THE SUMMER MEETING IN MINNEAPOLIS

The fifty-sixth Summer Meeting and thirty-second Colloquium of the American Mathematical Society were held at the University of Minnesota, Minneapolis, Minnesota, Tuesday to Friday, September 4–7, 1951, in conjunction with meetings of the Mathematical Association of America, the Institute of Mathematical Statistics, the Econometric Society, and Section A of the American Association for the Advancement of Science.

Over 500 people registered for the meetings, among whom were the following 342 members of the Society:

A. A. Albert, G. E. Albert, B. A. Amirà, A. B. Ammann, E. W. Anderson, R. D. Anderson, R. L. Anderson, T. W. Anderson, H. A. Antosiewicz, K. J. Arnold, M. A. Basoco, W. F. Bauer, H. P. Beard, E. G. Begle, J. H. Bell, Theodore Bennett, A. H. Berger, R. H. Bing, H. D. Block, J. W. Bradshaw, R. W. Brink, G. S. Bruton, R. C. Buck, C. E. Burgess, L. J. Burton, L. E. Bush, J. H. Bushey, Jewell H. Bushey, W. H. Bussey, Eugenio Calabi, E. A. Cameron, R. H. Cameron, E. J. Camp, C. S. Carlson, Elizabeth Carlson, K. H. Carlson, R. E. Carr, Evelyn Carroll-Rusk, Maria Castellani, Lamberto Cesari, Abraham Charnes, Harold Chatland, K. T. Chen, S. S. Chern, Herman Chernoff, E. W. Chittenden, F. M. Clarke, H. D. Colson, T. F. Cope, A. H. Copeland, L. M. Court, T. B. Curtz, J. F. Daly, M. E. Daniells, G. B. Dantzig, P. H. Daus, Allen Devinatz, B. K. Dickerson, C. E. Diesen, Flora Dinkines, M. H. Dipert, W. J. Dixon, C. L. Dolph, J. L. Doob, Aryeh Dvoretzky, P. S. Dwyer, J. M. Earl, W. F. Eberlein, B. J. Eisenstadt, M. E. Estill, H. P Evans, R. L. Evans, F. D. Faulkner, A. M. Feyerherm, D. T. Finkbeiner, I. C. Fischer, M. M. Flood, E. E. Floyd, J. S. Frame, Abraham Franck, H. D. Friedman, D. R. Fulkerson, W. B. Fulks, W. R. Fuller, R. E. Fullerton, Arie Gaalswyk, David Gale, H. M. Gehman, B. R. Gelbaum, Gladys Gibbens, P. W. Gilbert, S. A. Gilbert, W. M. Gilbert, R. E. Gilman, Wallace Givens, V. D. Gokhale, Michael Goldberg, D. B. Goodner, S. H. Gould, Arthur Grad, L. M. Graves, L. J. Green, L. W. Green, V. G. Grove, John Gurland, Edwin Halfar, M. E. Haller, P. C. Hammer, Frank Harary, T. E. Harris, O. G. Harrold, W. L. Hart, H. L. Harter, Charles Hatfield, Nickolas Heerema, Olaf Helmer, I. N. Herstein, M. R. Hestenes, E. L. Hill, J. L. Hodges, A. J. Hoffman, R. V. Hogg, F. E. Hohn, D. L. Holl, Carl Holtom, T. C. Holyoke, Harold Hotelling, E. M Hove, Ralph Hull, M. G. Humphreys, W. A. Hurwitz, Jack Indritz, S. L. Isaacson, W. H. Ito, E. T. Jabotinsky, L. K. Jackson, C. G. Jaeger, L. W. Johnson, B. W. Jones, R. V. Kadison, Shizuo Kakutani, W. C. Kalinowski, G. K. Kalisch, Irving Kaplansky, Leo Katz, W. H. Keen, M. E. Kellar, J. L. Kelley, L. M. Kells, D. E. Kibbey, J. C. Kiefer, W. M. Kincaid, V. L. Klee, Fulton Koehler, T. C. Koopmans, C. F. Kossack, M. S. Kramer, O. E. Lancaster, R. E. Langer, E. H. Larguier, W. G. Leavitt, J. R. Lee, Patrick Leehey, D. H Lehmer, F. C. Leone, Howard Levene, D. J. Lewis, M. T. Lewis, B. W. Lindgren, Lee Lorch, W. S. Loud, Dorothy McCoy, S. W. McCuskey, W. C. McDaniel, J. V. McKelvey, J. C. C. McKinsey, E. J. McShane, C. C. MacDuffee, G. W. Mackey, G. R. MacLane, Saunders MacLane, H. M. MacNeille, H. B. Mann, Morris Marden, E. C. Marth, Imanuel Marx, K. O. May, J. R. Mayor, L. E. Mehlenbacher, Paul Meier, B. E. Meserve, E. J. Mickle, A. N. Milgram, W. H. Mills, E. E. Moise, Deane Montgomery, J. C.

Moore, Frederick Mosteller, H. T. Muhly, Sigurd Mundhjeld, Lewis Nelson, W. J. Nemerever, E. D. Nering, John von Neumann, Jerzy Neyman, O. M. Nikodým, E. A. Nordhaus, M. J. Norris, Rufus Oldenburger, Arthur Ollivier, J. M. H. Olmsted, E. J. Olson, Alex Orden, T. G. Ostrom, E. H. Ostrow, M. P. Peisakoff, J. L. Penez, O. J. Peterson, B. J. Pettis, H. P. Pettit, P. E. Pfeiffer, C. G. Phipps, George Piranian, Everett Pitcher, J. C. Polley, George Pólya, J E. Powell, G. B. Price, F. M. Pulliam, A. L. Putnam, Gustave Rabson, Tibor Radó, Howard Raiffa, J. F Randolph, R. B. Rasmusen, L. T. Ratner, G. E. Raynor, M. O. Reade, W. T. Reid, Haim Reingold, H B Ribeiro, F. D. Rigby, L. A. Ringenberg, B. V. Ritchie, E. K. Ritter, J. H. Roberts, H. A. Robinson, Murray Rosenblatt, P. C. Rosenbloom, Arthur Rosenthal, A. E. Ross, E. H. Rothe, Herman Rubin, Walter Rudin, H. E. Salzer, Hans Samelson, James Sanders, L. R. Sario, A. C. Schaeffer, Henry Scheffé, E. V. Schenkman, Peter Scherk, E. R. Schneckenburger, R. R. Seeber, Jr., I. E. Segal, George Seifert, E. B. Shanks, L. S Shapley, L. W. Sheridan, Seymour Sherman, Harold Shniad, S. S. Shu, Annette Sinclair, I. M. Singer, M. L. Slater, D. M. Smiley, M. F. Smiley, A. J. Smith, F. C. Smith, Ernst Snapper, J. L. Snell, W. S. Snyder, Andrew Sobczyk, E. S. Sokolnikoff, J. J. Sopka, E. H. Spanier, E. J. Specht, D. C. Spencer, Abraham Spitzbart, M. D. Springer, R. C. Staley, W. L. Stamey, M. L. Stein, H. E. Stelson, Rothwell Stephens, D. M. Stone, M. H. Stone, Irwin Stoner, E. B. Stouffer, J. V. Talacko, H. P. Thielman, G. H. M. Thomas, J. M. Thomas, J. E. Thompson, Gerhard Tintner, E. W. Titt, C. B. Tompkins, Leonard Tornheim, J. I. Tracey, E. F. Trombley, A. W. Tucker, J. W. Tukey, H. L. Turrittin, E. P. Vance, A. H. Van Tuyl, V. J. Varineau, N. H. Vaughan, Bernard Vinograde, D. F. Votaw, H. C. Wang, J. A. Ward, S. E. Warschawski, M. T. Wechsler, H. F. Weinberger, Alexander Weinstein, B. A. Welch, W. J. Wells, M. D. Wetzel, G. W. Whitehead, W. F. Whitmore, H. H. Wicke, L. A. Wolf, Jacob Wolfowitz, G. N. Wollan, Y. K. Wong, M. A. Woodbury, J. L. Yarnell, L. C. Young, J. W. T. Youngs, Arthur Zeichner, R. A. Zemlin, Antoni Zygmund.

The Colloquium Lectures, on *Topological transformation groups*, were presented by Professor Deane Montgomery of the Institute for Advanced Study on Tuesday afternoon and Wednesday, Thursday, and Friday mornings. Presiding were, in order, Professor E. W. Chittenden, Professor M. H. Stone, Professor John von Neumann, and Professor Everett Pitcher.

At a joint session of Section A and the Society at 10:30 A.M., Wednesday, September 5, Professor E. J. McShane of the University of Virginia gave an address, *Order preserving mappings of partially ordered spaces*, as a retiring Vice President of the American Association for the Advancement of Science. Professor R. W. Brink presided.

The Committee to Select Hour Speakers for Annual and Summer Meetings invited two speakers. On Thursday, September 6, Professor R. H. Bing of the University of Wisconsin gave an address on *Partitioning continuous curves* at 2:00 P.M. Professor J. H. Roberts presided. On Friday at 2:00 P.M. Professor G. W. Whitehead of the Massachusetts Institute of Technology gave an address on *Homot-*

opy theory. Professor Hans Samelson presided.

The organized social events provided for the Society consisted of:

An informal reception on Tuesday evening at the Campus Club, Memorial Union. Members of the receiving line included: from the University of Minnesota: President James L. Morrill, Dean Theodore C. Blegen, Professor and Mrs. Raymond W. Brink, and Professor and Mrs. George C. Priester; from the College of St. Thomas: The Reverend Bernard J. Coughlin, Mrs. Lawrence W. Sheridan, and Professor L. Earle Bush.

A banquet on Wednesday evening in the main ballroom of Coffman Memorial Union. Professor W. A. Hurwitz acted as toastmaster, and there were speeches by Professor Saunders MacLane (main speaker and representative of the Mathematical Association of America), Professor John von Neumann (representing the American Mathematical Society), Professor Gerhard Tintner (representing the Econometric Society), and Professor Paul S. Dwyer (representing the Institute of Mathematics Statistics).

A piano recital on Thursday evening by Emilie Pray, concert pianist and instructor at Macalaster College in St. Paul.

Activities for the women included a visit to the Betty Crocker Kitchens of the General Mills Corporation on Tuesday afternoon, and a luncheon on Thursday at the country club house of the Automobile Club of Minneapolis.

The Council met at 8:00 P.M. on Tuesday, September 4.

The Secretary announced the election of the following sixty-one persons to ordinary membership in the Society:

Mr. Jack Morell Anderson, University of South Dakota;
Mr. Richard Jardine Arthur, Assistant, University of Illinois;
Mr. Albert Elliot Babbitt, Jr., Columbia University;
Mr. William Beck, Assistant, University of Kentucky;
Mr. Volodymyr Bohun-Chudyniv, New York, New York;
Professor Raj Chandra Bose, University of North Carolina;
Mr. Charles Bostick, Assistant, University of Illinois;
Mr. John Joseph Brady, Mathematician, Naval Ordnance Laboratory, White Oak, Maryland;
Mr. Barron Brainerd, University of Michigan;
Mr. Richard Kettel Brown, Teaching Assistant, Rutgers University;
Mr. Pablo Emilio Casas, Princeton University;
Mr. Justus Chancellor, III, Mathematician, International Business Machines Corporation, Detroit, Michigan;
Mr. George Y. Cherlin, Instructor, Rutgers University;
Mr. Joseph Charles Connell, Staff Member, Sandia Corporation, Albuquerque, New Mexico;
Mr. Stanley Frost Dice, Lecturer, University of Pittsburgh;
Mr. Donald Epstein, Teaching Assistant, Syracuse University;

Miss Mae Irene Fauth, Instructor, Chemistry Department, Pennsylvania State College;
Reverend Peter Federchuck, Instructor, St. Basil's Prep School, Stamford, Connecticut;
Mr. Walter Clemens Frey, Teaching Assistant, Rutgers University;
Mr. Henry David Friedman, Pennsylvania State College;
Mr. Seymour Ginsburg, University of Michigan;
Mr. Karl Goldberg, Brooklyn, New York;
Mr. George Gumas, Assistant Project Engineer, Sperry Gyroscope, Great Neck, Long Island, New York;
Mr. Theodore Edward Hagensee, Instructor, Chicago Jewish Academy;
Mr. Harold Joseph Hasenfus, Chief, Rocket Section, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland;
Miss Emilie Virginia Haynsworth, Fellow, University of North Carolina;
Mr. Owen Heller Hoke, University of Georgia;
Mr. Morton Roy Kenner, Mathematician, Bell Aircraft Corporation, Buffalo, New York;
Mr. Jack C. Kiefer, Research Associate, Cornell University;
Mr. Martin David Kruskal, Assistant, Institute for Mathematics and Mechanics, New York University, New York, New York;
Acting Assistant Professor Esayas George Kundert, University of Tennessee;
Mr. Hasell Thomas LaBorde, Instructor, University of North Carolina;
Mr. Christian Donald LaBudde, Mathematician, Air Weapons Research Center, Museum of Science and Industry, Chicago, Illinois;
Lieutenant Commander Patrick Leehey, Technical Officer, Mathematical Sciences Division, Office of Naval Research, Washington, D. C.;
Mr. Bernard Werner Levinger, New York, New York;
Mr. San Dao Liao, University of Chicago;
Mr. Robert Dale Lowe, Assistant, Northwestern University;
Mr. Harold V. McIntosh, Cornell University;
Professor Samuel Alexander McLeod, Lander College, Greenwood, South Carolina;
Mr. Pat Maxwell, Jr., Staff Member, Sandia Corporation, Albuquerque, New Mexico;
Mr. Rudolph Burton Merkel, Teacher, Sacramento City Unified School District, Sacramento, California;
Sister Ingonda Maria von Mezynski, Teacher, Holy Ghost College, Techny, Illinois;
Mr. Frederick David Miller, Teaching Fellow, University of Michigan;
Mr. Charles Thomas Molloy, Mathematics Section, Kellex Corporation, New York, New York;
Dr. Cathleen Synge Morawetz (Mrs.), Staff Member, Massachusetts Institute of Technology;
Mr. Sidney I. Neuwrith, Bio-Statistician, Schering Corporation, Bloomfield, New Jersey;
Mr. Robert Stephen Novosad, Instructor, Tulane University;
Mr. Martin Orr, Mathematician, Signal Corps Engineering Laboratories, Signal Corp Center and Fort Monmouth, New Jersey;
Mr. William Coe Orthwein, Aerophysics, Consolidated Vultee Aircraft Corporation, Fort Worth, Texas;
Mr. Nicholaas du Plessis, Senior Lecturer, Rhodes University, Grahamstown, South Africa;
Mr. John Joseph Randazzo, St. Louis University;

Mr. Marvin Shinbrot, Aeronautical Research Scientist, National Advisory Committee for Aeronautics, Moffet Field, California;
Miss Arienne Bernice Silverstein, Columbia University;
Mr. Louis Solomon, Harvard University;
Mr. Benjamin Ormond Van Hook, Mississippi Southern College;
Professor Orlando Eugenio Villamayer, Faculty of Engineering, University of Cuyo, San Juan, Argentina;
Mr. Milo Wesley Weaver, Instructor, University of Texas;
Mr. Edward B. West, Teaching Assistant, University of Missouri;
Assistant Professor Walter Whittier Wright, Physicist, State Engineering Experiment Station, Georgia Institute of Technology, Atlanta;
Mr. Chung-Tao Yang, Tulane University;
Mr. Joseph Zeu-tse Yao, University of Chicago;

It was reported that the following two persons had been elected as nominees of institutional members as indicated:

Cornell University: Mr. Daniel Burrill Ray.
Queens College: Mr. Alfred Gaetano Vassolotti.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Swiss Mathematical Society: Mr. Andrew Bernard Ammann, Research Associate, University of Chicago; Wiskundig Genootschap: Professor Lauwerens Kuipers, Technical Faculty, University of Indonesia;

The University of Arkansas, Fayetteville, Arkansas; The College of the Holy Cross, Worcester, Massachusetts; The University of Miami, Coral Gables, Florida; Mississippi State College, State College, Mississippi; and Montana State College, Bozeman, Montana have been elected to institutional membership.

