PARTITIONING CONTINUOUS CURVES

R. H. BING

I. INTRODUCTION

1. The continuous curve. By a continuous curve we mean a compact, locally connected, metric continuum. For definitions of locally connected, continuum, and other terms used in this paper, see either [18] or [21]. Those who like to visualize topology concretely may wish to think of a continuous curve as a chunk out of Euclidean 3-space—one that is connected (all in one piece), one that is bounded (lies on the interior of a sphere), and one that is locally connected (nearby points belong to small connected subsets). A wad of paper, an irregularly shaped rock, or the earth itself may be considered as examples. However, our remarks about continuous curves will apply equally well to those in Euclidean spaces of all dimensions and to those in a Hilbert space.

When Jordan first introduced the term continuous curve, he defined it analytically to be the image (in the plane) of a straight line interval under a continuous transformation. It was not until over twenty years later that it was discovered that any compact locally connected metric continuum was the image of a straight line interval under a continuous transformation and conversely. This interesting and unusual discovery adds spice to the study of mathematics [24, p. 12]. Another interesting aspect of this discovery is that it was made independently by two mathematicians, Hahn and Mazurkiewicz. Since Peano had shown earlier that a square plus its interior is the image of a straight line interval, a continuous curve is sometimes called a Peano continuum.

In this discussion we shall be interested in the continuous curve itself and not the continuous transformation of an interval. Hence, we use the definition in the first paragraph rather than the analytic one. In this discussion we shall be interested in the structure of a continuous curve.

2. Examples of continuous curves. A straight line interval, a square plus its interior in the plane, and a cube plus its interior in 3-space are examples of continuous curves. In fact, any closed n-cell or n-simplex is an example. One can get a less familiar continuous

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curve by taking a continuum which is the sum of a finite number of dissimilar continuous curves.

If M is any compact point set whatever, it is possible to obtain a continuous curve by adding to M a countable collection of arcs. We use this fact in constructing the following two examples of continuous curves which we shall use.
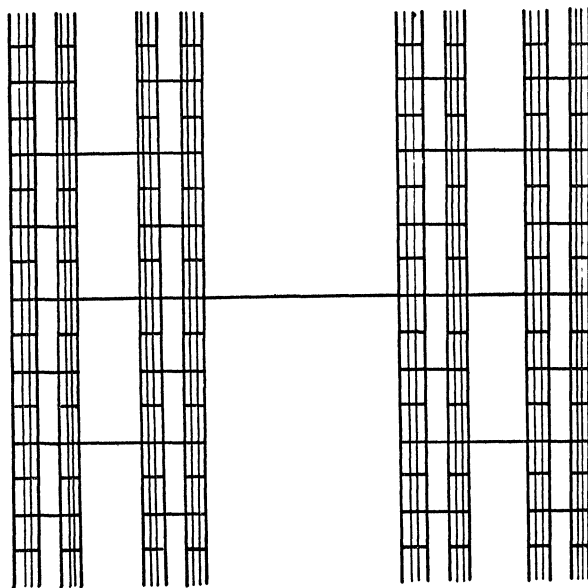

FIG. 1

EXAMPLE A. Let M be the cartesian product of a straight line interval and a Cantor set. Form a continuous curve by adding to M a decreasing sequence of horizontal straight line intervals as indicated in Figure 1. These intervals extend from $(0, 1/2)$ to $(1, 1/2)$, and $(0, 1/4)$ to $(1/3, 1/4)$, from $(2/3, 1/4)$ to $(1, 1/4)$, from $(0, 3/4)$ to $(1/3, 3/4)$, $\cdots$.

EXAMPLE B. It is shown in [5] that there is a bounded continuum S in Euclidean 3-space that is irreducible with respect to separating 3-space and has the property that if two of its subcontinua intersect, one is a subset of the other. The continuum S is 2-dimensional. It is called hereditarily indecomposable because no subcontinuum of it is the sum of two proper subcontinua of the subcontinuum. A continuous curve M may be formed by adding to S the sum of a decreasing sequence of mutually exclusive arcs. If K is a subset of M irreducible

with respect to separating M, each component of K is indecomposable.

3. **Partitioning.** An interval can be partitioned. This fact is used in defining the integral of a continuous function over an interval. ($\int_a^b f(x)dx = \lim \sum f(\xi_i) \Delta x_i$ where ξ_i is a point of and Δx_i is the length of the ith element of the partitioning of $[a, b]$.)

In studying a square plus its interior it is sometimes convenient to partition it into small pieces. For example, in integrating over a square region, we may partition it into small rectangles (if we use rectangular coordinates) or into pieces of other shapes (in polar coordinates for instance).
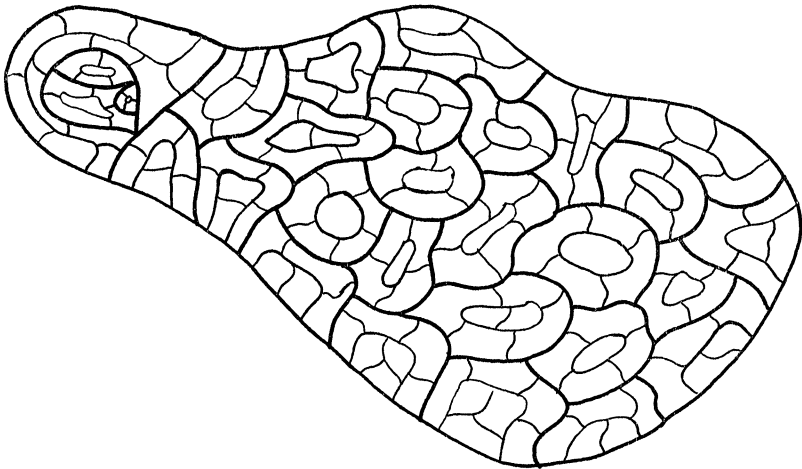

FIG. 2

A partitioning of a continuous curve M is a finite collection G of mutually exclusive connected open subsets of M whose sum is dense in M [6]. If the mesh of G is less than ϵ (each element of G is of diameter less than ϵ), G is called an ϵ-partitioning. The essential difference between a partitioning and a grating [13, p. 279] is that the elements of a partitioning are connected. It is much like a grille decomposition [15].

