.*

All solutions of $\xi^2=\alpha$ are found, when α is an order type of the form $\alpha=\tau\cdot\eta$, τ denumerable and scattered (Bull. Amer. Math. Soc. Abstract 58-4-330; Davis and Sierpinski, C. R. Acad. Sci. Paris vol. 235 (1952) pp. 850–852). Every order type μ is (essentially) uniquely representable as an ordered sum $\mu = \sum_{x \in M} \phi(x)$ where $\phi(x) \in R = \{\omega, \omega^*, \omega^*+\omega, 1, \cdots, n, n+1 \cdots\}$, and, if x immediately precedes x' in M, then $\phi(x)+\phi(x') \notin R$. For every type μ let $R(\mu)$ be the range of ϕ in this representation. Let A, B respectively denote the set of nonzero denumerable scattered types τ of the form $\tau = 1+\tau'$, $\tau = \tau''+1$. Let $S(\tau)$ be the set of all solutions of $\xi^2 = \tau \cdot \eta$. Results: (1) If $\tau \in A \cap B$, $S(\tau) = \{\tau \cdot \eta\}$. (2) If $\tau \in A-B$, or if $\tau \in B-A$, $S(\tau) = \{\tau \cdot \eta, \tau \cdot \eta + \tau\}$, or $S(\tau) = \{\tau \cdot \eta, \tau + \tau \cdot \eta\}$. (3) If $\tau \notin A$, $\tau \notin B$ then; (3a) If $R(\tau) = \{\omega^*+\omega\}$, $S(\tau)$ consists of $\tau \cdot \eta$ and of all $\xi = \sigma + \tau \cdot \eta + x$, where $x + \sigma = \tau$, $\sigma \notin A$, $x \notin B$, and $\xi \leqq S(\tau) \leqq \aleph_0$; (3b) If $\{\omega\} \subseteq R(\tau) \leqq \{\omega, \omega^*+\omega\}$, $S(\tau)$ consists of $\tau \cdot \eta$, $\tau \cdot \eta + \tau$, and of all $\zeta = \tau + \tau \cdot \eta + x$, where $x + \tau = \tau$, $x \notin B$, and $S(\tau) \in \{\zeta \cdot \aleph_0\}$; (3c) The case when $\{\omega^*\} \subseteq R \leqq \{\omega^*, \omega^*+\omega\}$ is symmetric with (3b); (3) If $n \in R(\tau)$, n finite, or if $\{\omega, \omega^*\} \subseteq R(\tau)$, $S(\tau) = \{\tau \cdot \eta, \tau + \tau \cdot \eta, \tau \cdot \eta + \tau\}$. (Received October 20, 1952.)

111. Harley Flanders: *The norm function of an algebraic field extension.*

Let K be an extension of finite degree n of an algebraic field k. With respect to a basis $\omega_1, \cdots, \omega_n$ of K over k, one may define the general norm function $N(X)$. It is a polynomial over k, homogeneous of degree n in the n variables $X = (X_1, \cdots, X_n)$. It is proved that $N(X)$ is equal to an irreducible polynomial raised to the power $e = n/m$, where m is the maximum of the field degrees $(k(\theta): k)$ for all θ in K. This result is applied to the problem of characterizing the field norm $N_{K/k}$ by means of its algebraic properties. (Received October 17, 1952.)

112*t*. Alfred Horn: *On the eigenvalues of a matrix with prescribed singular values.*

H. Weyl [Proc. Nat. Acad. Sci. U. S. A. vol. 35 (1949) pp. 408–411] has shown that if A is a matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$ and singular values $\alpha_1, \cdots, \alpha_n$ (the singular values are the non-negative square roots of the eigenvalues of A^*A), and if (1) $\alpha_1 \geqq \cdots \geqq \alpha_n \geqq 0$, then (2) $|\lambda_{i_1}, \cdots, \lambda_{i_k}| \leqq \alpha_1 \cdots \alpha_k$ whenever $1 \leqq i_1 < \cdots < i_k \leqq n$, $1 \leqq k \leqq n$, and (3) $|\lambda_1 \cdots \lambda_n| = \alpha_1 \cdots \alpha_n$. In this paper it is shown that the simultaneous relations (2), (3) are best possible. In other words, given any pair of n-tuples $(\lambda_1, \cdots, \lambda_n)$, $(\alpha_1, \cdots, \alpha_n)$ satisfying (1), (2), and (3), then there exists a matrix with the former as eigenvalues and the latter as singular values. This solves a question raised by Seymour Sherman. (Received October 10, 1952.)