The following appointments of representatives of the Society were reported: Professor F. C. Ogg at the inauguration of Asa Smalledge Knowles as President of the University of Toledo on May 9, 1951; Professor L. M. Blumenthal at the inauguration of Ralph L. Woodward as President of Central College on June 2, 1951; Professor E. M. Beesley at the inauguration of Malcolm A. Love as President of the University of Nevada on June 11, 1951; and Professor S. W. McCuskey at the 125th anniversary ceremonies of Western Reserve University on June 11, 1951.

The following additional appointments by the President were reported: as a committee to submit to the Policy Committee a list of nominees for the United States National Committee on Mathematics: E. G. Begle, W. T. Martin, and G. A. Hedlund. (This committee nominated Einar Hille, Marshall Stone, John von Neumann, and Norbert Wiener); as a committee to select Gibbs Lecturers for

1952 and 1953: L. V. Ahlfors (Chairman), Salomon Bochner, and Nathan Jacobson; as a committee on arrangements for the Providence meeting: F. M. Stewart (Chairman), C. R. Adams, A. A. Bennett, L. W. Cohen, H. M. Gehman, and E. H. Lee; as a committee on arrangements for the Southeast meeting: J. C. Eaves (Chairman), J. C. Currie, F. A. Lewis, E. P. Miles, W. V. Parker, W. A. Rutledge, and W. M. Whyburn; Professor K. O. Friedrichs was reappointed the Society's representative on the Advisory Board of the Applied Mechanics Reviews for a three-year term starting July 1, 1951; Professor Eric Reissner as Chairman of the Editorial Committee for the Applied Mathematics Symposium Proceedings for period July 1, 1951 to June 30, 1952; and Professor A. H. Taub reappointed a member of Editorial Committee for the Applied Mathematics Symposium Proceedings for period July 1, 1951 to June 30, 1954. (Committee now consists of: Eric Reissner (Chairman), R. V. Churchill, and A. H. Taub.)

The following dates of meetings have been approved by the Council: December 1, 1951, at California Institute of Technology; February 23, 1952, at Columbia University; April 25–26, 1952, at Columbia University; April 25–26, 1952, at the University of Chicago; May 3, 1952, at Fresno State College; June 21, 1952, at the University of Oregon; and September 2–5, 1952, at Michigan State College.

Invitations to give addresses in 1951 were announced: R. H. Bing, Summer Meeting, Minneapolis; J. C. Oxtoby for the October meeting in Washington, D.C.; E. H. Rothe for the October meeting in Norman, Oklahoma; J. L. Kelley and Herbert Robbins for the November meeting, Auburn, Alabama; and J. C. C. McKinsey for the December meeting in Pasadena.

The Council voted to accept an invitation to hold at the Carnegie Institute of Technology on June 16–17, 1952, the Fifth Symposium on Applied Mathematics.

The Executive Director reported that the postcards sent out with the programs of the April meetings showed that in many cases the programs did not reach members until as long as 18 days after mailing. Other procedures for distributing the programs are under con-consideration, but no definite plan has yet been decided on.

The headquarters of the Society will be moved from New York City to 80 Waterman Street, Providence, Rhode Island, during the week of September 17.

The bulk of the Society's library has now been moved to the University of Georgia.

The report of the Librarian was received. The Council voted to put the problem of exchanges of our periodicals in the hands of the

Editor of Mathematical Reviews and also voted to cooperate with the University of Georgia in requesting copies of dissertations to be deposited in the University of Georgia Library.

The Council voted to appoint Professor G. B. Hochschild a member of the Proceedings Editorial Committee during the absence from this country of Professor Jacobson.

The Council voted that when two or more meetings are set at dates sufficiently close so that the programs should be sent out in one mailing, the deadline for abstracts for all these meetings be set as the deadline for the earliest of these.

The Council voted to approve the following resolution, which was also approved by the Mathematical Association of America:

The Board of Governors of the Mathematical Association of America and the Council of the American Mathematical Society respectfully call to the attention of the Congress of the United States the following considerations, which in our view indicate the extreme importance to the national welfare of having sufficient funds provided now, through the National Science Foundation, for the support of basic research and of a program of fellowships for the training of more scientists.

Since the present emergency is not likely to end for quite a number of years, it is important to take a long range view of our needs for scientific research in connection with the national defense. The airplane manufacturers and the various research agencies under the Department of Defense are now unable to secure enough trained and experienced scientists and engineers to fill their requirements. This serious shortage is partly due to the diversion of students from the fields of science and engineering to the armed services and to other activities during the years 1941–45. It is also undoubtedly due in part to the fact that many young men with potential scientific ability did not have the funds to develop their talents. In order to provide an adequate supply of trained investigators for the near future, it is necessary to begin at once to provide such encouragement for the prospective scientists as is contemplated in the fellowship program of the National Science Foundation.

The intimate dependence of practical developments upon basic research in science has been shown repeatedly in recent years. The universities have been the centers where basic research most naturally flourishes. The provision of funds to help support investigations in the universities is also important for our defense and general welfare in the near future. The amount suggested in the budget of the National Science Foundation is small compared with the amounts currently provided by the universities themselves, but will nevertheless be an effective stimulus in accelerating basic research.

In view of the preceding, the Board of Governors of the Mathematical Association of America and the Council of the American Mathematical Society respectfully urge that the Congress appropriate adequate funds for the support of basic scientific research and of students taking scientific training, as provided in the budget of the National Science Foundation, and directs that the Secretaries of the Association and the Society forward copies of this statement to the members of the Committees on appropriation of the Senate and of the House of Representatives.

The Council voted to approve the recommendation of the Committee on Russian Translations that the preparation of these translations be continued.

The Council directed the Secretary to obtain assurance, whenever he feels it necessary, that at any scheduled meeting of the Society there will be no discrimination as to race, color, creed, or nationality.

As requested by the members present at the Annual Meeting in 1950 at Gainesville, the Council reconsidered its actions concerning the University of California. After a lengthy discussion, the Council voted to reaffirm its action of September 1, 1950, when it passed the following resolution:

The Council of the American Mathematical Society deplores the harm done to academic freedom and scientific progress by the recent action of the Regents of the University of California in imposing arbitrary and humiliating conditions of employment on the faculty. The Council notes that this action has already resulted in a great discontent and loss of morale in the California faculty, and in the consequent desire of many distinguished faculty members to move elsewhere. The future effects of this action upon the scientific and academic work of the California faculty and upon the standing of the University will be disastrous. The Council therefore urges that the Regents reconsider their action, so as to restore academic freedom and to insure the continued high standing of the University of California.

The Council also passed the following resolution:

Owing to the expressed reluctance of a large number of mathematicians to attend meetings of the American Mathematical Society at the University of California on account of the conditions condemned in a resolution adopted by the Council at its meeting of September 1, 1950, the Council hereby regretfully resolves that it will not consider holding any meetings of the Society at the University of California during the calendar years 1951, 1952, and 1953 unless those conditions have been alleviated in the meantime.

The Secretary was instructed to send copies of these resolutions to the editors of Science.

Abstracts of the papers read follow. Presiding officers at the sessions for contributed papers were Professors B. J. Pettis, R. E. Langer, E. H. Spanier, Tibor Radó, Dr. H. M. MacNeille, Professors J. L. Doob, Antoni Zygmund, S. E. Warschawski, O. G. Harrold, M. R. Hestenes.

Papers whose abstract numbers are followed by the letter "*t*" were read by title. Paper number 426 was presented by Professor Chatland, 459 by Dr. Kadison, 462 by Professor Gelbaum, 476 by Dr. Marx, 484 by Professor Piranian, 506 by Professor Charnes, 508 by Professor Hestenes, 518 by Dr. Hammer, 522 by Professor Sobczyk, and 533 by Professor Calabi.

Mr. Foster was introduced by Professor J. A. Ward, Professor Goffman by Professor J. W. T. Youngs, Professor Polansky by Professor S. E. Warschawski, and Dr. Yu by Professor Szolem Mandelbrojt.

ALGEBRA AND THEORY OF NUMBERS

421*t*. A. A. Albert: *On commutative division algebras.*

Let D be a division algebra with a unity quantity over its center F. If D is commutative and has degree two over F, then it is shown that F has characteristic two and that the subfields of D are all inseparable. An example showing the existence of such algebras is given. The result is also used in the proof of the theorem stating that every commutative power-associative finite division algebra is a field. The last and most important contribution of the paper is a construction of central division algebras of all odd dimensions n over any field F of characteristic not two such that there exists a cyclic extension of degree n over F. This result can be combined with a construction of L. E. Dickson to show that there exist finite commutative division algebras of every dimension $n>2$ over any finite field F of characteristic not two. (Received July 20, 1951).

422. A. A. Albert: *On nonassociative division algebras.*

This paper is a revision and expansion of the paper entitled *On commutative division algebras.* The main addition is a proof of the theorem stating that every finite power-associative division ring D of characteristic $p>5$ is a finite field. As in the commutative case the result holds for rings of characteristic 3 and 5 providing that it is assumed that all scalar extensions of the power-associative attached commutative algebra $D^{(+)}$ are also power-associative. (Received August 18, 1951.)

423*t*. Leonard Carlitz: *Diophantine approximation in fields of characteristic p.*

Let Φ denote the field consisting of the numbers $\alpha=\sum_{-\infty}^{m} c_i x^i$, $c_i \in GF(p^n)$; define $e(\alpha)=e^{2\pi i a_1/p}$ where $c_{-i}=a_i\theta^{n-1}+\cdots+a_n$, where θ defines the $GF(p^n)$. The following items are treated. 1. Kronecker's theorem for Φ. 2. Criteria for uniform distribution. 3. Estimates for "Weyl sums" $\sum e(\phi(A))$, $A\in GF[p^n, x]$, deg $A<m$, and $\phi(u)$ is a polynomial of degree less than p with coefficients in Φ. 4. Applications. 5. The sum $S_m(\alpha, \beta)=\sum e(A^2\alpha+2B\beta)$, deg $A<m$. The main tool here is the following analogue of the Hardy-Littlewood approximate functional equation: $S_m(\alpha, \beta)=\eta p^{na/2} \cdot S_{m-a}(1/\alpha, \beta/\alpha)$, where deg $\alpha=-a<0$, deg $\beta<0$, and η is a complex number of absolute value 1. 6. Upper and lower bounds for $S_m(\alpha, \beta)$. (Received July 2, 1951.)

424*t*. Leonard Carlitz: *Independence of arithmetic functions.*

Bellman and Shapiro (Duke Math. J. vol. 15 (1948) pp. 229–235) proved the algebraic independence of a certain set of arithmetic functions; a short direct proof was given by L. I. Wade (ibid. p. 237). In the present note the ordinary product of arithmetic functions is replaced by the Dirichlet problem. It is then proved that the functions $I_0, \cdots, I_r$ are algebraically independent, where $I_k(m)=m^k$; indeed if $\Phi(u_0, \cdots, u_r)$ denotes a power series which converges for $|u_i|<1+\epsilon$, $\epsilon>0$, then $\Phi(I_0, \cdots, I_r)=0$ implies the identical vanishing of Φ. If $Q_k(m)=m^k$ for m quadratfrei and 0 otherwise, then similar results hold for $I_0, \cdots, I_r, Q_0, \cdots, Q_s$. (Received July 2, 1951.)

425*t*. Leonard Carlitz: *Some applications of a theorem of Chevalley.*

The well known theorem of Chevalley on systems of equations with coefficients in a finite field (Abhandlungen aus dem Mathematischen Seminar der Hansischen

Universität vol. 11 (1936) pp. 73–75) is applied (1) to show that certain systems of equations with polynomial coefficients have nontrivial solutions. For example if $f(u_1, \cdots, u_s)$ is a polynomial of degree not greater than k and with coefficients in $GF[q, x]$, $f(0, \cdots, 0)=0$, then the equation $f(U_1, \cdots, U_s)=0$ with $U_i \in GF[q, x]$ always has a nontrivial solution provided $s \geqq k^2+1$; in general this value of s cannot be diminished. (2) Chevalley's theorem is applied to certain problems of approximation in the field Φ consisting of the numbers $\sum_{-\infty}^{m} c_i x^i$, $c_i \in GF(q)$. For example, if $\alpha \in \Phi$, $m \geqq 1$, $k \geqq 1$, there exist polynomials A, B, $A \neq 0$, deg $A \leqq km$, such that deg $(A^k\alpha - B) < -m$. (Received July 2, 1951.)