In defining the integral over a plane domain R ($\iint_R f(x, y)dS = \lim$ (mesh of partitioning $\to 0$) $\sum f(\xi_i, \eta_i)\Delta S_i$) we do not need to impose the conditions that the elements of the partitioning are either connected, mutually exclusive, or open. However, we shall be using partitionings in connections where these properties are important.

4. **Decreasing sequence of partitionings.** If G and H are two partitionings of the same continuous curve, G is called a refinement of H if each element of G is a subset of an element of H. In Figure 2, the partitioning represented by light lines is a refinement of the one represented by heavy lines. A sequence G_1, G_2, $\cdots$ of partitionings is called a decreasing sequence if, for each positive integer i, G_{i+1} is a refinement of G_i and the mesh of G_i approaches 0 as i increases without limit.
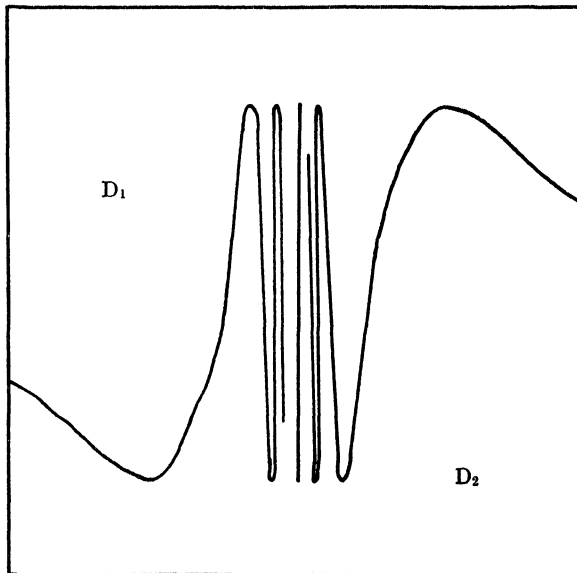

FIG. 3

While it is apparent that there is a decreasing sequence of partitionings of a square plus its interior, it is not so obvious that every continuous curve has such a sequence of partitionings. However, it is true that any continuous curve has such a sequence. This has given us new information about the continuous curve.

5. **Purpose of paper.** It is only within the last three years that it has been known that each continuous curve can be partitioned [6; 4; 15; 16]. It is the purpose of this paper to point out some recent developments, to show ways in which a continuous curve can be partitioned, and to point out analogies between the ways in which a square plus its interior can be partitioned and the ways in which an arbitrary continuous curve can be partitioned. Details of some proofs

are included for completeness, but some readers may prefer to skip them.

II. BREAKING A CONTINUOUS CURVE INTO TWO PIECES

1. The type of division needed. If one pursues the proper attack, it is not difficult to show that any continuous curve can be partitioned. However, if one gets off on the wrong track, considerable difficulty may be encountered.

Suppose it is desired to partition a continuous curve into pieces of diameters less than a fixed positive number ϵ. A straightforward method of attack is to divide the continuous curve into two pieces, divide each of the two pieces, and continue this procedure until an ϵ-partitioning results. However, unless certain precautions are taken, this method does not succeed.

If $\epsilon = 1$, M is a square plus its interior, and at the first stage it is partitioned into two pieces as shown in Figure 3, the above procedure will not lead to a partitioning. The common boundary of D_1 and D_2 is a part of the closure of the graph of $y = \sin 1/x$ and neither D_1 nor D_2 is the sum of a finite number of connected subsets each of diameter less than 1.

A set has property S if, for each positive number ϵ, the set is the sum of a finite number of connected subsets each of diameter less than ϵ. This notion was used by Sierpinski and when R. L. Moore used it later, he called it property S [19; 17; 18].

A precaution to take in partitioning a continuous curve is to divide it at each step so that the pieces have property S.

We shall be interested in the following result [6].

THEOREM 1. *If H and K are two mutually exclusive closed subsets of a continuous curve M, there are two mutually exclusive open subsets D_H and D_K of M containing H and K respectively such that* (a) *each of D_H and D_K has property S and* (b) $M = \overline{D}_H + \overline{D}_K$.

Before proving this theorem, we shall review the method of constructing a set with property S.

2. Gradual growing process. This is a method used to form sets with property S. Compare with [22; 21].

A set H is said to ϵ-grow into a set H' in a set M if M contains H', each point of H' belongs to a connected ϵ-subset (subset of diameter less than ϵ) of H' that intersects H, and for some positive number δ, H' contains all connected δ-subsets of M that intersect H.

Let M be a set with property S, H be a subset of M, and ϵ be a positive number. Suppose one wishes to obtain a subset of M with

property S which contains H but no point whose distance is more than ϵ from H. Such a set can be obtained by the following gradual growing process. Let ϵ_1, ϵ_2, $\cdots$ be a sequence of positive numbers whose sum is no more than ϵ. Let H ϵ_1-grow into a set D_1 in M, D_1 ϵ_2-grow into a set D_2 in M, $\cdots$, D_i ϵ_{i+1}-grow into a set D_{i+1} in M, $\cdots$. Then the sum D_0 of D_1, D_2, $\cdots$ is an open subset of M, has property S, contains H but no point at a distance more than ϵ from H.

3. An attempt to divide. One might try to prove Theorem 1 by letting each of the sets H and K grow gradually in M into mutually exclusive open subsets whose sum is dense in M. Some difficulty may be encountered in trying to keep the two subsets from intersecting and at the same time making their sum dense in M. This difficulty could be overcome if we could answer the following question in the affirmative.