113*t*. J. R. Jackson: *Extension of relations on finite sets to partial orderings.*

A *strict partial ordering* on set A is a transitive relation $\prec$ on A such that $a \prec a$ is false for $a \in A$. The result given here is easily translated into the language of "ordinary" partial orderings. Let R be a relation on finite set A, represented by a segment of integers: $A = \{1, 2, \cdots, N\}$. Define $N \times N$ matrix $P = (p_{mn})$ by setting $p_{mn} = 1$ if mRn, $p_{mn} = 0$ otherwise. Let $P + P^2 + \cdots + P^N = \overline{P} = (\overline{p}_{mn})$. Define relation $\overline{R}$ on A by including $m\overline{R}n$ if and only if $\overline{p}_{mn} > 0$ (this definition can be made more elegant by appropriately choosing the p_{mn} from the Boolean algebra with two elements). Then the following six conditions are equivalent: (i) $P^N = 0$; (ii) tr $\overline{P} = \overline{p}_{11} + \overline{p}_{22} + \cdots + \overline{p}_{NN} = 0$; (iii) there exists an $N \times N$ permutation matrix G such that all the nonzero entries of $G'PG$ are below the main diagonal; (iv) R can be extended to a strict partial ordering on A (and hence, in fact, to a complete ordering of A); (v) $\overline{R}$ is a strict partial ordering on A; (vi) $\overline{R}$ is the (unique) minimal extension of R

to a strict partial ordering on A. (Prepared under Contract Nonr 233(02), Office of Naval Research.) (Received October 14, 1952.)

114. Bjarni Jónsson and Alfred Tarski: *Factor relations over algebras*. Preliminary report.

Let $\mathfrak{A}=\langle A, f_0, f_1, \cdots \rangle$ be any algebra. (For notations see Jónsson-Tarski, Amer. J. Math. vol. 73, pp. 894 ff.) $S\perp S'$ means that S, S' are *complementary* congruence relations over $\mathfrak{A}$, i.e., $S|S'=A\times A$ and $S\cap S'$ is the identity relation on A. S is called a *factor relation*, $S\in \boldsymbol{FR}$, if $S\perp S'$ for some S'. The following conditions are equivalent: (α) $\boldsymbol{FR}$ is a Boolean algebra under $S|T$ and $S\cap T$; (β) $S|T=T|S$ and $(S|T)\cap S'\subseteq T$ whenever $S\perp S'$, $T\in\boldsymbol{FR}$; (γ) $(S|T|S)\cap S'\subseteq T$ whenever $S\perp S'$, $T\in \boldsymbol{FR}$. As is known, every algebra satisfying (α) has at most one direct decomposition into indecomposable factors. In particular, (α) holds for every $\mathfrak{A}=\langle A, +\rangle$ such that $x+x=x$ for every $x\in A$ and for any $x_1, x_2, x_3\in A$ there is a $z\in A$ with $x_i+z=z+x_i=z$ [or $x_i+z=z+x_i=x_i$], $i=1, 2, 3$. Under a modified definition of $S\perp S'$, (α)-(γ) remain equivalent for algebras in the wider sense; (α) holds for every $\mathfrak{A}=\langle A, R\rangle$ where R is a reflexive, antisymmetric relation and for any $x_1, x_2, x_3\in A$ there is a $z\in A$ with x_iRz [or zRx_i], $i=1, 2, 3$. This improves the known factorization theorems for lattices, doubly directed systems, etc. (Birkhoff, *Lattice theory*, 1948, pp. 25–27; Nakayama, Math. Japonicae vol. 1, p. 49; Tarski, *Cardinal algebras*, pp. 277, 308). (Received October 21, 1952.)

115t. Bjarni Jónsson and Alfred Tarski: *Decomposition functions on algebras*.

For notations see preceding abstract. Let $\mathfrak{A}=\langle A, +\rangle$. A function f on $A\times A$ to A is called a *decomposition function* (on $\mathfrak{A}$), $f\in\boldsymbol{DF}$, if $f(x, x)=x$, $f(x, f(y, z))=f(f(x, y), z)=f(x, z)$, and $f(x+z, y+u)=f(x, y)+f(z, u)$ for all $x, y, z, u\in A$. If $S\perp S'$ and, for all $x, y\in A$, $f(x, y)$ is defined as the only $z\in A$ with $xSzS'y$, then $f\in\boldsymbol{DF}$; conversely, if $f\in\boldsymbol{DF}$, and S and S' are defined by $xSy\leftrightarrow f(x, y)=y$, $xS'y\leftrightarrow f(x, y)=x$, then $S\perp S'$. Thus, the study of direct decompositions reduces to that of $\boldsymbol{DF}$. Given $u\in A$ and $f\in\boldsymbol{DF}$, let $f_u(x)=f(x, u)$ and $f'_u(x)=f(u, x)$ for $x\in A$. Then $f_u(x+y)=f_u(x'+y')$ whenever $f_u(x)=f_u(x')$ and $f_u(y)=f_u(y')$, and hence f_u is a homomorphism on $\mathfrak{A}$; if $u+u=u$, then $f_u(x+y)=f_u(x)+f_u(y)$, and f_u is an endomorphism; similarly for f'_u. Condition (α) of preceding abstract is equivalent to each of the following conditions: (ϵ) $f_ug_u=g_uf_u$ for all $f, g\in\boldsymbol{DF}$ and every $u\in A$; (ζ) $f_ug_u=g_uf_u$ for all $f, g\in\boldsymbol{DF}$ and some $u\in A$; (η) for all $f, g\in\boldsymbol{DF}$ and $x_1, x_2, x_3, x_4\in A$ there is a $u\in A$ such that $f_u(g_u(x_i))=g_u(f_u(x_i))$ for $i=1, 2, 3, 4$. (This theorem proves helpful in deriving some results of preceding abstract.) With appropriate modifications the results extend to arbitrary algebras in the wider sense. (Received October 16, 1952.)