426. Harold Chatland and H. B. Mann: *On polynomials reducible modulo every prime ideal.*

In a forthcoming paper H. B. Mann has shown that an irreducible polynomial over an algebraic number field cannot have a root in the ground field modulo every prime ideal. In this paper necessary and sufficient conditions are given that a polynomial factors modulo every prime ideal. It is shown that the polynomial, $f(x)$, of degree n, with coefficients in an algebraic number field, factors modulo every prime ideal if and only if the Galois group of $f(x)$ does not contain a cyclic permutation of degree n. More precisely, if g is the order of the group of $f(x)$ and $m = m(n_1, n_2, \cdots, n_r; f_1, f_2, \cdots, f_r)$ is the number of permutations consisting of n_1 cycles of degree f_1, n_2 cycles of degree f_2, and so forth, and if S is the set of primes modulo which $f(x)$ decomposes into n_1 irreducible factors of degree f_1, n_2 of degree f_2, and so forth, then $\sum_{\mathfrak{p} \subset S} 1/(N(\mathfrak{p}))^s = (m/g) \log (1/(s-1)) + O_{s\to 1}(s-1)$. It follows that an irreducible polynomial of prime degree over an algebraic number field cannot factor modulo every prime ideal. The sufficiency of the condition that the polynomial factor modulo every prime ideal holds also when the ground field is any field. It has been shown by H. B. Mann that the condition is not necessary in the general case. (Received July 25, 1951.)

427*t*. Eckford Cohen: *Arithmetic functions of polynomials.*

Let R be a fixed polynomial of degree r in $D = GF[p^n, x]$. A simple set of arithmetic functions denoted by $\epsilon_Z(A) = \epsilon(ZA, R)$ are defined in terms of pth roots of unity. It is shown that any arithmetic function defined for polynomials in D of degree less than r can be represented uniquely by a sum of these ϵ-functions. Applications of this result are made to the number of representations of polynomials by linear and bilinear sums. The methods of this paper have analogues in the rational case. (Received July 23, 1951.)

428*t*. Harvey Cohn: *Periodic algorithm for cubic forms.*

A reduction theorem of three-dimensional lattices is proved using as a starting point Minkowski's critical rectangular parallelepiped. The theorem states, in effect, that any three noncollinear points (ξ_i, η_i), where $i=1, 2, 3$, can be transformed by an integral unimodular transformation $\xi_i' = (a\xi_i + b\eta_i + c)/\lambda$, $\eta_i' = (a'\xi_i + b'\eta_i + c')/\lambda$ where $\lambda = a''\xi_i + b''\eta_i + c''$, det $(ab'c'') = 1$, into points (ξ_i', η_i') lying one in each of the regions R_1: $\{\xi' + \eta' + 1 \leqq 0, \xi' \leqq 0, \eta' \leqq 0\}$; R_2: $\{\xi' + \eta' + 1 \geqq 0, \xi' \leqq 0, \eta' \geqq 0\}$; R_3: $\{\xi' + \eta' + 1 \geqq 0, \xi' \geqq 0, \eta' \leqq 0\}$. In the preliminary report of this result (Bull. Amer. Math. Soc. Abstract 56-3-220) the theorem was promised in an equivalent form applicable to "continued fraction" constructions for three-dimensional lattices. (Received July 5, 1951.)

429*t*. Jesse Douglas: *On the basis theorem for finite abelian groups.*

Supplementing the proof given by the author in Proc. Nat. Acad. Sci. U.S.A vol. 37 (1951) pp. 359–362 (cf. Bull. Amer. Math. Soc. Abstract 57-4-274), this paper supplies two additional simple proofs of the "fundamental theorem of abelian groups." The detailed account will appear in later issues of Proc. Nat. Acad. Sci. U. S. A. vol. 37 (Aug., Sept., 1951). (Received July 11, 1951.)

430*t*. A. E. Foster: *A generalized harmonic conjugate for commutative algebras.* Preliminary report.

This paper extends the work of Wagner (Duke Math. J. vol. 15 (1948) p. 455). With every set of partial differential equations of the second order in the form $\partial^2 u/\partial x_r \partial x_j = \sum_{i=1}^{n} c_{rji}\ \partial^2 u/\partial x_i \partial x_j$ $(r=2, \cdots, n;\ j=r, \cdots, n)$ there can be associated a unique commutative distributive algebra over the field of the c's such that the set will be the generalized Laplace equations if the c_{rji} are such that the algebra is associative and of the Frobenius type. It is shown that if the algebra is of this type, it is always possible to determine the conjugates of a function satisfying the above set of equations. (Received July 23, 1951.)

431. J. S. Frame: *The classes and representations of the groups of* 27 *lines and* 28 *bitangents.*

The finite groups G of order 51840 and H of order 2,903,040, related to the exceptional Lie algebras E_6 and E_7, contain simple subgroups of index 2 denoted by G_0 and H_0. G is the group of automorphisms of the 27 lines on a general cubic surface, which is a subgroup of index 28 in the group H_0 of the automorphisms of the 28 bitangents to a plane quartic curve of genus three. In this paper matrices are given for a set of 36 conjugate hyperplane reflections S_i of which six generate a six-dimensional orthogonal representation of G, and it is shown that if T_k and U_k are respectively products of two or three permutable S_i's, then every element of G can be written either as a U_jU_k or as a U_jT_k. The 25 classes and irreducible characters of G are then determined. Classes are described by a symbol $1^\alpha 2^\beta 3^\gamma \cdots$ in which some of the integers α, β, γ may be negative. A complete set of six basic invariants for G are found. Then the 30 classes and irreducible characters of H_0 are found using induced representations and modular theory. Finally all large subgroups of H_0 are determined from the character table. (Received July 11, 1951.)

432*t*. Leonard Gillman: *On intervals of ordered sets.*

The following statements are considered. $\boldsymbol{P}(\aleph_\alpha)$: $\beta<\alpha$ implies $2^{\aleph_\beta}<\aleph_\alpha$; $\boldsymbol{P}'(\aleph_\alpha)$: $\beta<\omega_{cf(\alpha)}$ implies $2^{\aleph_\beta}<\aleph_\alpha$; $\boldsymbol{P}$: $\aleph_\alpha$ singular implies $\boldsymbol{P}(\aleph_\alpha)$; $\boldsymbol{Q}(\aleph_\delta, \aleph_\alpha)$: every ordered set of power $\aleph_\delta$ has a family of $\aleph_\alpha$ mutually disjoint intervals; $\boldsymbol{Q}(\aleph_\alpha)$: $\boldsymbol{Q}(\aleph_\alpha, \aleph_\alpha)$; $\boldsymbol{Q}$: $\aleph_\alpha$ singular implies $\boldsymbol{Q}(\aleph_\alpha)$. Obviously $\boldsymbol{Q}(\aleph_0)$ is true, while both $\boldsymbol{Q}(2^{\aleph_0})$ and $\boldsymbol{Q}(\aleph_1)$ are false. Results: (1) if $\aleph_\alpha$ is singular then $\boldsymbol{P}(\aleph_\alpha)$ implies $\boldsymbol{Q}(\aleph_\alpha)$; (2) for every α, $\boldsymbol{Q}(\aleph_\alpha)$ implies $\boldsymbol{P}'(\aleph_\alpha)$. It follows that: (3) $\boldsymbol{Q}(\aleph_\alpha)$ fails for every regular $\aleph_\alpha$ which is not strongly inaccessible; (4) $\boldsymbol{Q}(2^{\aleph_\alpha})$ fails for every α; (5) if ω_α is regular and $\aleph_{\omega_\alpha}$ is singular, then $\boldsymbol{P}(\aleph_{\omega_\alpha})$ and $\boldsymbol{Q}(\aleph_{\omega_\alpha})$ are equivalent; (6) $\boldsymbol{P}$ and $\boldsymbol{Q}$ are equivalent. If $\aleph_\alpha$ is strongly inaccessible then $\boldsymbol{Q}(\aleph_{\alpha+1}, \aleph_\alpha)$ holds, but the question of $\boldsymbol{Q}(\aleph_\alpha)$ itself is not settled in this case (for $\alpha>0$). (Received June 15, 1951.)

433. Wallace Givens: *Polarities and signature in continuous geometry.*

Let $\mathfrak{R}$ be an irreducible regular ring of order not less than 3, Z its center (necessarily a field) assumed not of characteristic two, and $L=\overline{R}_{\mathfrak{R}}$ the lattice of principal right ideals in $\mathfrak{R}$. [Notations agree with those of von Neumann (cf. his five papers in Proc. Nat. Acad. Sci. U.S.A. 1936–1937).] Assume $x \rightarrow x'$ an antiisomorphism of $\mathfrak{R}$ so $(x)_r \rightarrow (x'a)_l^r \equiv (x)_r^*$ is a dual endomorphism (correlation) in L. Excluding rank $a=1/n$ in the discrete case of order n, a necessary and sufficient condition that $(x)_r^{**} \supseteq (x)_r$ is $a'=\alpha a$, $\alpha \in Z$. For such a in $\mathfrak{R}$ the correlations are called polarities in agreement with the classical theory of linear complexes, quadrics and antiquadrics. As in the geometry of the quadric cone, one can reduce considerations to the case in which a^{-1} exists by introducing a complement of the vertex $(a)_l^r$. Theorems proved include: (1) dim $[(x)_r \cap (x)_r^*] = \dim (x)_r - \text{rank } (x'ax)$ so $(x)_r$ is totally isotropic if and only if $x'ax=0$; (2) if $(e)_r$ is totally isotropic, there exists a totally isotropic $(f)_r$ such that dim $(e)_r = \dim (f)_r$ and $(e)_r \cap (f)_r = (0)_r$; (3) a totally isotropic subspace of maximum dimension δ exists; and, (4) $a=e'af+f'ae+g'ag$ where $e+f+g=1$, e, f and g are suitable orthogonal idempotents, $(e)_r$ and $(f)_r$ are totally isotropic of dimension δ, and $g'ag$ is definite (that is, for $x \in g\mathfrak{R}g$, $x'ax=0 \rightarrow x=0$). (Received July 23, 1951.)

434*t*. Franklin Haimo: *Scripts and their centers.*

Scripts, as defined in Rosenbloom, *The elements of mathematical logic*, New York, 1950, cannot have centers with more than one element unless such scripts have only one prime. If this prime is not in the center, then the latter set is linearly ordered by divisibility. A script is the set sum of four of its sub-scripts. These components are defined in terms of the prime pre- and post-factors of their elements. Two by two, they intersect in the common intersection of all four, the sub-script generated by the finite powers of the primes of the script. (Received July 16, 1951.)

435. Nickolas Heerema: *An algebra determined by a binary cubic form.*

Consider the linear associative algebra $A=A(a, b, c, d)$ generated over a field F by R and S where $ax^3+bx^2y+cxy^2+dy^3 \equiv (Rx+Sy)^3$. As in the quadratic case, leading to Clifford algebras, R and S are assumed to commute with the elements of the coefficient field F, which is taken to have more than two elements. The approach is to select, by an economical process, a collection B of monomials which can be shown to contain a basis for A. Using representations by infinite matrices it is then possible to show that B is actually a basis. Some of the principal results are: (1) $A(a, b, c, d)$ is infinite-dimensional for any choice of a, b, c, and d, and an extensive class of bases for A is identified. (2) Every properly homomorphic image of A is finite-dimensional. Hence A is not a direct sum of nonzero algebras over F. (3) If $f(x, y)$ has nonproportional linear factors, the center of A is isomorphic to $F(u, v)/\{u^3+v^2+k^2v\}$ for $k \neq 0$ in F, where $F(u, v)$ designates the polynomial domain in two commuting indeterminants u and v. (Received July 23, 1951.)

436*t*. Nathan Jacobson and C. E. Rickart: *Homomorphisms of Jordan rings of self-adjoint elements.*

Let $\mathfrak{A}$ be an associative ring with an involution $a \rightarrow a^*$ and let $\mathfrak{H}$ denote the set of self-adjoint elements $h=h^*$. Then $\mathfrak{H}$ is a special Jordan ring. In the present paper the investigation of Jordan homomorphisms of rings begun in a previous paper [Trans. Amer. Math. Soc. vol. 69 (1950) pp. 479–502] is continued by studying Jordan homomorphisms of rings of type $\mathfrak{H}$. The principal result is that if $\mathfrak{A}$ is a general matrix ring

$\mathfrak{S}_n$, $n \geqq 3$, with an involution such that $e_{ii}^* = e_{ii}$ $(i = 1, \cdots, n)$ and every element of $\mathfrak{H}$ is of the form $a + a^*$, then any Jordan homomorphism of $\mathfrak{H}$ can be extended to an associative homomorphism of $\mathfrak{A}$. This result can be extended to locally matrix rings and in this form is applicable to involutorial simple rings with minimal one-sided ideals. Jordan isomorphisms of the Jordan ring of self-adjoint elements of an involutorial primitive ring with minimal one-sided ideals onto a second Jordan ring of the same type are also studied. (Received July 9, 1951.)

437. Irving Kaplansky: *Algebras of type I.*

In a previous paper (Ann. of Math., March, 1951) the author studied AW^*-algebras, an abstract analogue of W^*-algebras (weakly closed algebras of operators on Hilbert space). In this paper a structure theory is given for AW^*-algebras of type I. The main results are as follows. 1. Two such algebras, which are homogeneous in the appropriate sense, are isomorphic if and only if their centers are isomorphic. 2. An AW^*-algebra of type I is W^* if and only if its center is W^*. 3. A $*$-automorphism leaving the center elementwise fixed is inner by a unitary element. (Received July 13, 1951.)

