BOOK REVIEWS

Handbuch der Laplace-Transformation. Vol. 1. *Theorie der Laplace-Transformation.* By G. Doetsch. (Mathematische Reihe, vol. 14.) Basel, Birkhäuser, 1950. 15+581 pp.

The theory of the Laplace and of the Laplace-Stieltjes transforms is essentially a creation of the last thirty years though much of the theory may be said to be inspired by the somewhat older theory of Dirichlet series. The organization of the theory of Laplace-Stieltjes transforms some twenty odd years ago unified the theory of Dirichlet series with the much less developed theory of Laplace integrals and with the theory of analytic almost periodic functions. In the unified theory the Laplace transform

$$\mathfrak{L}\{F\} = f(s) = \int_0^\infty e^{-st}F(t)dt$$

plays a preferred role owing to its importance for the applications. It enters in a natural manner in the theory of boundary value problems and much of the Heaviside Ars conjectandi can be rationalized by a judicious use of Laplace transforms. There are also close contacts with analytic function theory and with the theory of Fourier transforms. The Laplace transform would seem to be assured of a place in the sun for some time to come merely on the score of its wide range of applications.

Gustav Doetsch has devoted almost thirty years of his life to the theory of Laplace transforms and their applications to boundary value problems, especially for the heat equation. Since his connection with the Technological Institute of Stuttgart in 1924–31 he has always paid much attention to the applications and he seems to have kept close and mutually profitable contacts with engineers and physicists. He writes with an eye to this larger audience and he wants to be understood by people who have an immediate use for the results. Naturally, these are not always the results having significance, for instance, in the theory of locally compact groups, supposedly the standard of excellence in up-to-date analysis.

This is the author's fourth book on Laplace transforms. It was preceded by *Theorie und Anwendung der Laplace-Transformation,* Berlin, Springer, 1937, *Tabellen zur Laplace-Transformation und Anleitung zum Gebrauch,* Berlin and Göttingen, Springer, 1947, jointly with H. Kniess and D. Voelker, and *Die zweidimensionale Laplace-Transformation,* Basel, Birkhäuser, 1950 (volume 12 of the same series), jointly with D. Voelker. In the first book, the author's concern with the applications and with the qualifications of the students of applied mathematics led him to dismiss the Lebesgue integral. This attitude has now changed radically, the Lebesgue integral is taken as the basic concept and the author makes a rather eloquent plea for its acceptance and use by applied mathematicians. To help the less qualified readers there is an excellent appendix containing what is needed of auxiliary theorems in real and complex function theory. There are also occasionally alternate proofs provided based on the Riemann integral.

The present treatise, of which the first volume is now before us, is written with skill and painstaking care. It provides a clear, easily readable and detailed introduction to the theory. Within the limits set by the author, it carries the reader practically to the borders of present day knowledge. Naturally there are omissions and, this being a critical review, some omissions will be pointed out below. But the main structure is there in all its splendor and there is an abundance of special results, nowhere else to be found.

While the Laplace-Stieltjes transform as well as the Mellin transform are mentioned and discussed in some detail, the treatise is concerned chiefly with the Laplace transform proper, both the unilateral and the bilateral cases. It is quite natural that the author should treat the Laplace-Stieltjes transform somewhat briefly; this corresponds both to his own predominant interests and to the fact that the Laplace-Stieltjes transform has been discussed rather fully in D. V. Widder's treatise. The author very carefully points out the differences in function theoretical behavior between Laplace-Stieltjes transforms and proper Laplace transforms. Perhaps the reader is apt to get a somewhat exaggerated view of these differences: after all any Dirichlet series can be converted into a Laplace transform by division by a logarithmico-exponential function of arbitrarily slow growth.

The theory is divided into five parts: Basic analytical and function theoretical properties of the Laplace transformation. Inversion of the Fourier and Laplace transformations, Parseval's identity and related questions. A generalization of the Laplace transformation. The Laplace transforms of special classes of functions. Abelian and Tauberian theorems.