116. Jan Kalicki and Dana Scott: *Some equationally complete algebras*. Preliminary report.

Continuing the investigations of Bull. Amer. Math. Soc. Abstract 58-6-582 it was proved that (1) $\mathfrak{A}=\langle A, '\rangle$ with one unary operation is equationally complete if and only if either $x'=x$ whenever $x\in A$ or $x=y'$ whenever $x, y\in A$; (2) a lattice is equationally complete if and only if it is distributive; (3) a group is equationally complete if and only if it is Abelian and each of its elements is of the same prime order. This applies to groups considered as systems $\langle G, \cdot\rangle$, $\langle G, \cdot, ^{-1}\rangle$, or $\langle G, \cdot, ^{-1}, 1\rangle$; (4) a ring $\mathfrak{R}=\langle R, \cdot, +, -, 0\rangle$ with an element $z\neq 0$ for which $z\cdot z=0$ is equationally complete if and only if, whenever $x, y\in R$, $x\cdot y=0$ and $px=0$ ($1x=x$, $(k+1)x=kx+x$, and p

is a prime); (5) every ring for which there is a prime p such that for every element x, $px=0$ and $x^p=x$ ($1x=x1=x$, $(k+1)x=kx+x$, $x^{k+1}=x^k\cdot x$) is equationally complete; (6) every Boolean algebra is equationally complete (this result is essentially known from the literature); (7) a Brouwerian algebra is equationally complete if and only if it is Boolean; (8) a closure algebra $\langle A, \cup, \cap, -, C\rangle$ is equationally complete if and only if $Cx=x$ whenever $x\in A$; (9) a relation algebra is equationally complete if and only if it is a Boolean relation algebra. (Received October 14, 1952.)

117. M. S. Klamkin: *On transcendental numbers generated by the method of Liouville.*

If the number $\sum_1^\infty 10^{-a_n}$ can be shown to be transcendental by application of Liouville's theorem on algebraic numbers, it is then shown that integral sequences $\{b_n\}$ of lower order than $\{a_n\}$ $[b_n=o(a_n)]$ can be chosen such that the number $\sum_1^\infty 10^{-b_n}$ is also transcendental. (Received October 14, 1952.)

118. L. D. Kovach: *On elementary nilpotent algebras.*

An elementary nilpotent algebra of degree n over a field $\mathfrak{F}$, denoted by $\mathfrak{C}_l^{(n)}$, is defined to be any isomorphic copy of the algebra of $n\times n$ matrices over $\mathfrak{F}$ with zeros on and above the main diagonal and beyond the lth column. Interest in this class of algebras stems from the fact that $\mathfrak{C}_l^{(n)}$ is a characteristic ideal of the total nilpotent algebra $\mathfrak{C}_{n-1}^{(n)}$. The latter is significant by virtue of a theorem proved by Dubisch and Perlis (*On total nilpotent algebras*, Amer. J. Math. vol. 73 (1951) pp. 439–452), namely, that the total nilpotent algebras together with their subalgebras constitute the totality of associative nilpotent algebras. In the present paper the results of Dubisch and Perlis are generalized to elementary nilpotent algebras. All automorphisms and all ideals of $\mathfrak{C}_l^{(n)}$ are explicitly determined and it is determined which ideals are characteristic. (Received July 2, 1952.)

119. Dana Scott: *Algebraic characterization of equational completeness.* Preliminary report.

For terminology see Bull. Amer. Math. Soc. Abstract 58-6-582. Let $P_n[\mathfrak{A}]$ be the algebra of polynomial functions of n arguments over the algebra $\mathfrak{A}$, i.e. the subalgebra of the infinite direct power of $\mathfrak{A}$ generated by the n identity functions $f_i(x_1, \cdots, x_n)=x_i$ for $i=1, \cdots, n$. $\mathfrak{A}$ is a *P_n-algebra* if $\mathfrak{A}$ is isomorphic to $P_n[\mathfrak{A}]$. $\mathfrak{A}$ is *P_n-simple* if there are no proper homomorphs of $\mathfrak{A}$ that are P_n-algebras. S is an *EC-extension* of $\mathfrak{A}$ if there is an algebra $\mathfrak{B}$ in the same similarity class as $\mathfrak{A}$ such that $\mathfrak{B}\in EC$, $Id(\mathfrak{A})\subseteq Id(\mathfrak{B})$, and $S=Id(\mathfrak{B})$. Results: (1) if $\mathfrak{A}\in EC$, then for every $n>1$, $P_n[\mathfrak{A}]$ is P_n-simple and $Id(\mathfrak{A})=Id(P_n[\mathfrak{A}])$; (2) if $\mathfrak{A}$ is a P_n-simple P_n-algebra with at least two elements and $n>1$, then $\mathfrak{A}\in EC$; (3) if $\mathfrak{A}$ is a finite algebra, then there are only finitely many EC-extensions of $\mathfrak{A}$. (Received October 14, 1952.)