438. H. B. Mann: *On identically solvable congruences.*

Let $F(x)$ be a polynomial with coefficients in a field Ω. The congruence $F(x) \equiv 0(p)$ is called identically solvable in Ω if it has a solution in Ω for every prime ideal p of Ω. The following theorem is proved: Let Ω be an algebraic number field and let S be the set of all prime ideals p for which the congruence $F(x) \equiv 0(p)$ has a solution in Ω. Let n be the order of the Galois group G of $F(x)$ and m the number of elements of G that leave at least one root of $F(x)$ fixed. Then $\lim_{s \to 1+} \sum_{p \in S} N(p)^{-s} / -\log (s-1) = m/n$. Moreover the congruence $F(x) \equiv 0(p)$ is identically solvable in Ω if and only if $m = n$. This theorem is obtained as a consequence of a theorem of Frobenius (Sitzber. Ber. Akad. (1896) pp. 689–705). In particular the congruence of the theorem is not identically solvable if $F(x)$ is irreducible or if $F(x) = x^k - a$, where a is not a kth power. The theorem need not hold if Ω is not an algebraic number field even if Ω has infinitely many discreet non-Archimedian valuations. (Received July 30, 1951.)

439*t*. L. J. Mordell: *On cubic equations $z^2 = f(x, y)$ with an infinity of integer solutions.*

It is shown that the equation $z^2 = p^2 + \lambda x + ax^2 + bxy + cy^2 + Ax^3 + Bx^2y + Cxy^2 + Dy^3$, where the coefficients are integers and where $p \neq 0$, $c = 2p$, and $p^2(b^2 - 4ac) + \lambda^2 c$ is positive and not a square, has an infinity of integer solutions. (Received July 16, 1951.)

440*t*. Leo Moser: *On small quadratic nonresidues.*

Some unpublished results of A. Brauer and some results of L. Rédei (Nieuw Archief voor Wiskunde vol. 23 (1950) pp. 150–162) on the distribution of small quadratic nonresidues are extended to smaller intervals. It is proved that for every $\epsilon > 0$ there exists an $N(\epsilon)$ such that for $p > N$, p prime, $p \equiv 1 \pmod 4$, the interval $[1, (p/2)^{1/2}]$ contains at least $(c - \epsilon)(p/2)^{1/2}$ nonresidues. $c = 1/2 - (1/2 - 3/\pi^2)^{1/2}$. For all primes $p > N$, the interval $[1, p^{1/2}]$ is shown to contain at least $(c - \epsilon)p^{1/2}$ nonresidues. Explicit results of this type, free of ϵ, are also obtained. (Received July 23, 1951.)

441*t*. T. G. Ostrom: *Some consequences of Hall's multiplier theorem.*

A set of integers $a_0, a_1, \cdots, a_n$ is called a difference set mod N if the set $\{a_i - a_j\}$ contains each nonzero residue exactly once. A multiplier is a number q such that the set $\{qa_i\}$, $i=0, 1, \cdots, n$, is the same as the set $\{a_j+s\}$, $j=0, 1, \cdots, n$, for some s. Let N_1 be a prime factor of N. We shall say that N_1 is of type I if there is some multiplier q such that the exponent to which q belongs mod N is greater than the exponent to which it belongs mod N_1. We shall say that N_1 is of type II if every multiplier q belongs to the same exponent mod N as it does mod N_1. This paper obtains the following consequences from a theorem of Hall: 1. Unless every factor of N is of type II, the existence of a difference set mod N necessitates the existence of a difference set mod one or more of its factors. It follows that a difference set with $n=x^r$, $(r, 3)=1$, cannot exist unless there is a difference set with $n=x$. 2. Unless every factor of N is of type I, it is necessary that $n \equiv p^r$ mod N, where p is a prime factor of n and $3r$ divides n or $n+1$. 3. If every prime factor of N is of type II, the number of multipliers divides n or $n+1$. (Received July 9, 1951.)

442t. Peter Scherk: *On complexes in abelian groups.* Preliminary report.

Given an additive abelian group G of order $g \leqq \infty$. Let $A=\{a_1, \cdots, a_r\}$, $B=\{b_0, b_1, \cdots, b_s\}$, $C=\{c_1, \cdots, c_t\}$ denote finite nonempty complexes in G. Then $A+B=\{a_i+b_k\}$, and $C-B$ is the set of all elements $d \subset G$ such that $d+B \subset C$. Theorem 1: Let $r+s<g$, $A+B=C$. Suppose that for each m the equation $(r+m)(b_k-b_0)=0$ has not more than m solutions $b_k \subset B$ ($k>0$; $m=0, 1, \cdots, s-1$). Then $t \geqq r+s$. Theorem 2: Let $s<t$, $A=C-B$. Suppose that for each m the equation $(t-m)(b_k-b_0)=0$ has not more than m solutions $b_k \subset B$ ($k>0$; $m=0, 1, \cdots, s-1$). Then $r \leqq t-s$. Theorem 1 generalizes a theorem by Cauchy and Davenport on cyclic groups of prime order. Its proof follows closely Davenport's pattern. The two theorems and their proofs are connected by a duality theorem essentially due to Hadwiger. (Received July 20, 1951.)

443. Peter Scherk: *On sets of integers.*

Let $n>0$ be fixed. Definitions: I = set of all non-negative integers not greater than n; $A=\{a\}$, $B=\{b\}, \cdots$ = subsets of I; $A+B=\{a+b\} \cap I$; $A-B$ = largest C such that $C+B \subset A$; $\tilde{A}$ = complement in I of $\{n-a\}$; $A(x)$ = number of all the $a \leqq x$. Khintchine's inversion formula can readily be generalized as follows: $A-B=(\tilde{A}+B)^{\sim}$, $A+B=(\tilde{A}-b)^{\sim}$. By means of these formulas the following result and its dual are established: Let $0 \leqq r$, $0 \leqq s$; $r+s<n$; $r \subset A$, $s \subset B$, $n \not\subset C=A+B$. Then there exists an m such that $C(n)-C(n-m) \geqq A(r+m)-A(r)+B(s+m)-B(s)$. The number m satisfies the following requirements: (1) $n-m \subset C$. (2) There is no decomposition $r+s+m=a+b$ where $r \leqq a \leqq r+m$, $s \leqq b \leqq s+m$; in particular $r+m \not\subset A$, $s+m \not\subset B$. (3) $m=n-r-s$ or $0<m \leqq (n-r-s)/2$. (Received July 16, 1951.)

444t. M. F. Smiley: *The lemma of Zassenhaus for half-loops.*

If H, K, L are subgroups of a group G, K is normal in G, and L is normal in H, then LK is normal in HK and $LK \cap H$ is normal in H and the identity mapping of G induces an isomorphism of the corresponding quotient groups. This isomorphism theorem is extended to *half-loops* (that is, multiplicative systems with a right unit and unique right solvability), and yields the usual Schreier theorem. Ore's generalization of Remak's theorem applies to half-loops and the isomorphism of the factors in two direct decompositions into indecomposable factors are induced by a single automorphism, as usual. (Received July 3, 1951.)

445. Leonard Tornheim: *Asymmetric minima of indefinite binary quadratic forms and asymmetric approximation by rationals.*

Let $f=ax^2+bxy+cy^2$ have determinant $b^2-4ac=1$. Let p be the greatest lower bound of the non-negative values of f for integral x, y and q the least upper bound of the negative values. For k a positive integer let $M^{-1/2}=\min(kp, -q)$. Markoff and others have found properties of the set of values of M when $k=1$. Segre has determined the smallest value of M for each value of k. This result is extended by finding the next smallest value and showing that it as well as all values close enough to it are condensation points. Analogous results hold for asymmetric approximation to a real number θ by rational numbers x/y. The problem is the determination of the largest value of N such that an infinite number of rationals x/y exist for which $-k/y^2N \leqq x/y-\theta \leqq 1/y^2N$. Proofs depend on the properties of continued fractions. (Received July 23, 1951.)

446. Bernard Vinograde: *The matrix equations $AX=XA'$ and $B^2=I$.*

Let A be nonsingular and nonderogatory with coefficients in any field of zero characteristic. Let $B=\sum_1^n x_iA^{i-1}$. In seeking a rational solution of $B^2=I$ one may proceed by replacing A by its companion matrix M and equating the coefficients of simplest structure. If one symmetrizes the quadratic relations which are thus obtained, a system of symmetric matrices $\{S_i\}$ of Hankel type result with the properties: (1) $MS_i=S_iM'$ and (2) the columns of S_i are the ith columns of the powers of M. Hence the transforms of the S_i's form a basis of the rational symmetric solutions of $AX=XA'$. The comparison of the two matrix equations is accomplished by (1) using the Kronecker product $M\times M^{-1}$ to display the structure of the S_i's and (2) comparing differentials. (Received July 23, 1951.)

ANALYSIS

447*t*. Lipman Bers: *New treatment of boundary value problems for minimal surfaces with singularities at infinity.* Preliminary report.

In a recent paper (Trans. Amer. Math. Soc. vol. 70 (1951) pp. 465–491) the author established an existence theorem for a class of boundary value problems for minimal surfaces with singularities at infinity. The boundary curve P was assumed to deviate but little from a convex curve. In the present paper this condition of "essential convexity" is removed. The new proof is based on an a priori estimate resulting from the theory of quasiconformal transformations. (Received July 23, 1951.)

448*t*. Lipman Bers and Shmuel Agmon: *The expansion theorem for pseudo-analytic functions.* Preliminary report.

Theorem. Let $w(z)$ be regular pseudo-analytic for $|z-z_0|<R$ (cf. L. Bers, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 130–136; vol. 37 (1951) pp. 42–47). Then $w(z)$ can be expanded in a series of global formal powers which converges for $|z-z_0|<R$. This strengthens the result announced in the notes quoted above. A similar theorem holds for formal Laurent series. (Received July 23, 1951.)

449*t*. D. G. Bourgin: *Some multiplicative functionals.*

Suppose G and G' refer to an Abelian compact group and to its character group respectively and let $L_2(G)$ be a ring with multiplication interpreted as convolution. The general form of a norm continuous multiplicative functional M on $L_2(G)$ to the complex field is found in terms of the Fourier transform of x. The determining set is a finite subset of G'. These and similar conclusions depend on the finiteness of determining sets of continuous multiplicative functionals on such spaces as for instance $L_p(S)$, $p \geqq 1$, and so on, where S is discrete. Other cases where the determining set is finite include $C(S)$, S compact, where weak replaces norm continuity for M. These results may be viewed as the special one-dimensional cases of a representation theory for the multiplicative semi-group parts of certain Banach algebras. (Received July 20, 1951.)

450*t*. A. P. Calderón and Antoni Zygmund: *Singular integrals in the theory of the potential.*

If P, Q are points of E^n, the length of the vector $P-Q$ will be written $|P-Q|$ Let $\Omega(P)$ be any function defined on the surface Σ of the n-dimensional unit sphere with center O, satisfying condition Lip α ($\alpha>0$), with $\int_\Sigma \Omega(P)dP=0$. Let $K(P-Q) = |P-Q|^{-n}\Omega[(P-Q)|P-Q|^{-1}]$, and let $K_\lambda(P-Q)=K(P-Q)$ for $|P-Q|\geqq 1/\lambda$, $K_\lambda(P-Q)=0$ otherwise. Let $\tilde{f}_\lambda(P)=\int_{E^n}K_\lambda(P-Q)f(Q)dQ$. Then, for $\lambda\to\infty$, (1) If $f\in L^p$, $1<p<\infty$, the function $\tilde{f}_\lambda(P)$ tends to a limit $\tilde{f}(P)$, in the metric L^p, almost everywhere, and majorized by a function from L^p. (2) If $|f|(1+\log^+|f|)$ is integrable, $\tilde{f}_\lambda(P)\to\tilde{f}(P)$ in the metric L^1 over every set of finite measure. (3) If $f(P)dP$ is replaced by $\mu(dP)$, where μ is a completely additive set function with $\int_{E^n}|\mu(dP)|<\infty$, $\tilde{f}_\lambda(P)$ tends to a finite limit almost everywhere. These results have applications to the problems of the differentiability of the potentials. Take, for example, $n=2$. (4) Let $f(s, t)$ vanish near the point at infinity and let $|f| \log^+|f|\in L$. Let $u(x, y) =\int_{E^2}f(s, t)\log(1/r)dsdt$. Then $u(x, y)=\int_{-\infty}^x\int_{-\infty}^y\tilde{f}(s, t)dsdt$, for some $\tilde{f}\in L$. The partial derivatives u_x, u_y are absolutely continuous on almost every line parallel to either axis. The derivatives u_{xx}, u_{xy}, u_{yy} exist almost everywhere and are given by the classical expressions. Moreover, u has p.p. a second differential. (5) Under the assumptions of (4), the Newtonian potential $U(x, y)=\iint_{E^2}f(s, t)r^{-1}dsdt$ is absolutely continuous on almost every line parallel to either axis. If $f\in L^q$, $q>2$, then U has p.p. a first differential. (Received July 16, 1951.)

451. R. E. Carr: *Pattern integration with Riemann double integrals.*

The proper Riemann double integral $\iint_R f(x, y)dA$ is defined to be the $\lim_{n\to\infty}\sum_{k=1}^n f(\xi_k^{(n)}, \eta_k^{(n)})A_k^{(n)}$, where $A_k^{(n)}$ is the kth subregion when R is divided into n parts, and $(\xi_k^{(n)}, \eta_k^{(n)})$ is any point in $A_k^{(n)}$. (It is assumed that as $n\to\infty$ the maximum diameter of each sub-region approaches zero.) The pattern integral is defined to be the corresponding limit, providing it exists, when the summation is restricted to a prescribed subset P of the set $N=\{k\}_1^n$. The case is considered in which R is the unit square $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, and in which the subdivision is made by the horizontal lines $x=i/m$, $i=1, 2, \cdots, m-1$, and the vertical lines $y=j/n$, $j=1, 2, \cdots, n-1$. The principal theorem shows that, if $\{\beta_{jk}\}$ is any prescribed double sequence of complex numbers where $|\beta_{jk}|<M$ for all j and k, and if $\lim_{m,n\to\infty}(1/mn)\sum_{j,k=1}^{m,n}\beta_{jk}=\beta$, then, from the existence of $\iint_R f(x, y)dA$, $\lim_{m,n\to\infty}(1/mn)\sum_{j,k=1}^{m,n}\beta_{jk}f(\xi_j^{(m,n)}, \eta_k^{(m,n)}) =\beta\iint_R f(x, y)dA$, where $(j-1)/m\leqq\xi_j^{(m,n)}\leqq j/m$, $(k-1)/n\leqq\eta_k^{(m,n)}\leqq k/n$. The extension of this theorem to multiple integrals of higher order is immediate. (Received May 21, 1951.)