QUESTION. *Does there exist a positive integer n such that the following result holds for each continuous curve M, each positive number ϵ, and each pair of mutually exclusive closed subsets H and K of M? If R is a finite subset of M such that each point of R belongs to an arc in M of diameter less than ϵ that intersects $H+K$, there are two collections A_H and A_K of arcs satisfying the following conditions:*

(a) *Each element of A_H intersects H but not K and each element of A_K intersects K but not H nor any element of A_H.*

(b) *Each element of R belongs to an element of $A_H + A_K$.*

(c) *Each element of $A_H + A_K$ is of diameter less than $n\epsilon$.*

We cannot let $n=1$ and always get such collections A_H and A_K, for let M be the sum of the interval from $(-10, -1)$ to $(10, 1)$ and the interval from $(-10, 1)$ to $(10, -1)$, $H = \{(-10, 1)\}$, $K = \{(10,1)\}$, $R = \{(-10, -1), (10, -1)\}$, and $\epsilon = 11$. Had the answer been in the affirmative for $n=1$, the attempts made in [15] to partition a continuous curve might have succeeded [16].

Since we do not know the answer to the above question, we shall get some other results before proving Theorem 1.

4. A crude division. Before proving Theorem 1, it is convenient to prove a weaker theorem.

THEOREM 1'. *Theorem 1 is true if condition (a) is replaced by the weaker conditions that D_H has property S and D_K has only a finite number of components.*

There are two essentially different methods of proving Theorem 1'. One method is due to Whyburn [22, Corollary, p. 137]. He considers an uncountable family of sets with property S such that of any two

of the sets, one contains the closure of the other. He shows that one of the sets can be modified so that it still has property S and the complement of its closure has a finite number of components.

An essentially different method is to let H grow gradually and K in a wild fashion to form sets D_H and D_K satisfying the conditions of Theorem 1' [6]. This method is similar to another attack we shall use later so we include the details.

Details of second method of proof of Theorem 1'. The open subsets D_H and D_K are formed by permitting H and K to grow so that H grows in a convergent fashion. The procedure is as follows.

We use $\rho(p, q)$ to denote the distance between p and q.

Suppose that, at the ith stage, H and K have been enlarged to sets H_i and K_i such that $\rho(H_i, K_i) > 0$ and each point of M belongs to an arc in M of diameter less than $1/2^i$ that intersects $H_i + K_i$. Let R_i be a finite subset of M such that each point of M belongs to an arc in M of diameter less than $1/2^{i+1}$ that intersects R_i. For each point p of R_i let A_p be an arc in $M - \overline{H}_i$ from p to a point of K_i if there is one; if there is no such arc A_p, let B_p be an arc of diameter less than $1/2^i$ in $M - \overline{K}_i$ from p to a point of H_i.

Use H_i' to denote H_i plus the sum of the B_p's, K_i' to denote K_i plus the sum of the A_p's, and δ_i to denote $\rho(H_i', K_i')$. Let $H_{i+1} = \{p \mid p$ lies on an arc of diameter less than $\delta_i/3$ in M that intersects $H_i'\}$ and $K_{i+1} = \{p \mid p$ lies on an arc of diameter less than $\delta_i/3$ in M that intersects $K_i'\}$.

Since H_i has ϵ_i-grown into H_{i+1} for $\epsilon_i = 7/(2^{i+1} \cdot 3)$ and the ϵ_i's have a finite sum, $\sum H_i = D_H$ has property S.

Since K_i has only a finite number of components and each component of K_{i+1} intersects K_i, $\sum K_i = D_K$ has only a finite number of components.

Since no point of M is at a distance of more than $1/2^i$ from $H_i + K_i$, $M = \overline{D}_H + \overline{D}_K$.

5. **Proof of Theorem 1.** By using Theorem 1', it is shown in Theorem 3' of the next section that for each positive number ϵ, there is an ϵ-partitioning of M. We use this result in establishing Theorem 1. For each integer i, let G_i be a $1/2^i$-partitioning of M.

The general idea is to let both H and K grow gradually into sets D_H and D_K. Suppose at the ith stage H and K have been enlarged to H_i and K_i so that $\rho(H_i, K_i) > 0$ and $H_i + K_i$ intersects each element of G_i. Let $\{p_j\}$ be a finite number of points such that each element of G_{i+1} contains an element of $\{p_j\}$ and each point of $\{p_j\}$ lies in an element of G_i.

If there is an arc in both $M - \overline{H}_i$ and an element of G_i from p_j to a

point of K_i, let A_j be such an arc; if there is no such arc, let B_j be an arc in both $M - \overline{K}_i$ and an element of G_i which is irreducible from p_j to $\overline{H}_i$.

Let $\rho(H_i', K_i') = \delta_i$ where H_i' is the sum of H_i and all the B_j's while K_i' is the sum of K_i and all the A_j's.

Suppose $H_{i+1} = \{p \mid$ there is an arc in M of diameter less than $\delta_i/3$ from p to $H_i' \}$ and $K_{i+1} = \{p \mid$ there is an arc in M of diameter less than $\delta_i/3$ from p to $K_i' \}$.

The sets H_i and K_i have ϵ_i-grown into H_{i+1} and K_{i+1} where $\epsilon_i = 7/(2^{i+1} \cdot 3)$. Hence, the sets $D_H = \sum H_i$ and $D_K = \sum K_i$ have property S. Since $H_i + K_i$ intersects each element of G_i, $M = \overline{D}_H + \overline{D}_K$.

III. PARTITIONING

1. **Treating S-sets as continuous curves.** It is necessary that a set M have property S in order to be partitionable. (A set M is partitionable if for each positive number ϵ, there is an ϵ-partitioning of it.) The converse is true but it is not so obvious. However this converse follows from the facts that a continuous curve is partitionable and that any connected set with property S can be imbedded in a continuous curve so that any partitioning of the continuous curve produces a partitioning of it with no larger mesh.