Analysis

120. M. G. Arsove: *A convergence theorem for δ-subharmonic functions.*

Let $\{w_n\}$ be a sequence of (quasi) δ-subharmonic functions on a region Ω, having mass distributions of uniformly bounded total variation. If the functions w_n all lie between two δ-subharmonic functions and converge point-wise (quasi everywhere) on Ω to a function w, then there exists a δ-subharmonic function w^* differing from w

at most on a set of capacity zero. Moreover, the mass distributions for $\{w_n\}$ converge vaguely to that for w^*. This theorem, which contains as a particular case the theorem of Brelot-Cartan for increasing sequences of subharmonic functions, is proved by introducing a special mode of convergence in the Banach space of functions of potential type and employing the following results: (i) every set of positive interior capacity supports a distribution of the unit mass having a continuous potential; (ii) (Choquet) all Borel sets are capacitable. (Without the Choquet theorem, the exceptional set can only be shown to be of zero *interior* capacity.) The derivation does not employ the Hilbert space technique of Cartan (or the intermediary of functions of finite energy), and thus affords an independent proof of the Brelot-Cartan theorem. (Received October 14, 1952.)

121. E. F. Beckenbach: *Some results in geometric function theory.*

Concerning functions $f(z)$ which are analytic in the unit circle $|z| = r < 1$, certain inequalities are known to hold only for restricted values of r. By means of properties of convex functions, bounds valid for all $r<1$ are now obtained in some of these situations. Thus it is known that if on each concentric circle of radius $r<1$ the mean of $|f(z)|$ is $\leqq 1$, then the mean of $|f(z)|^2$ is $\leqq 1$ for $r^2 \leqq 1/2$; it is now shown that the latter mean is $\leqq 1/[2(1-r^2)]$ for $1/2 \leqq r^2 < 1$. (Received November 26, 1952.)

122*t*. Ben Bernholtz: *Analytic functions in locally convex spaces.*

The theory of analytic functions with arguments and values in a Banach space is generalized to the case of functions with arguments and values in complete locally convex complex linear topological spaces (l.t.s.) L and L' respectively. A function $f(x)$ on L to L' is analytic in the domain $D \supset L$ if it is single-valued, continuous, and G-differentiable in D. An analytic function is F-differentiable and can be expanded in an infinite series of F-differentials. This theory rests heavily on that for analytic functions $x(\zeta)$ of a complex variable ζ with values in a complete locally convex complex l.t.s. L. In the latter case, $x(\zeta)$ analytic means $x^*[x(\zeta)]$ is analytic in the classical sense for all x^* in L^*, the space of continuous linear functionals on L. Series expansions for these analytic functions are obtained and from them expansions for G-differentiable functions are deduced. (Received September 16, 1952.)

123. Rufus Isaacs: *Pursuit games.*

The two players each control certain aspects of the motion of two points; P, the pursuer, and E, the evader. Either motion may suffer restrictions as to velocity, acceleration, curvature, etc., and each player may enjoy full or partial knowledge of the other's position. A variety of games may be formulated with such pay-off's as time of capture. A strategy consists in specifying a player's navigation for every possible position. Some such games have been solved; others embody formidable difficulties. (Received October 15, 1952.)

124. R. B. Leipnik: *A generalization of the Hahn-Banach theorem.*

Let S be an abelian group, S' a boundedly complete lattice-ordered abelian group. A pair (f, g) is called an outside gauge if $f(0)=0$, $f(x-y) \leqq f(x) - g(y)$, and an inside gauge if $g(0)=0$, $g(x-y) \geqq g(x) - f(y)$, where f and g are functions on S to S'. In case (f, g) is both an inside and an outside gauge, it is called a dual guage. If h satisfies $f(z) \geqq h(z) \geqq g(z)$ for each z in domain h, then h is said to be gauged by (f, g). Theorem: If a convex (concave) or linear h is gauged by an outside (inside) gauge (f, g), then h can be extended to a function H on domain $f \cap$ domain g which is convex (concave) or linear and is gauged by (f, g). If S and S' are real vector spaces, then sub (super)-homogeneity of h is retained in H. If (f, g) is taken to be a dual gauge, then this reduces to the Hahn-Banach Theorem. (Received October 16, 1952.)

125. S. F. Mack: *The discontinuity of the second normal derivative of the potential.* Preliminary report.

O. D. Kellogg (*Derivatives of harmonic functions*, Trans. Amer. Math. Soc. vol. 33 (1931) pp. 486–510) gave sufficient conditions with respect to smoothness of surface boundary and boundary values in order that derivatives of harmonic functions take on limiting values on the surface. The following statement applies to discontinuities across the surface boundary even when the derivatives do not exist at the boundary. Let the surface S in the neighborhood of a point q be of class C_2 with reference to the tangent plane at q, and z, $-z$ be equal distances along the normal on either side of the tangent plane; and let $u(p)$ be the potential of a density distribution σ which is continuous at q. Then on the normal at q, $\lim_{z\to 0}\{(d^2u/dn^2)_z-(d^2u/dn^2)_{-z}\}=-8\pi k\sigma$, where k is the mean curvature of S at q. The proof is obtained by referring S to a surface tangent to S at q, the normal sections of which are circles osculating the normal sections of S. For a closed surface conductor the above theorem is equivalent to $d^2u/dn_+^2=2k(q)du/dn_+$. (Received October 14, 1952.)