452. Lamberto Cesari: *Some connections between Lebesgue area and measure theory.*

Let S be any Fréchet surface in E_3 of finite Lebesgue area $L(S)$, and $T:x=x(w)$, $w\in Q$, any representation of S on a simple closed Jordan region Q. Let $\mathfrak{G}$ be the collection of all maximal continua g of constancy for T on Q (proper continua and single points); $\mathfrak{M}$, $\mathfrak{O}$, $\mathfrak{B}$ respectively the collections of all subsets of Q, of all subsets of Q open in Q, of all Borel subsets of Q; $\mathfrak{M}_0$, $\mathfrak{O}_0$, $\mathfrak{B}_0$ the subcollections of $\mathfrak{M}$, $\mathfrak{O}$, $\mathfrak{B}$ of those sets which are sums of continua $g\in\mathfrak{G}$. Let $G(O, T)$ be the Geocze area of the part of S defined by T on any set $O\in\mathfrak{O}$. Utilizing methods previously introduced [Memorie Acad. Italia vol. 13 (1943) pp. 1323–1481] it is proved that *for any sequence O_i of sets $O_i\in\mathfrak{O}_0$ we have $G(\sum O_i, T)\leqq\sum G(O_i, T)$*. By making use also of the general theory of set functions [H. Hahn and A. Rosenthal, *Set functions*, 1948] it is shown that *the function $G(O, T)$, $O\in\mathfrak{O}_0$, can be extended to a function $\phi(M, T)$ defined for all $M\in\mathfrak{M}_0$, coinciding with $G(O, T)$ for all $O\in\mathfrak{O}_0$, and ϕ is a regular ordinary measure function, is a content function, is totally additive in $\mathfrak{B}_0$, and is continuous.* Further results are obtained for restricted classes of representations. (Received July 16, 1951.)

453*t*. V. F. Cowling: *On analytic continuation by quasi-Hausdorff methods.*

In this paper regions in the complex plane in which a Taylor series is summed to its "proper" value by means of quasi-Hausdorff methods are determined (see G. H. Hardy, *Divergent series*, Oxford University Press, 1949, pp. 277–288). The method is similar to that employed by R. P. Agnew in his determination of regions in the complex plane in which a Taylor series is summed by Hausdorff methods. The lengthy nature of the results prevents their being stated here even in the special case of the geometric series. (Received July 23, 1951.)

454*t*. Albert Edrei: *On entire and meromorphic functions which have only real zeros and real ones.*

In this paper, the author proves that, if $f(z)$ is an entire function, and if the roots of each of the two equations $f(z)=a$, $f(z)=b$ $(a\neq b)$ are all real, then the order of $f(z)$ is finite and does not exceed one. The statement becomes false if the word entire is replaced by meromorphic. However, there exist analogous theorems for meromorphic functions which are stated and proved in the paper. As an application of the simplest of the theorems, the author proves a conjecture of Schoenberg on the generating function of a totally positive sequence. (Received August 2, 1951.)

455. B. J. Eisenstadt: *The component of the identity of the space of continuous functions into the circle.*

Let $R_{2q}(X)$ denote the component of the identity of the space of continuous functions from a topological space X into the reals mod $2q$. $R_{2q}(X)$ can be made, in a natural way, into a metric, abelian group. In this paper, necessary and sufficient conditions are put on a metric, abelian group so that it be equivalent to $R_{2q}(X)$ for some q and for some compact, connected space X. A pseudo multiplication by real scalars can be introduced on $R_{2q}(X)$ and the points of X define pseudo linear functionals from $R_{2q}(X)$ to the reals mod $2q$. Thus, in the characterization, conditions are put on the abstract group to ensure the existence of a pseudo multiplication and of sufficiently many pseudo linear functionals. The looked for space X is then obtained as a subset of the set of all pseudo linear functionals. In picking out the points of X, certain Banach spaces associated with the group are investigated, a new linear functional is discussed, and a new characterization of the Banach space of real, continuous functions is given. (Received August 1, 1951.)

456t. R. S. Finn: *Isolated singularities of nonlinear partial differential equations.*

Equations of the form (1) $(\sigma\phi_x)_x+(\tau\phi_y)_y=0$ are considered. Let $\sigma=f(\phi_x, \phi_y)$, $\tau=g(\phi_x, \phi_y)$, and let (1) be of elliptic type. Let $\phi(x, y)$ be a solution single-valued in a neighborhood of an isolated singular point q. Then if near q, $(\sigma^2+\tau^2)^{1/2}|\nabla\phi| =o(1/r)$ where r is the distance to q, the singularity at q is removable. Previous results of Bers on the minimal surface equation (Ann. of Math. vol. 53 (1951) pp. 364–386) and of the author (Bull. Amer. Math. Soc. Abstract 57-3-212) appear as special cases. If $\sigma=\tau =f(\phi_x^2+\phi_y^2)$, every solution with n-valued gradient at q may be represented in the form $\phi(x, y)=c$ arc tan $y/x+\chi(x, y)$, where $\chi(x, y)$ is n-valued and bounded at q. Also considered are nonlinear equations of the form $\sum_1^n(\sigma_\nu\phi_{x_\nu})_{x_\nu}=0$. If near an isolated singular point q, $(\sum_1^n \sigma_\nu^2)^{1/2}|\nabla\phi| =(1/r^{n-1})$, $\phi(x, y)$ is bounded at q. The proofs depend on repeated application of several forms of Green's identity, and on an integration process defined over level surfaces of a function. (Received July 23, 1951.)

457t. Evelyn Frank: *On certain determinantal equations.*

The nature of the roots of certain determinantal equations is derived by the use of continued fractions. Also, conditions on determinantal equations are found which give the number of roots with positive and negative real parts. The determinantal equation $|A-zI| =0$ has all roots pure imaginary or zero if the elements a_{jk} and $-a_{kj}$ are conjugate complex constants and the a_{jj} are pure imaginary or zero, $j, k=1, 2, \cdots, n$. (Received May 21, 1951.)

458t. M. S. Friberg: *A method for the effective determination of conformal maps.*

A method similar to that of Theodorsen is derived for conformal mapping of nearly circular regions. Let C: $\rho=\rho(\theta)$, $0\leqq\theta\leqq 2\pi$, ($\rho''(\theta)$ continuous) be a closed Jordan curve and $\beta(\theta)$ the angle between the radius vector and the external normal at the point $w=\rho(\theta)e^{i\theta}$. Suppose $w=f(z)$ maps $|z|<1$ conformally onto the interior of C such that $f(0)=0$, $f'(0)>0$, and let $\theta(\phi)=\arg f(e^{i\phi})$. Then $\theta(\phi)$ and $\theta'(\phi)$ satisfy the integro-differential equation (*) $\log \theta'(\phi)= \log\cos \beta[\theta(\phi)]+(1/2\pi)\int_0^\pi\{\beta[\theta(\phi+t)] -\beta[\theta(\phi-t)]\} \cot (t/2)dt\equiv T[\theta(\phi)]$. Under suitable assumptions regarding $\beta(\theta)$ the author solves (*) by successive approximations. If $\theta_0(\phi)=\phi$, $\theta_n(\phi)=T[\theta_{n-1}(\phi)]$, explicit estimates are obtained for $|\theta_n-\theta|$ and $|\theta_n' -\theta'|$, which show the convergence to be *geometric*. The actual *numerical* solution of (*) is obtained by replacing (*) by an analogous vector equation (in p-dimensional space). This vector equation is solved by iteration and the difference between the nth iterate and $\theta_n(\phi)$ is estimated. The numerical procedure is similar to the one employed by A. Ostrowski in connection with the Theodorsen-Garrick method (National Bureau of Standards, 1949). (Received July 25, 1951.)

459. Bent Fuglede and R. V. Kadison: *Determinant theory in finite factors.*

A determinant is defined in factors of type II_1 and is shown to have the usual desirable properties on the regular operators. Actually the "determinant" which is defined reduces, in the n-dimensional case, to the absolute value of the nth root of the actual determinant. With the aid of the theory developed, it is proved that the trace of an arbitrary operator in a factor of type II_1 lies in the convex hull of the spectrum of the operator. (Received May 2, 1951.)

460*t*. P. R. Garabedian: *Orthogonal harmonic polynomials.*

It is shown that the harmonic polynomials $\cos h\phi\int_0^\pi P_n'(z+i\rho\cos t)\cos htdt$ $\sin h\phi\int_0^\pi P_n'(z+i\rho\cos t)\cos htdt$ are orthogonal over all the prolate spheroids $z^2\mathrm{ch}^{-2}\alpha+\rho^2\mathrm{sh}^{-2}\alpha<1$ in the sense of the norm $\int\int\int f^2\rho d\rho d\phi dz$ for harmonic functions f. Here ρ, ϕ, z are cylindrical coordinates, and P_n is the Legendre polynomial of degree n. It follows that the same harmonic polynomials are orthogonal over the surface of each of these spheroids with respect to the weight function $|1-(z+i\rho)^2|^{1/2}$. A similar complete orthogonal system of harmonic polynomials is found for the oblate spheroids $z^2\mathrm{sh}^{-2}\alpha+\rho^2\mathrm{ch}^{-2}\alpha<1$, and analogous results are obtained with the Dirichlet integral as norm in both the prolate and the oblate cases. The kernels for the complete orthonormal systems thus defined are set up, and in terms of them expressions are given for the Green's functions of the Laplace and biharmonic equations in a spheroid. The results can be extended to the case of a shell region between two confocal spheroids. (Received July 13, 1951.)

461*t*. P. R. Garabedian and D. C. Spencer. *Complex boundary value problems.*

In a cell M of the $2k$-dimensional Euclidean space of the variables x_j, y_j, differential forms ϕ are discussed which are pure in the sense that they can be expressed in terms of the dz_j only, $z_j=x_j+iy_j$. For these pure forms, complex operators d and δ are introduced which are an analogue of Hodge's operators d and δ in k-dimensional space. A boundary value problem for the system of partial differential equations $\delta d\phi=0$ in M is posed, and the corresponding existence theorem is developed by a method which combines the Fredholm integral equation and orthogonal projection. The Green's and Neumann's forms G and N for the system $\delta d\phi=0$ are defined, and the kernel form K for forms satisfying the generalized Cauchy-Riemann equations $d\beta=\delta\beta=0$ is introduced. The fundamental identity $K=dG+\delta N$ is proved for forms of each order p, $1\leqq p\leqq k$. When $p=k$ the kernel form K reduces essentially to the Bergman kernel function in k complex variables. Thus the present theory yields a generalization of the methods of classical potential theory which applies, in particular, to the study of analytic functions of several complex variables. (Received July 13, 1951.)

462. B. R. Gelbaum and G. K. Kalisch: *Measure in semigroups.*

This investigation concerns conditions implying that a semigroup S with a cancellation law (no identity or commutativity, unless so stated) with bounded invariant measure is a group. (1) If the measure in S satisfies a weakened version of "shearing in $S\times S$ is measurability preserving" (see P. R. Halmos, *Measure theory*, New York, 1950) then S is a group. A counterexample shows that the shearing condition may not be omitted entirely. (2) If S is locally compact with an invariant, bounded Borel measure, then S is a group. (3) In this case, if S is commutative or satisfies the above shearing condition, its topology may be weakened so that S becomes a separated compact topological group whose Haar measure coincides with the original measure.

(4) In the abelian case, if S has no topology relevant to the measure and no shearing condition is assumed, the measure on S can be extended to its quotient group with boundedness and invariance preserved. Here the hypothesis of boundedness may be replaced by the condition: if xF is measurable ($x \in S$, $F \subset S$), then so is F. Then the measure is again invariantly extensible to the quotient group. (Received January 11, 1951.)

463t. Casper Goffman: *A generalization of the Riemann integral.*

Every property $\mathfrak{P}$ of a set relative to an interval, which is such that (a) if $(S, I)\mathfrak{P}$ (the set S has the property $\mathfrak{P}$ relative to the interval I), and $T \subset S$, then $(T, I)\mathfrak{P}$, and (b) if $(S, I)\mathfrak{P}$, then $(CS, I)\mathfrak{P}$ (the complement CS of S does not have the property $\mathfrak{P}$ relative to I), yields upper and lower integrals—the upper and lower $\mathfrak{P}$-integrals—for every bounded real function defined on the interval $[0, 1]$. In particular, if $(S, I)\mathfrak{P}$ means that $S \cap I$ is empty, the $\mathfrak{P}$-integral is the Riemann integral, and if $(S, I)\mathfrak{P}$ means that $S \cap I$ may be covered by a sequence of intervals the sum of whose lengths is less than 1/2 the length of I, then the $\mathfrak{P}$-integral is the Lebesgue integral. It is shown that not all $\mathfrak{P}$-integrals are additive, but a condition is found which assures additivity. (Received July 12, 1951.)

464t. A. W. Goodman: *Inaccessible boundary points.*

By the use of infinite products an $f(z)$ is constructed which is analytic in $|z| < 1$ and maps that region conformally onto a region with inaccessible boundary points. (Received July 23, 1951.)

465t. P. R. Halmos: *Commutators of operators.*

(All italic capital letters in what follows denote bounded linear transformations of an infinite-dimensional complex Hilbert space into itself.) If $C = PQ - QP$, then C is called a *commutator*; if $C = P^*P - PP^*$, then C is called a *self-commutator*. Wintner has asked whether or not it is true that if C is a commutator, then the infimum of $|(Cx, x)|$, extended over all unit vectors x, is equal to zero. The main purpose of this note is to prove that the answer is no. The positive results from which this negative answer is easily deduced are the following ones. Theorem 1: if A is Hermitian, then A is the sum of two self-commutators Theorem 2: if A is Hermitian, then there exists an Hermitian B such that $A + iB$ is a commutator. As easy consequences of Theorems 1 and 2 one obtains the assertions (i) every (not necessarily Hermitian) A is the sum of four commutators, and (ii) every additive functional (trace), that vanishes on all commutators, vanishes identically. (Received May 31, 1951.)