If G is a partitioning of a continuous curve, the pieces of G are not continuous curves. However, if these pieces have property S, they can be satisfactorily dealt with.

Suppose M is a connected set with property S and $R(p, q)$ is the relative distance for M, that is, $R(p, q)$ is the greatest lower bound of the diameters of all connected subsets of M containing $p + q$. It may be shown that $R(p, q)$ preserves the topology of M and the diameter of any connected set is not increased under $R(p, q)$. Hence, any ϵ-partitioning of M under the metric $R(p, q)$ is an ϵ-partitioning of it under its original metric.

If we add only enough points to M to make it complete under $R(p, q)$, the resulting set M' is a continuous curve. Any ϵ-partitioning of M' induces an ϵ-partitioning of M. In fact we have the following result [6, Lemma 1; 21, pp. 154–158].

THEOREM 2. *If M is a connected set with property S with a relative distance metric, M' is the complete enclosure of M, and D is a connected open subset of M', then $M \cdot D$ is a connected open subset of M.*

2. **Crude partitioning.** Before showing that a continuous curve can be partitioned in various fashions, we show that it can be partitioned in some manner.

THEOREM 3'. *For each positive number ϵ and each continuous curve M, there is an ϵ-partitioning G of M.*

We use the following corollary which results from applying Theorems 1' and 2 to each component of M.

COROLLARY OF THEOREM 1'. *Suppose that the set M has property S and that H and K are two of its subsets such that $\rho(H, K) > 0$. Then M contains two mutually exclusive open subsets D_H and D_K containing H and K respectively such that $D_H + D_K$ is dense in M, D_H has property S, and D_K has only a finite number of components.*

The general idea in proving Theorem 3' is to use Theorem 1' and divide M into two pieces D_H and D_K so that D_H has property S, D_K has only a finite number of components and is of diameter less than ϵ. We apply the corollary of Theorem 1' to D_H and divide it into two pieces one of which has property S and the other of which has only a finite number of components each of diameter less than ϵ. The procedure is continued so as to get an ϵ-partitioning of M.

DETAILS OF PROOF OF THEOREM 3'. Let $(p_1, p_2, \cdots, p_n)$ be a finite subset of M such that no point of M is at a distance as much as $\epsilon/4$ from $\{p_i\}$. Let K_i be the set of all points of M that are at a distance from p_i of less than $\epsilon/4$ and H_i be the set of those that are at a distance from p_i of more than $\epsilon/3$.

Applying Theorem 1' to M, we find that there are mutually exclusive subsets D_{H_1} and D_{K_1} of M containing H_1 and K_1 respectively such that $\overline{D}_{H_1} + \overline{D}_{K_1} = M$, D_{H_1} has property S, and D_{K_1} has only a finite number of components. The components of D_{K_1} are of diameters less than ϵ and are elements of G. We divide the components of D_{H_1} further.

If D_{H_1} intersects both H_2 and K_2, we apply the corollary of Theorem 1' to D_{H_1} and find that there are open subsets D_{H_2} and D_{K_2} of D_{H_1} containing $H_2 \cdot D_{H_1}$ and $K_2 \cdot D_{H_1}$ respectively such that $D_{H_2} + D_{K_2}$ is dense in D_{H_1}, D_{H_2} has property S, and D_{K_2} has only a finite number of components. The components of D_{K_2} are elements of G. If K_2 does not intersect D_{H_1}, we do not add elements to G at this stage but let $D_{H_2} = D_{H_1}$.

The process is continued. For each positive integer i less than $n-1$ for which D_{H_i} intersects both H_{i+1} and K_{i+1} we apply the corollary to D_{H_i} and get mutually exclusive open subsets $D_{H_{i+1}}$, $D_{K_{i+1}}$ of D_{H_i} containing $H_{i+1} \cdot D_{H_i}$ and $K_{i+1} \cdot D_{H_i}$ respectively such that $D_{H_{i+1}} + D_{K_{i+1}}$ is dense in D_{H_i}, $D_{H_{i+1}}$ has property S, and $D_{K_{i+1}}$ has only a finite number of components. If D_{H_i} does not intersect K_{i+1}, we suppose $D_{K_{i+1}} = 0$ and $D_{H_{i+1}} = D_{H_i}$.

The elements of G are the components of D_{K_1}, D_{K_2}, $\cdots$, $D_{K_{n-1}}$, and $D_{H_{n-1}}$.

3. A more useful partitioning. If G is an arbitrary partitioning of a continuous curve as described in Theorem 3', it may not be the member of any decreasing sequence of partitionings because some element of it may not have property S. However, if we use Theorem 1 instead of Theorem 1', we obtain the following extension of Theorem 3'.

THEOREM 3. *For each continuous curve M and each positive number ϵ there is an ϵ-partitioning G of M such that each element of G has property S.*

A partitioning whose pieces have property S is called an S-partitioning. If its mesh is less than ϵ, it is an ϵ-S-partitioning.

4. Decreasing sequence of partitionings. Theorem 2 may be applied to any set with property S to get the following corollary of Theorem 3.

COROLLARY OF THEOREM 3. *For each positive number ϵ and each set M with property S, there is an ϵ-S-partitioning of M.*

Using the corollary we find that if G is an S-partitioning of a continuous curve and ϵ is a positive number, each piece of G may be ϵ-S-partitioned. Hence, we have the following useful result [6].

THEOREM 4. *Each continuous curve has a decreasing sequence of partitionings.*

IV. CORE REFINEMENTS

In Figure 2, the partitioning represented by light lines is a core refinement of the one represented by heavy lines.

A partitioning is regular if each of its elements is the interior of its closure. This prevents pieces from having unnecessary boundary points.

If the partitioning H is a refinement of the partitioning G, the elements of H which have a boundary point in common with the boundary of an element of G are called border elements of H. Other elements are called interior elements.