126. R. S. Phillips: *An inversion formula for Laplace transforms and semi-groups of linear operators.*

This paper is primarily concerned with the problem of determining necessary and sufficient conditions that a closed linear operator be the infinitesimal generator of a semi-group of bounded linear transformations. The principal tool is an inversion formula for Laplace transforms defined as follows: Suppose $\int_0^\infty \exp(-\omega\sigma)|f(\sigma)|d\sigma<\infty$. Set $F(\lambda)=\int_0^\infty \exp(-\lambda\sigma)f(\sigma)d\sigma$ and $f_\lambda(s)=\exp(-\lambda s)\sum(-1)^n(\lambda^2 s)^{n+1}[n!(n+1)!]^{-1}F^{(n)}(\lambda)$. Then $\lim_{\lambda\to\infty}f_\lambda(s)=f(s)$ a.e. Now let $T(s)$ be a semi-group of transformations such that $\int_0^1\|T(\sigma)\|d\sigma<\infty$. Then $T(s)$ is said to belong to class $(1, A)$ or $(1, C)$ according as it is strongly Abel or Cesaro summable to the identity at $s=0$. For $T(s)$ of class $(1, A)$, the infinitesimal generator A is not in general closed. However the smallest closed extension $\overline{A}$ (called c.i.g.) exists and the Laplace transform of $T(s)$ is the resolvent of $\overline{A}$. Necessary and sufficient conditions that a closed linear operator be the c.i.g. of a semi-group of class $(1, A)$ (or $(1, C)$) are found. The foregoing results are then applied to a perturbation theory for semi-groups. It is shown that if $\overline{A}$ is the c.i.g. of a semi-group of class $(1, A)$ (or $(1, C)$) and if B is a linear bounded operator, then $\overline{A}+B$ is likewise the c.i.g. of a semi-group of class $(1, A)$ (or $(1, C)$). (Received August 26, 1952.)

127. R. M. Redheffer: *An inequality for entire functions of finite type.*

Let $F(z)$ be an entire function such that lim sup $[\log|F(z)|]/|z|\leqq 1$, as $|z|\to\infty$; and suppose $F(x)$ is real, $-\infty<x<\infty$. Then the inequality $|F(x)|^2\leqq A$ implies $|F(x)|^2+|F'(x)|^2\leqq A$, $|F'(x)|^2+|F''(x)|^2\leqq A$, and so on. If any of these relations becomes an equality at a point where the higher of the two derivatives is not zero, then $F(z)=\pm A\cos(\pm z+B)$. (Received November 21, 1952.)

128t. R. M. Redheffer: *Completeness of* $\{e^{i\lambda_n x}\}$.

Let $\{\lambda_n\}$ be a set of positive numbers, $\lambda_{n+1}\geqq\lambda_n$, let $\Lambda(u)$ be the number of λ_n's $\leqq u$, and assume $\Lambda(u)/u\leqq M$. If $\limsup_{x\to\infty}(\Lambda[x+x(\log x)^a]-\Lambda(x))/x(\log x)^a\leqq d$

for $a<-1$, then the set $\{e^{\pm i\lambda_n x}\}$ cannot be complete L^2 on an interval of length greater than $2\pi d$. This improves a previous result of the author, and verifies a conjecture of Beurling. When $\lambda_{n+1}-\lambda_n \geqq c$ and no further assumption is made, the set $\{e^{i\lambda_n x}\}$ cannot be complete on an interval of length $>2\pi/c$. If a set $\{e^{i\lambda_n x}\}$ is complete on a closed interval of length d, and remains complete when N terms but not when $N+1$ terms are removed, Paley and Wiener call N the *excess* of the set. If $|\Lambda(u)-du| \leqq A$, then the excess does not exceed $[4A-3/2]$; and there is a sequence with $|\Lambda(u)-du| \leqq A$ for which the excess is as large as $[4A-2]$. These statements improve and complete a result of Paley and Wiener. (Received November 21, 1952.)