466t. Einar Hille: *A note on Cauchy's problem.*

A Cauchy problem is formulated for the functional equation $y'(t) = U[y(t)]$ or, more generally, $y^{(n)}(t) = U^n[y(t)]$, where U is a linear operator on a B-space Y to itself. A solution defined for $t > 0$ such that $\lim\sup_{t\to\infty} t^{-1} \log \|y(t)\| < \infty$ is of normal type. Such solutions are uniquely determined by their initial values provided U is closed, the resolvent $R(\lambda; U)$ exists for $\lambda > \lambda_0$, and $\lambda\|R(\lambda; U)\|$ stays bounded when $\lambda \to \infty$. If U generates a semi-group $\{T(t)\}$ of linear bounded operators, then $T(t)[y_0]$ gives the only solution of normal type when $y_0 \in D[U]$, the domain of U. Conversely, if U satisfies the uniqueness conditions and if the equation has a solution $y(t) = y(t; y_0)$ for $y_0 \in D[U]$ and $\|y(t; y_0)\| \leq Me^{\alpha t}\|y_0\|$, then U generates a semi-group $\{T(t)\}$ and $y(t; y_0) = T(t)[y_0]$. Applications are given to Cauchy's problem for linear partial differential equations. (Received August 8, 1951.)

467*t*. Einar Hille: *Behavior of solutions of linear second order differential equations.*

A study is made of the equation $w''=\lambda F(x)w$ where $F(x)$ is positive and continuous for $0\leqq x<\infty$, $xF(x)\notin L(0, \infty)$, and $\lambda=\mu+\nu i$ is a complex parameter with $\nu\neq 0$ if $\mu<0$. It is shown that certain fundamental solutions describe spirals in the complex plane as $x\rightarrow\infty$ and properties of monotony and convexity of absolute value and argument are determined as well as integrability properties over $(0, \infty)$. The main problem is the determination of sufficient conditions for $\mu\leqq 0$ under which the subdominant tends to zero when $x\rightarrow\infty$ and $F(x)$ times this solution is in $L(0, \infty)$. Extremal properties for the rate of growth of solutions of this class of differential equations are indicated. (Received June 22, 1951.)

468*t*. Einar Hille: *On the generation of semi-groups and the theory of conjugate functions.*

A new proof, based on elementary identities satisfied by the resolvent, is given of the theorem according to which a linear closed operator U of domain dense in the complex B-space X such that $\lambda R(\lambda; U)$ is a contraction operator for $\lambda>0$, is the infinitesimal generator of a semi-group $T(\xi)$ given by the strong limit of $[(n/\xi)R(n/\xi; U)]^n$. In an application to conjugate functions it is shown that the derivative of the conjugate satisfies the indicated conditions if $X=L_p(-\infty, \infty)$, $1<p<\infty$, and is the infinitesimal generator of the semi-group of Poisson transforms for the upper half-plane. Only for the case $p=2$ had the author previously published a proof based on the spectral theorem. (Received June 22, 1951.)

469. W. H. Ito: *On conjugate functions.*

The paper is concerned with an extension of the classical theorem of M. Riesz on the pth means of conjugate functions in the unit circle (Math. Zeit. vol. 27 (1927)), to conjugate functions harmonic in more general regions. 1. Let C be a simple, closed, Jordan curve with a continuously turning tangent, containing $z=0$ in its interior, $I(C)$, and let $F(z)=u(z)+iv(z)$ be analytic in $I(C)$, satisfy $v(0)=0$, and be continuous on $I(C)+C$. Then $[\int_C|v|^p ds]^{1/p}\leqq A_p(C)[\int_C|u|^p ds]^{1/p}$, $1<p<\infty$, where $A_p(C)$ depends on only p and C. 2. If, instead of the continuously turning tangent condition, we substitute the hypothesis that C be convex, the above inequality also holds in the range $2\leqq p<\infty$. We can relax the assumption of continuity of $F(z)$ in favor of a less restrictive hypothesis in both theorems 1 and 2. A lemma on conjugate functions on the unit circle due to A. Zygmund (Fund. Math. vol. 13 (1929) Lemma α) is extended to the case of the boundary values of functions conjugate within a convex contour. (Received July 18, 1951.)

470*t*. Meyer Jerison: *On algebras associated with a compact group.* Preliminary report.

If G is a compact group and R is a Banach algebra, then the set of continuous mappings of G into R, with multiplication defined by convolution and the norm defined as the supremum of the norms of the images under the mapping, is a Banach algebra which will be denoted by $C(G, R)$. It is clear that without some restriction on G or R, neither G nor R will be determined by the algebra $C(G, R)$. However, the following theorem can be proved: If $C(G, R)$ is (isometric and) isomorphic to $C(G', R')$, G and G' are compact abelian groups, and the only idempotents of R and R' are their respective identities, then G is isomorphic to G' and R is isomorphic to R'. This result is obtained by proving first that if $R=R'$ is the field of complex numbers, then G and G' are isomorphic. In the latter theorem, it is not necessary to assume that the groups are abelian. (Received July 11, 1951.)

471. J. L. Kelley: *Banach spaces with the extension property.*

A Banach space B has the *extension property* if: if F is a bounded linear function on a subspace of a Banach space C, the values of F being in B, then F has an extension F' to all of C such that $\|F\|=\|F'\|$. *Theorem.* A Banach space with the extension property is equivalent to the space of continuous real-valued functions on an extremally disconnected compact Hausdorff space. The converse, as well as a weakened form of the above theorem, has been established by L. Nachbin (Trans. Amer. Math. Soc. vol. 68 (1950) pp. 28–46) and by D. B. Goodner (ibid. vol. 69 (1950) pp. 89–108). (Received July 23, 1951.)

472. V. L. Klee: *Convex sets in linear spaces.* II.

This is a sequel to an earlier paper bearing the same title (Duke Math. J. vol. 18 (1951) pp. 443–466). It is demonstrated that for a convex subset X of an arbitrary linear system L, the basic questions concerning polygonal connectedness of $L\sim X$ have the same answers as in the case of a two-dimensional L. Some characterizations of hyperplanes are given. In answer to a question of Erdös, it is proved that no separable Banach space can be covered by fewer than c hyperplanes. Finally, it is proved that every non-reflexive separable Banach space contains a pair of disjoint bounded closed convex sets which cannot be separated by a hyperplane. This extends a result of Dieudonné and, when combined with a theorem of Tukey, provides a new characterization of reflexivity. (Received July 20, 1951.)

473*t*. A. J. Lohwater: *A uniqueness theorem for a class of harmonic functions.*

Let $u(r, \theta)$ be harmonic in $|z|<1$, $z=re^{i\theta}$, and let the integral $\int_0^{2\pi}|u(r, \theta)|d\theta$ be bounded independently of r. It is proved that, if $u^*(\theta)=\lim_{r\to 1} u(r, \theta)=0$ for almost all θ and if $u^*(\theta)$ is infinite only for $\theta=\theta_k$, where $0<\theta_k<\theta_{k+1}<2\pi$ and $\lim_{k\to\infty}\theta_k=\theta^*<2\pi$, then there exist constants c^* and c_k, $c_k\neq 0$ $(k=1, 2, \cdots)$ with $\sum|c_k|<\infty$, such that $u(r, \theta)\equiv c^*K(r, \theta-\theta^*)+\sum_{k=1}^{\infty} c_kK(r, \theta-\theta_k)$, where $K(r, \theta-\alpha)$ is the Poisson kernel $(1-r^2)/[1+r^2-2r\cos(\theta-\alpha)]$. If $u^*(\theta)=0$ for almost all θ, and if $u^*(\theta)$, wherever else it may exist, is finite, then $u(r, \theta)$ is identically zero in $|z|<1$. (Received July 18, 1951.)

474*t*. A. J. Lohwater: *On the Schwarz reflection principle.*

Let $w=f(z)$ be meromorphic in $|z|<1$ and let $|f(re^{i\theta})|$, $z=re^{i\theta}$, have radial limit 1 for almost all θ belonging to an arc A $[0\leqq a_1<\theta<a_2\leqq 2\pi]$ of $|z|=1$. It is shown that if P is a singular point of $f(z)$ on A, then, given an arbitrary point $e^{i\lambda}$ of $|w|=1$, either $e^{i\lambda}$ is in the range $R(P)$ of $f(z)$ at P, or else there exists a Jordan arc L in $|z|<1$ terminating at a point ζ of $|z|=1$ arbitrarily close to P, such that as $z\to\zeta$ along L, $f(z)\to e^{i\lambda}$ (that is, $e^{i\lambda}$ is an asymptotic value of $f(z)$ "near P"). If, furthermore, $f(z)$ has a finite number of zeros and poles in a neighborhood of the singular point P, the union of $R(P)$ and the set of asymptotic values of $f(z)$ "near P" is precisely one of the sets $|w|\leqq 1$, $|w|\geqq 1$, or the extended plane. These theorems extend results of Carathéodory [Comment. Math. Helv. vol. 19 (1947) p. 266], Nevanlinna [Annales Academiae Scientiarum Fennicae (A) vol. 32 (1929) no. 7, p. 28], and Seidel [Trans.

Amer. Math. Soc. vol. 36 (1934) p. 208]. (Received July 18, 1951.)

475. G. R. MacLane: *Approximation by the derivatives of an entire function.*

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n/n!$ in $|z| < R$. Then the family $\{f^{(m)}(z)\}$ is normal in some neighborhood of $z=0$, and all its limit functions are holomorphic there (that is, none is the constant ∞) if and only if $|a_n| \leqq A$ for some constant A and all n. If this condition is satisfied, then $f(z)$ is entire and the family of derivatives is normal in the z-plane. If there are exactly p distinct limit functions, then $f(z) = g(z) + \sum b_n z^n/n!$, where $b_n \to 0$ (as $n \to \infty$) and $g(z) = \sum_{n=0}^{p-1} A_n \exp(w^n z)$, with $w = \exp(2\pi i/p)$ and g.c.d. $\{p, 1 \text{ sgn } |A_1|, 2 \text{ sgn } |A_2|, \cdots, (p-1) \text{ sgn } |A_{p-1}|\} = 1$. Now $|a_n| \leqq A$ implies $|f(z)| = O(\exp |z|)$, but the reverse implication is false. A result in the opposite direction is: there exists an entire function $F(z)$ such that (1) $|F(z)| = O(\exp (1+\epsilon)|z|)$, for any $\epsilon > 0$, and (2) if D is any simply-connected domain of the z-plane and $q(z)$ is any function holomorphic in D, then there exists a subsequence, n_k, of the positive integers such that $F^{(n_k)}(z) \to q(z)$ uniformly in any closed subset of D. (Received July 20, 1951.)

476. Imanuel Marx and George Piranian: *On the characterization of Lipschitz functions.*

Let $f(t)$ be a continuous function of t. The function $\alpha[f; t] = \liminf_{h\to 0} \log |f(t+h) - f(t)| / \log |h|$ may be called the Lipschitz function of $f(t)$. For any constant α, $0 < \alpha < 1$, W. S. Loud [Proceedings of the American Mathematical Society vol. 2 (1951)] has constructed a function $f(t)$ which has α as Lipschitz function in a sense somewhat stricter than that of the statement above. The authors show that if $\alpha(t)$ is a Lipschitz function, it is the limit inferior of a sequence of continuous functions of t. Conversely, if $0 \leqq \alpha(t) \leqq 1$ and $\alpha(t)$ is the limit inferior of a sequence of continuous functions, there exists a function $f(t)$ such that $\alpha[f; t] = \alpha(t)$. (Received July 20, 1951.)

477*t*. T. S. Motzkin and J. L. Walsh: *On the derivative of a polynomial and the Chebyshev polynomial.*

If $\mu(z)$ is positive on the point set E: $(z_1, z_2, \cdots, z_n)$, the problem of determining the polynomial $T_{n-1}(z) \equiv z^{n-1} + A_1 z^{n-2} + \cdots + A_{n-1}$ which minimizes max $[\mu(z)|T_{n-1}(z)|, z \text{ on } E]$ is essentially equivalent to the problem of determining the polynomial $t_{n-2}(z)$ which for given $f(z)$ minimizes max $[\mu(z)|f(z) - t_{n-2}(z)|, z \text{ on } E]$. Let $f_1(z) \equiv B_0 z^{n-1} + B_1 z^{n-2} + \cdots + B_{n-1}$ be the polynomial of degree $n-1$ which coincides with $f(z)$ on E. Suppose $B_0 \neq 0$. *The value of* arg $\{[f(z) - t_{n-2}(z)]/B_0\}$ $=$ arg $[T_{n-1}(z)]$ *in each point of E is independent of $\mu(z)$ and $f(z)$. The origin lies in the convex hull of the points $[f(z_i) - t_{n-2}(z_i)]$. All zeros of the polynomial $f_1(z) - t_{n-2}(z)$ lie in the convex hull of E.* Previous related work is due to Fejér, Fekete, and von Neumann. (Received July 23, 1951.)

478*t*. D. J. Newman: *A new proof of a conjecture of Pólya.*

Pólya has a conjectured that an entire function of zero type cannot be bounded over the integers unless it is a constant. A proof of this statement is given in Paley and Wiener, *Fourier transforms in the complex domain*. This proof, however, is somewhat involved and uses rather deep theorems on Fourier transforms. In this note the author gives a simpler proof of the conjecture of Pólya without any recourse to

Fourier transforms. The author uses Wigert's theorem which states that if $f(z)$ is an entire function of zero type then the function $g(z)=\sum_{n=0}^{\infty} f(n)z^n$, convergent for $|z|<1$, is an entire function of $1/(1-z)$; he also uses the fact that $\sum_{n=-\infty}^{-1} f(n)z^n$ represents the analytic continuation of $g(z)$ to $|z|>1$. These two facts imply that $g(z)(1-|z|)$ is bounded over the entire complex z-plane, if $f(n)$ is to be bounded. It is then shown that, since the only singularity of $g(z)$ is at $z=1$, and since $g(z)(1-|z|)$ is bounded, it follows that $g(z)=a+c/(1-z)$. From this it follows, by comparing coefficients, that $f(n)=c$, or that $f(z)-c$ is an entire function of zero type which vanishes at the integers. Such a function, however, must be identically zero, and the theorem is proved. (Received June 1, 1951.)

479*t*. D. J. Newman: *On series of the form* $\sum \pm x^n/n!$.