The partitioning H is called a core refinement of the partitioning G if the following conditions are satisfied:

(a) Each border element of H is adjacent to an interior element.

(b) For each element g of G, the closure of the sum of the interior elements of H in g is connected.

(c) Both G and H are regular partitionings.

In [8] additional conditions were used in defining a core refinement for brick partitionings but we shall not need them here. We regard the interior of the closure of the sum of the interior elements of H in g as the core of g.

In some of the applications of partitionings, use is made of the following result.

THEOREM 5. *For each continuous curve M there is a decreasing sequence G_1, G_2, $\cdots$ of partitionings of M such that G_{i+1} is a core refinement of G_i.*

DETAILS OF PROOF. We prove this theorem by showing that if G is a regular S-partitioning of M and ϵ is a positive number, there is an ϵ-S-partitioning H which is a core refinement of G.

For each element g of G, let H' be a regular ϵ-S-partitioning of g. There is a dendron T in g that intersects each element of H'. Let H'' be a regular S-partitioning of g that refines H' and has a mesh less than the distance between T and the boundary of g.

Each element h'' of H'' with a boundary point on the boundary of g is a subset of a border element of H. If h' is the element of H' containing h'', the border element h of H containing h'' is the interior of the closure of the sum of all elements k of H'' such that there is an arc in h' from h'' to k that does not intersect the closure of any element of H'' whose closure intersects T. All elements of H'' not in such a border element of H are interior elements of H.

V. THE CONVEXIFICATION PROBLEM

1. **The method of solution.** A metric is a convex metric for a continuous curve M if for each pair of points of M there is a point of M halfway between them. In 1928 Menger raised [14] the question as to whether or not each continuous curve has a convex metric that preserves its topology. During the following twenty years a number of contributions [1; 3; 9; 10; 12] were made toward the solution of this question but it was not until the notion of partitioning was introduced that the final answer was given [6; 15].

THEOREM 6. *Each continuous curve has a convex metric.*

The fact that each continuous curve has a decreasing sequence G_1, G_2, $\cdots$ of partitionings is used in showing that each continuous curve has a convex metric. Each piece of each G_i is assigned a size and the distance between two points p, q is defined in terms of the sizes of the chains from p to q whose links are elements of the G_i's.

2. **Definition of convex metric.** Let G_1, G_2, $\cdots$ be a sequence of partitionings of M such that G_{i+1} is a core refinement of G_i and the mesh of G_{i+1} is less than one third the distance between any two nonadjacent elements of G_i.

The elements of G_1 are assigned a size of $1/2$. Each border element of G_{i+1} is assigned a size equal to one half the size of the element of G_i containing it. Each interior element of G_{i+1} is given a size equal to $1/(2^{i+1} \cdot N_{i+1})$ where N_{i+1} is the number of elements in G_{i+1}.

If K is a continuum which is equal to the closure of the sum of the elements of G_i, the ith size of K is the sum of the sizes of the elements of G_i in K. We define $E_i(p, q)$ as the minimum of the ith sizes of all continua containing $p+q$. It may be noted that if $E_i(p, q)$ is the ith size of K, then K is the sum of a chain C_1, C_2, $\cdots$, C_n of continua such that C_1 contains p, C_n contains q, C_j intersects C_k if and only if $|j-k| \leq 1$ and C_j is the closure of an element of G_i. The convex metric for M is $D(p, q) = \lim E_i(p, q)$.

We note that $D(p, p) = 0$ and $D(p, q) \geq 0$. Also $D(p, q) = D(q, p)$ because $E_i(p, q) = E_i(q, p)$.

If p, q, r are three points, $E_i(p, q) + E_i(q, r) \geq E_i(p, r)$ because if K_{pq} and K_{qr} are continua with ith sizes containing $p+q$ and $q+r$ respectively, the ith size of the continuum $K_{pq} + K_{qr}$ is no more than the sum of the ith sizes of K_{pq} and K_{qr}. Hence, $D(x, y)$ satisfies the triangle condition.

We now show that if p is a limit point of the set A, then $D(p, A) = 0$. Since $E_i(p, q) + 1/2^{i+1} \geq E_{i+1}(p, q)$, $D(p, q) \leq E_i(p, q) + 1/2^i$. However, there is a point q of A such that $p+q$ belongs to the closure of the same element of G_i. Then $D(p, q) \leq 1/2^{i-1}$ and $D(p, a)$ (a element of A) is not bounded from 0.

We now show that if p is not a point of the closed set A, $D(p, A) > 0$. Suppose i is an integer such that no element of G_i is adjacent to both an element of G_i whose closure contains p and also to an element of G_i whose closure intersects A. If ϵ is the least of the sizes of the elements of G_i, we shall show that $D(p, A) \geq \epsilon$.

If K is a continuum containing p, intersecting A, and having an $(i+k)$th size, K contains 2^k elements g of G_{i+k} such that g is a border element of G_{i+k} and lies in border elements of G_{i+1}, G_{i+2}, $\cdots$, G_{i+k-1}. The size of g is as much as $\epsilon/2^k$ and the $(i+k)$th size of K is more than ϵ. Hence, for each point a of A, $E_{i+k}(p, a) > \epsilon$ and $D(p, a) \geq \epsilon$.

We show that the metric $D(x, y)$ is convex by showing that if p and q are two points, there is a point halfway between them. Let K be a continuum containing $p+q$ of ith size $E_i(p, q)$. Then K is the sum of two subcontinua K_p and K_q containing p and q respec-

tively each of whose ith sizes differ from one half the ith size of K by less than $1/2^i$. There is a point r_i in $K_p \cdot K_q$ such that neither $E_i(p, r_i)$ nor $E_i(q, r_i)$ differs from $E_i(p, q)/2$ by more than $1/2^i$. If r is a limit point of $r_1, r_2, \cdots$, it is halfway between p and q under the metric $D(x, y)$.