Part I starts out with some generalities concerning linear transformations and the basic concepts of functional analysis. Having defined the various forms of transforms to be studied in the treatise, the author goes very fully into the question of how operations on the elements of the domain of the Laplace transform are reflected in the range. This technique is highly useful in the applications for much of the power of the Laplace transformation is derived from the fact that it translates transcendental operations on the domain, such as differentiation, integration, convolution, and Hankel's transform, into algebraic operations on the range such as multiplication of the transform by a power or by another transform or a simple change of variables. This part ends with the discussion of the various half-planes of convergence, boundedness, and holomorphism. Here the notion of order of boundedness is a concept introduced by the author.

Part II is concerned chiefly with the inversion formulas: the complex integral formula, the Phragmén operator, and the operators of Post, Widder, and Boas, as well as inversion by series expansions. In connection with the latter, there are some results of W. B. Caton and the reviewer which escaped the author's attention owing to war conditions though the paper appears in the Bibliography. Chapter 5 contains new material; it is concerned with the representation of the integral over the interval $(0, T)$ by a contour integral, involving the transform, with applications to Dirichlet series. Chapter 6, on Parseval's identity, contains a number of interesting results. The following strikes the reviewer as rather peculiar (Theorem 3, p. 257): If $f_n(s) = \mathfrak{L}[F_n]$ and $e^{-x_0 t}F_n(t) \in L_1(0, \infty) \cap L_2(0, \infty)$, then $f_1(s)$ and $f_2(s)$ are orthogonal to each other over every vertical line $x = \xi$ with $\xi > x_0$.

Part III is comparatively brief, just one chapter devoted to the (C, k)-summability of Laplace integrals which is discussed in some detail and contains original material. One misses a reference to the papers of M. Riesz in Acta Litt. ac Scient., Szeged, volumes 1 and 2, 1923–1924, where general Laplace-Stieltjes transforms are considered. A proof of the convexity of the abscissa of summability as function of the index would also be relevant. There is no discussion of other methods of summability; the reviewer naturally misses a reference to the work of Hille and Tamarkin where the problem of summability is subsumed under that of representing a function as the quotient of two Laplace integrals.

Part IV is concerned with the Laplace transforms of (1) entire functions of exponential type, (2) functions analytic in a strip (bilateral transforms), or in a sector (Mellin transforms), and (3) functions in $L_2(0, \infty)$. These are among the most important special classes to which the Laplace transform applies. Since the book is so full of special results, it is really a pity that most of the results of

Hausdorff concerning logarithmico-exponential functions are omitted. They are both elegant and useful.

The final part V deals with Abelian and Tauberian theorems and ends with the Ikehara theorem. The Wiener theorem is mentioned but not proved in detail and most of the work is based on the more elementary methods of Hardy, Littlewood, and Karamata. This part also contains original material and is basic for the applications to asymptotic representations.

The book contains at least fifteen research problems inserted in the text, relating to open questions in the theory. Some are fairly easy to solve on the basis of available results, others will require a more substantial effort.

References occupy a section of twelve pages and there is an excellent bibliography. The press work is beautiful and misprints very rare.

EINAR HILLE

Leçons de logique algébrique. By H. B. Curry. (Collection de Logique Mathématique, Série A, Monographies réunies par Mme Destouches-Fevrier, no. II.) Paris, Gauthier-Villars, 1952. 183 pp. $4.86.

This expository monograph is the outcome of a course given at Louvain in 1950–51. Although the important theoretical results have appeared in the works of the author and others, it is a valuable contribution to the literature by reason of its lucid architectonic treatment of elementary mathematical logic from the viewpoint of modern mathematics.

Chapter 1 treats of the nature and methodology of formal systems as previously proposed [*A theory of formal deducibility*, Notre Dame Mathematical Lectures, no. 6, University of Notre Dame, Notre Dame, Ind., 1950]. Mathematics is defined as the study of formal systems and mathematical logic is concerned with those formal systems which have some connection with philosophical logic.

These lectures are restricted to the logical algebras as characterized in Chapter II. Algebras are formal systems with free but no bound variables and with a fundamental transitive relation. Where this relation is reflexive, the algebra is called relational or in its reduced form, logistic. Logical algebras are relational or logistic algebras with two binary operators (sum and product) and the idempotent laws (tautology). Those considered of interest are also (at least) general lattices. A variety of interpretations are given, some propositional.

Postulates for semi-lattices (group logics) and lattices are intro-