The author discusses the effect of introducing $\pm$ signs into various series. A typical question which is answered is, "Can $\pm$ signs be introduced into the series $\sum x^n/n!$ such that the resulting function tends to 0 as $x\rightarrow\pm\infty$?" It is shown in fact that the only way in which $\sum \pm x^n/n!$ can tend to zero at $+\infty$ is for the signs to be alternating and this answers the previous question in the negative. The general tendency of the results seems to indicate that $\sum \pm x^n/n!$ is large at $+\infty$ for most choices of the sequence of $\pm$ signs, but it is shown however that almost all of the series involved are $o(e^x)$ as $x\rightarrow+\infty$. The major result of the paper is the following: The only time $\sum \pm x^n/n!$ will be $O(e^{(1-\epsilon)x})$ is when the $\pm$ signs involved are periodic after a while, that is after a finite number of exceptional signs, also the period must be not greater than $2\pi/\cos^{-1}(1-\epsilon)$. Series directly related to the exponential such as $\sum x^{an}/(an)!$ are also considered, it is shown that for $a>2$ it can never be made into a bounded function over the positive axis by an introduction of $\pm$ signs. (Received June 1, 1951.)

480*t*. H. W. Oliver: *An existence theorem for nth Peano differentials.*

A real-valued function of two variables $f(x, y)$ is said to have an nth Peano differential at (x, y), $n=1, 2, \cdots$, if there exist numbers $f_{pq}(x, y)$, $1\leqq p+q\leqq n$, such that (1) $f(x+h, y+k)=f(x, y)+\sum_{1\leqq p+q\leqq n} h^pk^q/p!q!f_{pq}(x, y)+o(\rho^n)$, $\rho\rightarrow 0$, where $\rho=(h^2+k^2)^{1/2}$. The following theorem is proved: if f is measurable in a domain and if, for (x, y) in the measurable set E, $f(x+h, y+k)=f(x, y)+\sum_{1\leqq p+q\leqq n-1} h^pk^q/p!q!f_{pq}(x, y)+O(\rho^n)$, as $\rho\rightarrow 0$, then pp. in E, there exist numbers $f_{pq}(x, y)$, $p+q=n$, such that (1) holds. This is an extension, on the one hand, of a result of Rademacher and of Stepanoff for the first differentials; and, on the other, of a result due to Denjoy and to Marcinkiewicz and Zygmund for the nth Peano derivatives of a function of one variable. (Received July 20, 1951.)

481*t*. H. W. Oliver: *Borel derivative and exact Peano derivatives.*

A real-valued function $f(x)$ has an nth Peano derivative $f_n(x)$ at x if there exist numbers $f_k(x)$ such that $f(x+h)=f(x)+\sum_{k=1}^{n} h^k/k!f_k(x)+o(h^n)$, $h\rightarrow 0$; it has a (Borel) B_n-derivative $B_nf'(x)$ if $B_nf'(x)=\lim_{h\rightarrow 0} 1/h\int_0^h 1/t_1\int_0^{t_1}\cdots 1/t_{n-1}\int_0^{t_{n-1}}\{(f(x+t_n)-f(x))/t_n\}dt_n\cdots dt_1$. An exact nth Peano derivative, e.n.P.d., is one which exists at every point of an interval. It is shown that (f_n denotes an e.n.P.d.): (a) f_n has the property of Darboux; (b) mean-value theorems analogous to the ordinary mean-value theorem hold; (c) if f_n is bounded either above or below, it is the ordinary nth derivative at every point; (d) if $f_n\equiv 0$, f is a polynomial of degree $n-1$; (e) there is an everywhere dense, open set where f_n is the ordinary nth derivative; (f) the set $\{x;\ \alpha<f_n(x)<\beta\}$ is, for all $\alpha<\beta$, either void or of positive measure; (g) a monotone function has an ordinary first derivative wherever it has a B_n-derivative;

(h) under proper restrictions on f, for example, continuity, the $(n+1)$th Peano derivative of the nth integral of f exists if and only if $B_n f'(x)$ exists. (Received July 20, 1951.)

482t. B. J. Pettis: *On a vector space construction by Hausdorff.*

In 1932 Hausdorff constructed in any Banach space a second category linear subspace that is not a Banach space. His construction method can be used for a variety of *Gegenbeispiel* purposes. For example, in n-dimensional real vector space there exists an additive proper subgroup that is closed under multiplication by rational scalars, has vacuous interior, and is second category, everywhere dense, non-Baire, and non-measurable, and has both it and its complement having for measurable subsets only sets of measure zero and for Baire subsets only sets of the first category. Another application shows that in any second category linear topological space there exists a continuum of pairwise disjoint everywhere dense linear manifolds. These examples include the characteristics of examples due to Hamel, Kodaira, Halperin, Tukey, and Klee. They also show that in some recent theorems in topological groups certain hypotheses can not be dropped. (Received July 12, 1951.)

483. P. E. Pfeiffer: *Equivalence of measures on infinite product spaces.*

The measures considered are general probability measures on infinite product spaces. Necessary and sufficient conditions for equivalence are developed in terms of certain quasi-equivalence relations on suitable subclasses of the measurable sets. The class F (known to be a sigma-algebra of sets) is defined to be the class of those sets which are either $n+1$, $n+2$, $\cdots$-cylinder sets for each positive integer n or the empty set (for notation, see Halmos: *Measure theory*). A relation between certain conditional probabilities plus equivalence on F is shown to be necessary and sufficient for equivalence. Examples are given to exhibit the roles of these conditions. Comparison is made with (1) a theorem by Y. Kawada (Math. Japonicae vol. 1 pp. 170–177, Theorem 7) and (2) a theorem of S. Kakutani (Ann. of Math. vol. 49, p. 218). With the aid of Theorem 3 of Kawada's paper, new proofs of (2) and a modified form of (1) are given. Methods of the paper involve use of standard theorems of measure theory, Kawada's "projection" of measures, properties of the class F, and properties of conditional probabilities. (Received July 9, 1951).

484. Deborah Piranian and George Piranian: *Uniformly accessible Jordan curves with large sets of relative harmonic measure zero.*

Lohwater and Seidel [Duke Math. J. vol. 15 (1948) pp. 137–143] have constructed a Jordan region R whose boundary passes through a collinear set E of positive Lebesgue measure in such a way that E has harmonic measure zero relative to R. Also, Lohwater and G Piranian [Michigan Mathematical Contributions vol. 1 (1951) pp. 1–4] have exhibited a Jordan region R whose boundary passes through a set which has positive two-dimensional Lebesgue measure and has harmonic measure zero relative to R. The present authors simplify the previous constructions in such a way that the additional condition indicated in the title is satisfied. The proofs no longer depend on Lavrentieff's theorem on finite accessibility [Rec. Math. (Mat. Sbornik) vol. 36 (1929) pp. 112–115], but are based on the principle of Montel and Lavrentieff [C R. Acad. Sci. Paris vol. 184 (1927) pp. 1407–1409]. (Received July 12, 1951.)

485*t*. Everett Pitcher: *The projection property of quadratic functions.*

The following abstract approach to characteristic roots is developed. Let (A) $B(\eta, \phi, \lambda)$ define a real-valued, symmetric additive, homogeneous function of $\eta, \phi \in Y$, which is differentiable in λ for $0 \leqq \lambda < b$. Set $I(\eta, \lambda) = B(\eta, \eta, \lambda)$. Assume (B), (C), (D) of Bull. Amer. Math. Soc. Abstract 55-1-24, where the *index property* (that the index of $I(\cdot, \lambda)$ be finite and equal to the number of characteristic roots less than λ) and the *minimum property* are defined. The *projection property* is the property that with any interval $0 \leqq \lambda \leqq e < b$ there is a family of projections π_λ of Y onto a subspace Z of finite dimension such that (i) if $I(\eta, \lambda) < 0$ then $I(\pi_\lambda \eta, \lambda) < 0$ and (ii) if $\pi_\lambda \zeta = 0$ then $B(\pi_\lambda \eta, \zeta, \lambda) = 0$. *The projection property is equivalent to the index and minimum properties.* For quadratic functions of the calculus of variations, projections are readily defined by use of broken extremals in the manner which has been developed by Morse (see his exposition in *The calculus of variations in the large*, Amer. Math. Soc. Colloquium Publications, vol. 18). (Received July 23, 1951.)

486. F. J. Polansky: *On the conformal mapping of variable regions.*

Let D and E denote two bounded simply-connected domains which contain the origin in the complex w plane. Let $f(z)$ and $g(z)$ be the functions which map D and E conformally onto the domain $|z| < 1$ in the complex z plane in such a way that the origins correspond and $f'(0)$ and $g'(0)$ are positive. A "distance" e between the boundaries is defined and we prove that, uniformly in $0 < r < 1$, $\int_{C_r} |f(t) - g(t)| \, |dt| \leqq K(A_1^{1/2} + A_2^{1/2})\epsilon^{1/8}$, $\int_{C_r} |f'(t) - g'(t)| \, |dt| \leqq \phi(e, \delta)$, where C_r is the circle $|t| = r < 1$. Here K is an absolute constant, A_1 and A_2 are the areas of the regions. For the second inequality the boundaries are presumed to be rectifiable Jordan curves of length l_1 and l_2 and $\delta = |l_1 - l_2|$. The function ϕ depends on e, δ and certain other parameters which are directly characterized by geometric properties of the curves. Furthermore $\phi \rightarrow 0$ as $e, \delta \rightarrow 0$. (Received July 23, 1951.)

487. Gustave Rabson: *Fejer's theorem on $SU(2)$.*

The characters $\chi_n(x)$ of the special unitary group $SU(2)$ of two by two matrices are known explicitly. In terms of these the Dirichlet kernels, $D_n(x) = \sum_1^n k\chi_k(x)$, may be defined, and the Fejer p-kernels may be constructed from these using (C, p) summation. It may then be shown that (1) $\int |D_n(x)| \, dx$ is of order n; (2) $\int |F_n^1(x)| \, dx$ is of order $\log n$; (3) $\int |F_n^2(x)| \, dx$ is bounded; and (4) $F_n^3(x)$ is positive (this is an immediate consequence of a result due to Fejer in J. London Math. Soc. vol. 8 (1933) pp. 53–62). From (1) and (2) it follows from a result due to Lebesgue that there are continuous functions whose Fourier series diverge even in the sense of $(C, 1)$. From (3) it follows, as in the classical proof of Fejer's theorem, that the Fourier series of a continuous function will converge in the sense of $(C, 2)$. It may also be shown that the Fourier series of any function in L^2 will converge in the sense of $(C, 3)$ almost everywhere. From this and from an unpublished result due to J. Korevaar it follows that if $n_{k+1}/n_k > \lambda > 1$ then the sequence of n_kth partial sums of the Fourier series of a function in L^2 will converge to it almost everywhere. This result is due to Kolmogorov in the classical case. (Received July 23, 1951.)

488*t*. L. T. Ratner: *A note on semi-linear spaces.*

In the present note we begin by considering abstract semi-linear spaces (which differ from linear spaces in omission of one of the distributive laws: $(a+b)x \neq ax + bx$); later we introduce a topology or norm. Various hyperspaces of semi-linear spaces are themselves semi-linear spaces, and it is these hyperspaces which are the subject of primary interest in this study. Problems arising in connection with the study of multi-valued transformations in linear and semi-linear spaces are treated by hyperspace methods. Results are obtained on a "minimum hypothesis" basis. (Received July 26, 1951.)

489. M. O. Reade: *On certain conformal mappings of the exterior of the unit circle.* Preliminary report.

Let $w=f(z)=z+a_1/z+a_2/z^2+\cdots$ be regular and schlicht for $\infty>|z|>1$, and continuous for $\infty>|z|\geqq 1$; let the image of $|z|=1$ be a Jordan curve C. The principle results are the following ones. (i) If C is convex, then the image of each circle $|z|=r>1$ is convex, and (ii) if C is star-shaped, with respect to $w=0$, then the image of each circle $|z|=r>1$ is star-shaped, with respect to $w=0$. Integral representations of each type of mapping function are discussed. The derivations follow Study, *Vorlesungen über ausgewählte Gegenstände der Geometrie*, Leipzig, Teubner, 1913. (Received July 23, 1951.)

490. P. C. Rosenbloom: *The distribution of values of entire functions.*

Let $\eta(w)$ be any entire function which satisfies a differential equation of the form $P\eta'+Q\eta=1$, where P and Q are polynomials in w. Then if $f(z)$ is any entire function, we can obtain very precise relations between the distribution of the zeros of $\eta(f(z))$ and the growth of $f(z)$. For example, if η is a polynomial with no repeated roots, then we obtain a very simple proof of Nevanlinna's second fundamental theorem. If $\eta(w)=\exp(w/2\pi i)-1$, then we obtain very precise results on the distribution of the points where the function $f(z)$ takes on integral values. (Received June 22, 1951.)

491. Walter Rudin: *Inversion of second order generalized differential operators.*

Let $Ly=y''+py'+qy$, where p and q are continuous on $[a, b]$. Suppose that the system $Ly=0$, $y(a)=y(b)=0$, is incompatible, and let $K(x, t)$ be the corresponding Green's function. The operator L is generalized as follows. If F is defined in (a, b), and $a<x<b$, let $y_h(t)$ be such that $Ly_h=0$, $y_h(x+h)=F(x+h)$, $y_h(x-h)=F(x-h)$. Put $\Delta_h F(x)=y_n(x)-F(x)$, and $\Lambda F(x)=\lim_{h\to 0} 2\Delta_h F(x)/h^2$. $\Lambda^* F(x)$, $\Lambda_* F(x)$ are defined likewise, with lim sup and lim inf in place of lim. If $F''(x)$ exists, then $\Lambda F(x)=LF(x)$. The following theorem generalizes a result well known if $F\in C''$. Suppose (1) F is continuous and bounded on (a, b); (2) $\Lambda^*F(x)>-\infty$, $\Lambda_*F(x)<+\infty$ on (a, b), except possibly on countable sets E_1 and E_2; (3) $\limsup_{h\to 0}\Delta_h F(x)/h\geqq 0$ on E_1, $\liminf_{h\to 0}\Delta_h F(x)\leqq 0$ on E_2; (4) there exists a measurable function g such that $g(x)\leqq \Lambda^*F(x)$ and $\int_a^b(x-a)(b-x)|g(x)|dx<\infty$. Then $\Lambda F(x)$ exists p.p., and $F(x)=-\int_a^b K(x, t)\Lambda F(t)dt+y(x)$ for all x on (a, b), where $Ly=0$, and $y(a)=F(a+)$, $y(b)=F(b-)$. (The existence of $F(a+)$, $F(b-)$ is part of the conclusion.) The theorem can be applied to uniqueness problems for Sturm-Liouville series. (Received July 18, 1951.)