Although a solution to the convexification problem has been given by means of partitionings, it would still be interesting to see the answer given by other methods.

3. **Unique segments.** If a circle is given a convex metric, for each point p there is a point q and two points r_1, r_2 such that both r_1 and r_2 are halfway between p and q. There is not a unique segment from p to q.

Although under a convex metric there is not a unique segment between each pair of points of a circle, there is a dense subset of a circle such that each pair of points of this dense subset is joined by a unique segment. In another paper I shall show that each continuous curve has such a convex metric. The result may be stated as follows.

THEOREM 7. *Each continuous curve has a dense subset R and a convex metric D such that each pair of points of R are the end points of a unique segment under D.*

For the surface of a sphere, there is a convex metric and a positive number ϵ such that each pair of points whose distance apart is less than ϵ are end points of a unique segment. It would be interesting to know a topological characterization of such sets.

A closed n-cell in a Euclidean space has a convex metric such that if two segments intersect in more than one point, the sum of these two segments is a segment. It would be interesting to obtain a topological characterization of continuous curves that can be given such a convex metric.

Each continuous curve with a convex metric that gives unique segments can be shrunk to each of its points in a very nice fashion. One might wonder if those continuous curves with such nice convex metrics could be characterized by the fashion in which they can be shrunk to points.

VI. MAPPING CONTINUOUS CURVES ONTO INTERVALS

1. **Introduction.** We have seen that if H and K are two mutually exclusive closed subsets of a continuous curve M, M can be divided into two mutually exclusive open subsets D_H and D_K containing H and K respectively such that each of the sets D_H, D_K has property S. We shall now see how uncountably many such divisions can be made.

We map M onto a straight line interval $[0, 1]$ so that H goes into 0, K goes into 1, and the inverse of each nondegenerate connected subset of $[0, 1]$ has property S [4]. Then except for a countable number of points x of $[0, 1]$, the collection of components of $T^{-1}[0, x)$ and $T^{-1}(x, 1]$ is an S-partitioning of M. We use $(a, b]$ to denote the closed interval $[a, b]$ minus the point a.

2. **Details of the transformation.** To prove the theorem regarding the mapping of M onto $[0, 1]$ we first prove the following lemma.

LEMMA FOR THEOREM 8. *Suppose C_1 and C_2 are mutually exclusive closed subsets of a continuous curve M, $M - C_i$ $(i = 1, 2)$ has property S, and ϵ is a positive number such that C_j $(j \neq i)$ can ϵ-grow into $M - C_i$. Then for each positive number δ, M contains mutually exclusive open subsets D_1 and D_2 satisfying the following conditions:*
   (a) *C_i can δ-grow into D_i and $(\epsilon + \delta)$-grow into $M - D_j$.*
   (b) *$M - (D_1 + D_2)$ can δ-grow into $M - (C_1 + C_2)$.*
   (c) *D_i and $M - D_i$ have property S.*

PROOF OF LEMMA. Let G be a γ-S-partitioning of $M - (C_1 + C_2)$ where γ is less than either δ or $\rho(C_1, C_2)/3$.

Let A_i be a finite collection of arcs in $M - C_j$ $(i \neq j)$ such that each of these arcs is of diameter less than ϵ, each intersects C_i, and each element of G intersects both an arc of A_1 and also an arc of A_2.

Let G' be a γ'-S-partitioning of $M - (C_1 + C_2)$ which refines G, where γ' is less than the distance between C_i and any arc of A_j $(i \neq j)$. Let R' be the closure of the sum of all elements g of G' such that g either lies in an element of G not adjacent to $C_1 + C_2$ or g lies in an element of G adjacent to C_i and $\bar{g}$ intersects an arc in A_j $(i \neq j)$. Then R is the sum of R' and the closure of all elements g' of G' of the following sort: g' lies in an element g of G which is adjacent to C_i but R' separates g' from C_i in $g + C_i$.

We define D_i to be the interior of the closure of the sum of C_i and all elements of G' which do not lie in R but lie in an element of G adjacent to C_i.

Each point of $D_i - C_i$ belongs to an arc intersecting C_i and lying in both D_i and also in the closure of an element of G. This shows the first part of condition (a) is satisfied. An arc showing that the second part of this condition is satisfied may be obtained by going out an element of A_i and then continuing in the closure of an element of G.

To see that condition (b) is satisfied, note that each point of $M - (C_1 + C_2)$ belongs to an arc that intersects R and lies in both $M - (C_1 + C_2)$ and also in the closure of an element of G.

Condition (c) is satisfied because G' is an S-partitioning.

THEOREM 8. *If H and K are mutually exclusive closed subsets of the continuous curve M, there is a continuous transformation T of M onto the interval $[0, 1]$ such that* (a) *$T(H) = 0$ and $T(K) = 1$,* (b) *if W is a nondegenerate connected subset of $[0, 1]$, $T^{-1}(W)$ has property* S, (d) *if 0 belongs to W, each component of $T^{-1}(W)$ intersects H and if 1 belongs to W, each component of $T^{-1}(W)$ intersects K.*

We prove this result by repeated applications of the preceding lemma. To get a start we obtain $C_1 = T^{-1}(0)$ and $C_2 = T^{-1}(1)$. Let G be an ϵ-S-partitioning of M where $\epsilon = \rho(H, K)/3$. Then C_1 is the sum of the closures of all elements of G whose closures intersect H plus all points that this sum separates from K. Similarly, C_2 is the sum of the closures of all elements of G whose closures intersect K plus all points that this sum separates from H.

In our first application of the lemma we use $T^{-1}(0)$, $T^{-1}(1)$, $1/2$ for C_1, C_2, δ. The sets $M - (D_1 + D_2)$, D_1, D_2 we obtain are called $T^{-1}(1/2)$, $T^{-1}[0, 1/2)$, $T^{-1}(1/2, 1]$.