129*t*. R. M. Redheffer: *Entire functions with zeros having a density.*

Let $F(x)=\Pi(1-x^2/\lambda_n^2)$, where $\lambda_{n+1}\geqq\lambda_n$ are positive numbers, let $\Lambda(u)$ represent the number of λ_n which are $\leqq u$, and assume $\Lambda(u)/u\leqq M$. If $\lim_{x\to\infty}\{\Lambda(vx)/vx-\Lambda(x/v)/x/v\}=0$ whenever $0<v<1$, then $\lim_{x\to\infty}\log^+|F(x)|/x=0$; and therefore, by the Phragmen-Lindelöf theorem, $|F(re^{i\theta})|=0(e^{\pi d r|\sin\theta|+\epsilon r})$, $\epsilon>0$, $d\leqq M$. In particular the result holds whenever $\lim \Lambda(u)/u=d$ exists. This generalizes a theorem of Carlson, which gives the same conclusion under the hypothesis $\lambda_{n+1}-\lambda_n\geqq c>0$, $\lim\Lambda(u)/u=d$. The converse is false; in fact if a and b are any numbers with $0\leqq a\leqq b$, there exists a sequence $\{\lambda_n\}$ such that $\Lambda(u)/u\leqq M$, $\limsup\{\Lambda(vx)/vx-\Lambda(x/v)/(x/v)\}=b$, $\liminf=a$, for a fixed v less than a preassigned δ, yet such that $\log^+|F(x)|/x\to 0$. It is possible to have arbitrarily large gaps free of zeros; if $M>0$, there is a sequence $\{\lambda_n\}$ such that $\Lambda(u)/u<M$, $\lambda_{n+1}-\lambda_n\geqq c$ for some $c>0$, $\Lambda(x_i)=\Lambda(y_i)$ for an infinite sequence $x_i>y_i\to\infty$, $x_i/y_i\to\infty$, and yet $\log^+|F(x)|/x\to 0$. (Received November 21, 1952.)

130*t*. R. M. Redheffer: *Entire functions with zeros uniformly distributed.*

In the notation of the previous abstract, if $|\Lambda(u)-du|\leqq A$ for $A\geqq 1/2$, $d>0$, then there is a constant C such that $|F(x)|<Cx^{4A-2}$. This improves a result of Paley and Wiener, who obtained the exponent $4A-1$. Moreover, there exists a set $\{\lambda_n\}$ such that $\lambda_{n+1}-\lambda_n\geqq 1/d$, $|\Lambda(u)-du|\leqq A$, and $|F(x)|>C_1x^{4A-2}$ for an infinite sequence $x=x_i\to\infty$. One can give estimates for the constants C in terms of λ_0, d, and A. If $a(u)$ is a positive function such that $a'(u)$ tends monotonically to 0 as $u\to\infty$, then the condition $|\Lambda(u)-du|<a(u)$ implies $\log^+|F(x)|<g(x)$, where $g(x)\sim[2a(x)-1]\log[x^2(d/\lambda_0)a(x)]+2a(x)$. Similar results allow comparison of two functions $F_1(x)$, $F_2(x)$; for example if $|\Lambda_1(u)-\Lambda_2(u)|\leqq A$, then $|F_1(x)-F_2(x)|\leqq Cx^{4A-2}$. Let K denote the class of entire functions which are $o(e^{|z|})$ as $|z|\to\infty$, and which belong to L^2 on $(-\infty,\infty)$. Let a be fixed and satisfy $0<a<2\pi$. Then the class of functions $G'(z)$, where G ranges over K, is identical with the class of functions $H(z+a)-H(z)$, where H ranges over K; but if $a\geqq 2\pi$, the latter class is a proper subset of the former. (Received November 21, 1952.)

131. R. E. Roberson: *The polynomial transform.*

The definition of a "polynomial transform" of a number sequence given by F. W. Bubb is generalized by the introduction of a multiplicative function $\phi(x)$ whose form can be chosen with some latitude. It is shown that such a transform exists if and only if the terms of the sequence can be bounded by a function which is transformable in the bilateral Laplace sense. Properties are given for the generalized form of the transform. Provided certain restrictions be placed on the choice of $\phi(x)$ (which are implicit in the particular choice of Bubb's definition), the polynomial transform is shown to be precisely the bilateral Laplace transform of a function constructed from the number sequence by means of an interpolating function $\Phi(t) = \mathfrak{L}_{II}^{-1}\{\phi(e^{-s})\}$. (Received October 9, 1952.)

Applied Mathematics

132t. G. W. Evans, II. *Applications of the Mauro Picone theorem for heat conduction.*

Use is made of a restricted statement of a theorem due to M. Picone (Math. Ann. vol. 101 (1929) pp. 701–712), which essentially asserts that a nonconstant solution of the heat conduction equation must have its maximum and minimum values along the boundary of the half-open xt-region of description, to prove the uniqueness of the solution $u(x, t)$ of the differential equation $\alpha u_{xx} = u_t - Q(t)$ for $0 \leqq x < a$, and the differential equation $\alpha u_{xx} = u_t$ for $a < x \leqq L$. The initial condition is $u(x, 0) = f(x)$ and the boundary conditions are $u_x(0, t) = 0$, $u_x(a+, t) = u_x(a-, t) = u_x(a, t)$, $u(a+, t) = u(a-, t) = u(a, t)$, and $ku_x(L, t) = h(t)[u(L, t) - g(t)]$, where $Q(t)$, $h(t)$, and $g(t)$ are known functions. The Picone theorem is also used to analyze the action of a thermal controlling unit, which consists of two thermocouples, for an oven. (Received September 22, 1952.)