In our next application of the lemma we let $T^{-1}(0)$, $T^{-1}[1/2, 1]$ $= T^{-1}(1/2) + T^{-1}(1/2, 1]$, $1/2$, $1/4$ be the C_1, C_2, ϵ, δ of the lemma and denote by $T^{-1}(1/4)$, $T^{-1}[0, 1/4)$, $T^{-1}(1/4, 1]$ the sets $M - (D_1 + D_2)$, D_1, D_2 obtained.

In general, suppose that $T^{-1}[0, i/2^j]$ and $T^{-1}[(i+1)/2^j, 1]$ have been defined. We apply the lemma using $T^{-1}[0, i/2^j]$, $T^{-1}[(i+1)/2^j, 1]$, $1/2^j$, $1/2^{j+1}$ for C_1, C_2, ϵ, δ and denote by $T^{-1}[(2i+1)/2^{j+1}]$, $T^{-1}[0, (2i+1)/2^{j+1})$, $T^{-1}((2i+1)/2^{j+1}, 1]$ the sets $M - (D_1 + D_2)$, D_1, D_2 obtained.

After $T^{-1}(x)$ has been defined for all numbers of $[0, 1]$ of the form $x = i/2^j$, the definition of T is continued in a continuous fashion— that is, $T^{-1}(x)$ is the intersection of all closed sets $T^{-1}[i/2^j, (i+1)/2^j]$ for which $i/2^j \leq x \leq (i+1)/2^j$.

In the next section we need to use the following information about the transformation T.

THEOREM 9. *If T is a continuous transformation mentioned in Theorem 8, then except for a countable number of values x $(0 \leq x \leq 1)$, both $T^{-1}[0, x)$ and $T^{-1}(x, 1]$ are uniformly locally connected. Furthermore, if M is the closure of a uniformly locally connected domain D and B is a subset of the boundary of D, except for a countable number of values x, each point of $T^{-1}(x) \cdot B$ is a limit point of both $T^{-1}[0, x) \cdot B$ and $T^{-1}(x, 1] \cdot B$.*

A set D is uniformly locally connected if for each positive number ϵ there is a positive number δ such that each pair of points of D

whose distance apart is no more than δ belongs to a connected subset of D of diameter less than ϵ.

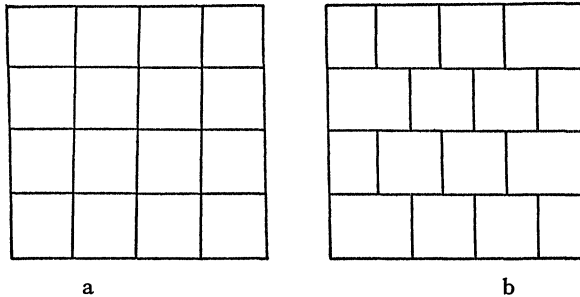The proof of Theorem 9 is contained in the proof of Theorem 5 of [4].

a

b

FIG. 4

VII. BRICK PARTITIONING

Figure 4 shows two ways in which a square plus its interior may be partitioned. The partitioning represented in Figure 4b is called a brick partitioning. A partitioning G is a brick partitioning if each of its elements is uniformly locally connected and equal to the interior of its closure while the interior of the sum of the closures of each pair of elements of G is uniformly locally connected. If G is a brick partitioning, not only can the elements of G be subdivided so as to form a partitioning G' which refines G but also adjacent elements of G and part of their common boundary can be added so as to form a partitioning G'' such that G refines G''.

In Figure 2, the left-most piece of the partitioning shown by heavy lines is not uniformly locally connected. Also, the interior of the sum of the closures of the two left-most pieces is not uniformly locally connected. If the three left-most pieces are replaced by the interior of the sum of their closures, a brick partitioning results.

Wilder showed [24, Theorem 3.4, p. 79] that if p is a point of a continuous curve M and D is a domain containing p, there is a uniformly locally connected domain E containing p and lying in D. However, to get a brick partitioning, we need to know that there is such a domain E such that $M - \bar{E}$ is uniformly locally connected. We get this result from Theorems 8 and 9.

It would be interesting to know if a modification of the methods used by Wilder would give this result. It appears that Moise had this in mind in [15].

Repeated applications of Theorems 8 and 9 show the following result [4].

THEOREM 10. *Each continuous curve has a decreasing sequence of brick partitionings.*

The first part of Theorem 9 is used in getting the elements of the partitionings to be uniformly locally connected and the second part is used to get the interior of the sum of the closures of two elements to be uniformly locally connected.

VIII. THE BOUNDARIES OF THE PIECES

1. Locally connected boundaries. In partitioning a square plus its interior as shown in Figure 4, the boundaries of the pieces are continuous curves and can be partitioned. If M is a finite complex and G is the collection of simplexes of M which are not part of the boundary of any other simplex, the elements of G have boundaries which are the sums of a finite number of continuous curves. These boundaries can be partitioned. (G may not be a brick partitioning.)

One might wonder if each continuous curve can be partitioned so that the boundaries of the pieces are locally connected. However, Example A (Figure 1) described in §I shows that this is not always possible. If the continuous curve is partitioned into pieces of diameters less than 1, each piece which is not a subset of a horizontal line has a boundary with uncountably many components.

However, in the partitioning of Example A, each component of the boundary of a piece of a partitioning is either a point or an interval and is therefore locally connected. One might wonder if each continuous curve can be partitioned so that the components of the boundaries of the pieces are locally connected. Example B of §I shows that this is not always possible.

It would be interesting to be able to characterize the types of continuous curves that can be partitioned into pieces with nice boundaries.

2. Dimensionally regular partitionings. Suppose that M is an n-dimensional continuous curve. It is known that for each positive number ϵ there is an open ϵ-covering H of M such that no point of M is covered by more than $n+1$ elements of H [11].

A partitioning G of the n-dimensional continuous curve M is called dimensionally regular if it is regular (each piece is the interior of its closure) and if the following inequality is satisfied when $j \leq n+1$.

Dimension (intersection of any j boundaries) $\leq n + 1 - j$.

In another paper I shall prove the following result.

THEOREM 11. *There is a decreasing sequence of dimensionally regular partitionings for each n-dimensional continuous curve.*

It may be noted that the boundaries of the pieces will be of dimension less than n.

IX. APPLICATIONS OF PARTITIONINGS

The study of partitionings throws light on the structure of continuous curves and is of interest aside from its applications. In fact, I regard the result that any continuous curve is partitionable as more interesting and topologically important than any of its applications.

The first use made of the fact that any continuous curve is partitionable was to solve the convexification problem as mentioned in §V.

In studying a continuous curve M, one may consider a decreasing sequence of finite open coverings of M. A decreasing sequence of partitionings might be used instead. For example, in showing that each connected open subset of a continuous curve is arcwise connected, one may use chains whose links are connected open sets to define the arcs [18]. Chains whose links are pieces of partitionings could be used instead. This was done in the following application.

1. **The Kline sphere characterization.** A nondegenerate continuous curve M is topologically equivalent to the surface of a sphere if and only if it satisfies the following two conditions:

(a) No pair of points separates M.

(b) Each simple closed curve separates M.

Several mathematicians contributed to the proving of this result ([25] and other references in [7]). The final link was in showing that conditions (a) and (b) also imply that no arc separates M [7]. This was done by showing that if there is an arc which separates M, there is a simple closed curve that does not. The simple closed curve was constructed by means of chains whose links are open sets. By using the notion of brick partitioning, the above result was strengthened [4] by showing that condition (b) could be replaced by the following weaker condition:

(b') Each simple closed curve whose complement has property S separates M.

2. **Basis with connected intersections.** A countable basis for a continuous curve is a countable collection $D_1, D_2, \cdots$ of open subsets such that if p is a point and E is an open set containing p there is an integer $i(p, E)$ such that D_i contains p and lies in E. If S is a square plus its interior, there is a countable basis for S whose elements are

the intersections of S with interiors of circles with rational radii and rational centers. The intersection of any two elements of such a basis is connected. One might wonder if each continuous curve has such a basis.

E. E. Floyd noticed that if G is a brick partitioning of M and H is the collection of all open sets h such that for some point p, h is the interior of the sum of the closures of all elements of G whose closures contain p, then the intersection of any two elements of H is connected. Consequently, Floyd and I showed [8] that any continuous curve has a countable basis such that the intersections of the elements of this basis are connected and uniformly locally connected.

3. **Characterizing 3-space.** Suppose H and K are two sets such that for each positive number ϵ there are ϵ-partitionings G_H and G_K of H and K respectively and a 1-1 correspondence between the elements of G_H and the elements of G_K such that two elements of G_H have a common boundary point if and only if the corresponding elements of G_K have a common boundary point. One might wonder if this implies that H is homeomorphic to K. It does not because if H is a square plus its interior and K is the sum of H and a straight line interval that intersects H in just one point, there are such partitionings G_H and G_K. However, it is possible to use the notion of partitionings to characterize certain sets.

The two-manifold can be partitioned so that the boundary of the elements of the partitionings are simple closed curves. Some 3-manifolds can be partitioned so that the boundaries of the pieces are simple surfaces (topologically equivalent to the surface of a sphere). It is an open question as to whether or not all compact 3-manifolds without boundary have this property. The 3-manifold obtained by adding an ideal point at infinity to ordinary Euclidean 3-space is called a 3-sphere. In fact any set homeomorphic to it is called a 3-sphere—for example, the surface of a sphere in Euclidean 4-space. Using the notion of partitioning, we can get the following characterization of a 3-sphere [2].

A necessary and sufficient condition that a continuous curve M be a 3-sphere is that one of its decreasing sequences of regular partitionings $G_1, G_2, \cdots$ have the following properties:

(a) *The boundary of each element of G_i is a simple surface.*

(b) *If the boundaries of two elements of $\sum G_i$ intersect, this intersection is a 2-cell.*

(c) *The intersection of the boundaries of three elements of G_i is one-dimensional at each of its points.*

(d) *If g is an element of G_{i-1} ($g = M$ if $i = 1$) the elements of G_i may be ordered $g_1, g_2, \cdots, g_n$ so that the boundary of g_j ($j = 1, 2, \cdots, n$) intersects the boundary of g plus the boundary of $g_1 + g_2 + \cdots + g_{j-1}$ in a connected set.*

4. Simultaneous partitionings. In a forthcoming paper [20] Garth Thomas shows that if the continuous curve M_1 is imbedded in the continuous curve M_2, there is a partitioning G_2 of M_2 such that the intersections of M_1 and the elements of G_2 give a partitioning G_1 of M_1.

X. QUESTIONS

We have already raised certain questions in the preceding sections but we mention some more here.

1. If M is locally simply connected, can it be partitioned into simply connected pieces?

2. If M is a chain from H to K where each of the sets H and K contains only one point, is there an open transformation satisfying the conditions of Theorem 8? Whyburn shows such a transformation in [23] if M is 1-dimensional. If the dimension of M is finite, is there such a transformation such that $T^{-1}(x)$ is of dimension less than the dimension of M?

3. Is there a decreasing sequence of partitionings $G_1, G_2, \cdots$ of M and a boundary operation carrying subsets of G_i into subsets of G_{i+1} that gives a homology theory?

4. Suppose M_1 and M_2 are two continuous curves such that one can be given a convex metric with unique segments but the other cannot. Is there an essential difference between the types of partitionings that M_1 and M_2 can have?

Many other things could be said concerning partitioning, but there is much to learn. We summarize the results of this paper in one sentence: Any continuous curve can be partitioned in somewhat the same way that a square plus its interior can be partitioned.

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