133. P. G. Hodge: *On plastic strains in slabs with cutouts.*

The complete elastic-plastic problem is set up for a circular slab with a central circular cutout, subjected to uniform external tension. The analysis is carried out under the assumption of generalized plane stress but with possibly finite deformations. The material is assumed to be isotropic, homogeneous and incompressible, to yield according to the Tresca yield condition, and to satisfy Hooke's law in the elastic domain and the plastic potential law in the plastic domain. The equations are solved by a perturbation method based on the ratio of the maximum shear stress to the shear modulus, in which each of the significant quantities, stress, displacement, and slab thickness, is expanded in a power series in this ratio. A similar technique is employed to solve the fully plastic flow problem. It is shown that for cutout radii greater than one percent of the radius of the slab, the "first approximations" obtained by neglecting all but the leading term in each series are satisfactory, up to loads at which the slab becomes wholly plastic. The ratio of the maximum strain in the just fully plastic slab to that in the completely elastic slab is computed. It is found that this ratio is less than about 6 if the cutout radius is at least one percent of the radius of the slab. Some observations are advanced on the case where the cutout is very small compared to the slab. (The results presented in this paper were obtained in the course of research conducted under Contract N7onr-35810 between the Office of Naval Research and Brown University.) (Received September 2, 1952.)

134. Edgar Reich: *A random walk related to the capacitance of the circular-plate condenser.*

Let C be the capacitance of a circular parallel-plate condenser of radius a, and plate separation ta. It follows from the integral equation of E. R. Love that $C = a/\pi$ times the mean number of steps of the following random walk: The walk starts at a random point in $[-1/t, 1/t]$; each step is chosen from a Cauchy distribution with unit semi-interquartile range; the walk ends when a step crosses a boundary point of $[-1/t, 1/t]$. Since this discrete random walk can be looked upon as a continuous Cauchy "Brownian" motion (such as studied by M. Kac and H. Pollard), *observed* at discrete times, a simple interpretation for the value of "stray" capacitance is possible. (Received October 10, 1952.)

135. W. R. Wasow: *On the differential equation for the stability of plane Couette flow.*

The differential equation $u^{(4)}-2\alpha^2u''+\alpha^4u+\lambda^2z(u''-\alpha^2u)=0$, where α is a real constant, is solved asymptotically for large values of the complex parameter λ, when z ranges over a full complex neighborhood of the origin. The results are used to reduce the problem whether plane Couette flow is stable at large Reynolds numbers to the question whether all roots of the equation $\int_{-\infty}^{z}Ai(-t)dt=0$ lie in the sector $|\arg z| < \pi/6$. Here $Ai(z)$ denotes Airy's integral. It is also proved that small disturbances of Couette flow do not possess the inner viscous layer occurring for other flow profiles. This is shown to be a rather special property of the differential equation under consideration, not shared, e.g., by the otherwise very similar differential equation $u^{(4)}+\lambda^2x(u''+u)=0$. (Received September 12, 1952.)

Geometry

136. J. W. Green: *Curves encircling a cylinder.*

Let C be a curve encircling a right cylinder of unit radius in such a manner that each element of the cylinder contains a single point of C. It is further required that the maximum distance between points of C not exceed 2. Under these conditions bounds are found for the length L and the height H of C, where by H is meant the minimum distance between two planes normal to the cylinder and containing C between them. These bounds are respectively: $2\pi \leqq L \leqq 2^{3/2}\pi$ and $0 \leqq H \leqq 2^{1/2}$. The bounds are the best possible and are all attainable, with the possible exception of the upper bound for length. (Received October 15, 1952.)

137. N. D. Lane and Peter Scherk: *Differentiable points in conformal plane.*

The authors define the differentiability of a point on an arc in conformal plane by means of two conditions involving pencils of circles. An analysis of these conditions yields a classification of the differentiable points. (Received October 14, 1952.)

138*t*. T. K. Pan: *A proof of a sufficient condition that two surfaces be applicable.*

On page 176 of his book *Differential geometry* (Macmillan, 1947), W. C. Graustein stated that two surfaces are applicable if and only if they can be mapped geodesically so that the total curvatures in corresponding points are equal. That the condition is necessary is obvious. The proof of the sufficiency of the condition was divided into two cases. Graustein proved one case and omitted the other, saying that it can be shown though not without difficulty. This note offers a simple complete proof with the use of the tensor calculus. (Received October 14, 1952.)

139. T. K. Pan: *Variation of congruences of curves of an orthogonal ennuple in a Riemannian space.*

Consider any three congruences of an orthogonal ennuple at a point of a Riemannian space. When one congruence is moved by local and a second congruence is moved by parallel displacement in the direction of the third congruence, the rate of change of cosine of the angle between the first two congruences is well known as Ricci's coefficient of rotation and has been considerably studied. This paper investigates the corresponding rate of change when the third congruence is replaced by an arbitrary one. Variation of one congruence with respect to another is defined and in terms of which equidistance and parallelism of congruences are expressed. (Received October 14, 1952.)

Logic and Foundations

140. Ernest Adams: *A representation theorem for classical rigid body mechanics.*

A representation theorem is proved for systems of classical rigid body mechanics, as axiomatized jointly by Herman Rubin and the author. This theorem establishes that for every system of rigid body mechanics there exists an isomorphic system of pseudo-rigid bodies: where by a psuedo-rigid body is meant a sub-system of a system of classical particle mechanics (Bull. Amer. Math. Soc. Abstract 57-6-524 of McKinsey, Sugar, and Suppes) such that the distance between any two particles of the sub-system remains constant over time. (Received October 15, 1952.)

141. L. A. Henkin: *On certain subsystems of the second-order functional calculus.*

One formulation of the second-order calculus employs two axiom schemata of the following form: $(\alpha)A \supset B$. In schema (i), α is any individual variable and B is formed from the wff A by replacing all free occurrences of α by some other symbol of the same type. In (ii), α is any functional variable and B is obtained from A by the rather complicated rule of substitution for such variables. Following an old suggestion of Leśniewski, it is shown that if (i) is strengthened to (i′) by allowing "α" to range over variables of any type, then (ii) may be replaced by the simpler schema (iii): $(\exists\gamma)(\beta_1)\cdots(\beta_n)(\gamma(\beta_1, \cdots, \beta_n)\equiv A)$, where A is any wff not containing the n-ary function symbol γ. Let Σ be the formal system obtained from this formulation of the second-order calculus by omitting (iii). Then the formal theorems of Σ are precisely those wffs which are valid with respect to all models in which the range of n-ary functional variables may be an arbitrary set of n-ary relations of individuals. Interesting subsystems of second-order calculus are thus obtained by adding to Σ a form of (iii) in which "A" is restricted to some special class of wffs; in particular, if the class of formulas of first order calculus is chosen, the resulting system is closely connected to the Gödel set-theory. (Received October 16, 1952.)

Statistics and Probability

142t. Aryeh Dvoretzky, Jack Kiefer, and Jacob Wolfowitz: *Sequential decision problems for processes with continuous time parameter. Estimation of parameters.*

The problem considered is that of sequential point and interval estimation (through continuous observation in time) for continuous time parameter stochastic processes. For example, for certain weight functions and a variety of cost functions, minimax sequential procedures are obtained for estimating the unknown parameter of the Wiener, Poisson, negative binomial, and gamma processes. The results for the last three processes contain as special cases new results on sequential estimation with discrete time parameter. (Received September 12, 1952.)

143*t*. Aryeh Dvoretzky, Jack Kiefer, and Jacob Wolfowitz: *Sequential decision problems for processes with continuous time parameter. Testing hypotheses.*

The problem considered is that of testing sequentially (through continuous observation in time) which of two laws represents the true distribution of a stochastic process with continuous time parameter. Results on Bayes' solutions, the optimum property of the sequential probability ratio test, Wald's equation, and fundamental identity, as well as other results from the discrete time case, carry over to the present case. In many examples, several of which are treated in detail, the *exact* operating characteristic function and the average sample number function can be found. The theory of general sequential decision problems may also be extended to the continuous time parameter case. (Received September 12, 1952.)

144*t*. Emanuel Parzen: *Invariance principle for certain uniform probability limit theorems.*

Erdös, Kac, and Donsker (Memoirs of the American Mathematical Society, no. 6, 1951) have established an "invariance principle" for certain probability limit theorems. The author has recently obtained a uniform continuity theorem, determining conditions in terms of characteristic functions that a sequence of distribution functions converge uniformly in a parameter, and has applied this theorem to determine conditions that a sequence of independent random variables X_n whose distribution functions $G_n(x, \theta)$ contain an unknown parameter θ satisfy the central limit theorem uniformly in θ, that the maximum likelihood estimate be uniformly asymptotically normally distributed, etc. These results are used to adapt Donsker's proof of the invariance principle to show that if the X_n satisfy the central limit theorem uniformly in θ then the distributions of various functions of the consecutive sums of the sequence also approach their limiting distributions uniformly in θ. (Received October 16, 1952.)

145*t*. Jacob Wolfowitz: *Consistent estimators of the parameters of a linear structural relation.*

Let $X=\xi+u$, $Y=\alpha+\beta\xi+v$, where ξ, u, v are chance variables such that u and v are jointly normally distributed with zero means and unknown covariance matrix, ξ is distributed independently of (u, v), and the distribution of ξ is not normal. If $F(x)$ and $G(x)$ are two distribution functions, we define the distance $\delta(F, G)$ between them as $\sup_x |F(x)-G(x)|$. Let N^* be the class of all normal distributions with mean zero, and $\delta(F, N^*)=\inf_{N\in N^*}\delta(F, N)$. Let (x_i, y_i), $i=1, \cdots, n$, be a sequence of independent observations on (X, Y). Let $A(x|c_1, c_2)$ be the empiric distribution function of $\{y_i-c_1-c_2x_i\}$, $i=1, \cdots, n$. Let $a_n(x_1, \cdots, x_n, y_1, \cdots, y_n)$ and $b_n(x_1, \cdots, x_n, y_1, \cdots, y_n)$ be Borel measurable functions of the arguments exhibited such that $\delta(A(x|a_n, b_n), N^*)<1/n+\inf_{c_1,c_2}\delta(A(x|c_1, c_2), N^*)$. Then a_n and b_n converge stochastically to α and β, respectively. The method can be generalized to more than two variables, to non-normal variables, and to other problems in estimation and testing hypotheses. Other definitions of distance can be used. (Received March 24, 1952.)

J. W. Green,
Associate Secretary