492. L. R. Sario: *Principal functions on Riemann surfaces.*

Let R be an arbitrary Riemann surface, closed or open, of finite or infinite genus. Consider a subregion D of R, relatively bounded by k. A functional L will be called a *normal linear operator* in D if it associates to every harmonic function v on k a harmonic function Lv in D satisfying the following conditions: (1) $Lv=v$ on k, (2) $\int_k d\overline{L}v=0$, (3) $L1=1$, (4) $Lv\geqq 0$ if $v\geqq 0$ on k, (5) $L(c_1v_1+c_2v_2)=c_1Lv_1+c_2Lv_2$. Consider on R

the analytic relative boundary l of a compact or noncompact subregion of R. In $S=R-l$ let s be a single-valued real function such that both branches permit a harmonic continuation across l. *If* $\int_{l^+ + l^-} \bar{d}s = 0$, *then there exists on* R *a single-valued real function* p, *harmonic on* l *and such that* $p-s=L(p-s)$ *in* S. *The function* $p=p(z;\, s,\, L)$ *is unique up to an additive constant and will be called the principal function for given* s, L *on* R. *If* $s \not\equiv Ls$, *then* p *is not constant*. By selecting different initial functions s and different operators L the corresponding principal function $p(z;\, s,\, L)$ furnishes the solution to different mapping, existence, uniqueness, and boundary value problems (Received July 17, 1951.)

493t. J. T. Schwartz: *Elliptic differential equations.* I. *Compact manifolds.*

If t is a generic covariant tensor of order p on a real analytic manifold M, then a notion of ellipticity for linear systems of differential equations of the form $L(t)=t'$ can be defined. If the coefficient functions occurring in L are analytic, the equation $L(T)=0$ has meaning for an arbitrary current of covariant order p. It is shown that a current satisfying such an equation is an analytic function. If L is self-adjoint in the formal sense, and M is compact, it is shown that L is essentially self-adjoint as an operator in Hilbert space. The inverse $(L-i)^{-1}$ is compact. The proof involves a generalization to elliptic systems of F. John's recent construction of a fundamental solution. (Received August 21, 1951.)

494t. J. T. Schwartz: *Elliptic differential equations.* II. *Noncompact manifolds.*

In the case where the real analytic manifold M is not compact, the space of solutions of an elliptic system $L(t)=0$ is discussed from the point of view of compactness and reproducing-kernel properties. If L is a formally positive elliptic operator, then its boundary-value theory on a noncompact manifold is discussed, use being made of the theory of semi-bounded operators. This discussion applies to arbitrary boundary conditions which are related in a suitable formal way to the positivity of L. (Received August 21, 1951.)

495. Seymour Sherman: *On a conjecture of Gelfand and Neumark.*

In their abstract characterization of uniformly closed subalgebras of bounded operators on hilbert space as Banach * algebras R satisfying six axioms, Gelfand and Neumark [*On the imbedding of normed rings into the ring of operators in hilbert space*, Rec. Math. (Mat. Sbornik) N.S. vol. 12 (1943) pp. 197–213] have conjectured that axiom 6, x^*x+e possesses a two-sided inverse element for each $x\in R$, is redundant. If the self-adjoint elements in R are ordered by letting $x\geqq 0$ mean $x=\sum_{k=1}^{n} z_k^* z_k$, $z_k\in R$, then it follows that, in the presence of the first five axioms, axiom 6 is equivalent to the *monotonicity of the norm* in the sense that for each $x, y\in R$, $0\leqq x\leqq y$ implies $\|x\|\leqq\|y\|$. Thus the suspected axiom, couched in algebraic terms, is replaced by an equivalent axiom phrased in the concepts of partially-ordered, normed, linear spaces It is also shown that if the norm of a C^* algebra is monotone with respect to a partial order whose cone of non-negative elements contains the set of positive semi-definite operators of the algebra, then this cone is just the set of positive semi-definite operators. (Received February 13, 1951)

496. I. M. Singer: *Maximal abelian algebras of finite factors.*

Let M and M' be a finite factor and its commutator, respectively, for which $Tr_M(I) = Tr_{M'}(I)$. Let $I_{M,M'}$ be the anti-isomorphism of M onto M' determined by $Tg = I_{M,M'}(T) \cdot g$, $T \in M$, and depending upon a fixed choice of a uniformly distributed vector g [Murray and von Neumann, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 208–248]. For any maximal abelian subalgebra A of M, the multiplicity of the weakly closed algebra $\bar{A}$ generated by A and $I_{M,M'}(A)$ is an invariant of A under the group of automorphisms of M. Any multiplicity can occur if M is approximately finite. If A is isomorphic to the continuous functions on a compact Hausdorff space X, then the uniform closure of the algebra generated by A and $I_{M,M'}(A)$ is naturally isomorphic to the continuous functions on $X \times X$ while the complete Boolean algebra of projections in $\bar{A}$ is naturally isomorphic to a complete Boolean algebra of subsets $X \times X$. If $\bar{A}_1$ and $\bar{A}_2$ have multiplicity one, then a necessary and sufficient condition that an isomorphism Φ carrying A_1 onto A_2 can be extended to an automorphism of M is that Φ can be extended to an isomorphism of $\bar{A}_1$ onto $\bar{A}_2$. (Received July 23, 1951.)

497. M. L. Slater: *Some generalizations of a lemma of Cesari.*

In a paper entitled *On the absolute extremals for the integrals on parametric surfaces* (Bull. Amer. Math. Soc Abstract 57-1-26) Cesari used the following lemma: Let $f(t)$ be continuous and nonincreasing in $0 \leqq t \leqq a$, greater than 0 in $0 \leqq t < a$, and satisfy $K[f(t)]^2 \geqq \int_t^a f(t)dt$. Then $a \leqq 2Kf(0)$. The author gives the following generalizations: Let $K_{1,\phi}$ $(0, a)$ be the set of non-negative functions $f(t) \in L(0, a)$ which are >0 a.e. in $[0, a]$ and satisfy $\phi(f(t)) \geqq \int_t^a f(t)dt$ throughout the closed interval $[0, a]$. $K_{1,\phi}$ is nonvoid and there exists $f_* \in K_{1,\phi}$ such that $f_*(t) \leqq f(t)$ for all $f \in K_{1,\phi}$ and all $t \in [0, a]$ if $\phi(x)$ $(x \geqq 0)$ has the following properties: (1) $\phi(0) = 0$ and $\phi(x)$ is strictly increasing, (2) $\phi \in C'$, (3) $\lim_{x\to\infty} \phi(x)/x = \infty$, (4) $\phi'(x)/x \in L(0, b)$ for any $b > 0$. $f_*(t)$ will satisfy $\phi(f_*(t)) = \int_t^a f_*(t)dt$. If in particular $\phi(x) = Kx^\alpha$ $(\alpha > 1)$ then $f_*(t) = ((\alpha-1)/\alpha K)^{1/(\alpha-1)}(a-t)^{1/(\alpha-1)}$. Again, let $K_{2,\phi}$ $(0, a; 0, b)$ be the set of non-negative functions $f(t, u) \in L(0, a; 0, b)$ which are greater than 0 a.e. in $[0, a; 0, b]$ and satisfy $\phi(f(t, u)) \geqq \int_t^a \int_u^b f(t, u)dtdu$ throughout the closed rectangle $[0, a; 0, b]$. $K_{2,\phi}$ is nonvoid and there exists $f_* \in K_{2,\phi}$ such that $f_*(t, u) \leqq f(t, u)$ for all $f \in K_{2,\phi}$ and all $(t, u) \in [0, a; 0, b]$ if $\phi(x)$ satisfies (1), (2), and (3) above and $\phi'(x)$ is in one of the classes $O(x^\alpha)$ $(\alpha > 0)$, $O((\log 1/x)^{-\beta})$ $(\beta > 2)$, $O((\log 1/x)^{-2}(\log \log 1/x)^{-\gamma})$ $(\gamma > 2), \cdots$, as $x \to 0$. $f_*(t, u)$ will satisfy $\phi(f_*(t, u)) = \int_t^a \int_u^b f_*(t, u)\,dtdu$. If in particular $\phi(x) = Kx^\alpha$ $(\alpha > 1)$, then $f_*(t, u) = (1/K)^{1/(\alpha-1)}((\alpha-1)/\alpha)^{2/(\alpha-1)}((a-t)(b-u))^{1/(\alpha-1)}$. (Received July 20, 1951.)

498. J. E. Thompson: *Some studies of conformal mapping on the boundary.*

This paper is concerned with the conformal mapping of infinite strip regions onto parallel strips. "Normal strip regions" are defined and the Ahlfors-Dufresnoy-Ferrand inequalities are modified and applied to obtain quite general distortion theorems concerning the mapping functions of such strips. A special type of normal strip region, called a quasi L strip, is introduced and more precise results are proved concerning its mapping function, generalizing some results of S. E. Warschawski. The paper concludes with a discussion of conditions under which the mapping function of Jordan regions satisfies a uniform Lip α condition on the boundary. Direct methods based on estimating the Dirichlet integral are used to obtain fairly general results of this type. More precise results are obtained by using the above-mentioned strip methods. For example, if $f(z)$ is the function which maps $|z| < 1$ conformally onto a Jordan region and if $0<\alpha\leqq 1$, conditions are given under which $f(z)$ is uniformly Lip $(\alpha-\epsilon)$ on $|z|=1$. Also for a certain class of curves a necessary and sufficient condition that $f(z)$ be uniformly Lip α on $|z|=1$ is given. (Received July 19, 1951.)

499t. Leonard Tornheim: *Normed fields over the complex field.*

A normed field F over the complex field C is a field containing C and having a norm satisfying $\|\alpha\|>0$ if $\alpha\neq 0$, $\|\alpha+\beta\|\leqq\|\alpha\|+\|\beta\|$, $\|\alpha\beta\|\leqq\|\alpha\|\,\|\beta\|$, and $\|c\alpha\|=|c|\,\|\alpha\|$ for any α, β in F and c in C. Then $F=C$, a result closely related to the Gelfand-Mazur theorem. A proof is presented based upon these simple facts. If $\alpha-c$ has an inverse for every complex c, then $\|1/(\alpha-c)\|$ attains a maximum M for c on a bounded closed nonvacuous set C_0 in the complex plane. Also for such an α, the sum S_n of $1/n(\alpha-c_0-ur)$, where u ranges over all nth roots of unity and $r<\|1/(\alpha-c_0)\|^{-1}$, has limit $1/(\alpha-c_0)$, whereas for c_0 a boundary point of C_0 and r properly chosen, $\|S_n\|$ is bounded away from M. This contradiction implies that $\alpha-c$ has no inverse for some c and, since F is a field, $\alpha-c=0$; hence $\alpha=c$. (Received July 30, 1951.)

500t. J. L. Walsh and D. M. Young: *On the accuracy of the numerical solution of the Dirichlet problem by finite differences.*

A solution of the difference equation analogue of Laplace's equation is discrete harmonic (d.h.). For d.h. functions the authors use analogues of harmonic measure and the Schwarz alternating process, as well as the explicit solution of the difference equation for the rectangle, to estimate errors for a d.h. function as an approximate solution of the Dirichlet problem for rectangular and other regions. The error is expressed in terms of either the derivatives or the modulus of continuity $\omega(\delta)$ on the boundary of the solution $u(x, y)$ of the Dirichlet problem. For example for the unit square the absolute value of the error in the closed region is bounded by the maximum of A and B where $A=2\omega[2^{1/2}(h^{2/7}+h)]+8Mh^{2/7}$ and $B=145\,Mh^{2/7}+24\omega(2h)+192\,Mh$ and where $2M$ is the oscillation of $u(x, y)$ on the boundary and h is the mesh size. (Received August 16, 1951.)

501t. H. C. Wang: *A theorem on fixed points under a group of homeomorphisms.*

In a recent paper, A. D. Wallace proved that if a cyclic group Z of homeomorphisms of a Peano continuum leaves fixed an end point, then Z must leave fixed another point, and in a later paper [Fund. Math. vol. 36], he asked the question whether this still holds if Z is assumed to be compact instead of cyclic. This note aims at answering his question by proving the following theorem: *Let E be an arcwise connected normal space, and G a group of homeomorphisms of E leaving fixed an end point p. G has no other fixed point if and only if, for each neighbourhood N of p, $G(N)=E$.* As a direct consequence of this theorem, it follows that the answer to the above question of Wallace is affirmative. (Received July 25, 1951.)

502t. J. G. Wendel: *Left centralizers and isomorphisms of group algebras.* Preliminary report.

A bounded linear operator A on a (real or complex) group algebra $L(G)$ is called a *left centralizer* in case it commutes with all left multiplications: $A(xy)=x(Ay)$, x, $y\in L(G)$. Using the Riesz-Kakutani theorem it is shown that to each such A there corresponds a unique regular (real or complex) bounded measure μ such that $Ax=x^*\mu$, where $(x^*\mu)(g)=\int x(gh^{-1})d\mu(h)$. Every right translation on $L(G)$ is an *isometric* left centralizer; it is shown that this property characterizes the right translations, up to scalar factors of unit modulus, and that the strong closure of the manifold spanned by the right translations in the space of all bounded operators on $L(G)$ coincides with the set of left centralizers. Now let T be an isomorphism of $L(G)$ upon a second group algebra $L(\Gamma)$, $\|T\| \leqq 1$. It is proved that G and Γ are bicontinuously isomorphic, and that T is induced by a point mapping followed by multiplication by a character. This generalizes a previous theorem (Pacific Journal of Mathematics vol. 1 (1951)) wherein the same conclusions were obtained for isometric T; but in the apparently more general case T turns out to be isometric after all. (Received July 18, 1951.)

503*t*. Bertram Yood: *Properties of linear transformations preserved under addition of a completely continuous transformation.*

In this paper a study is made of various properties of bounded linear transformations on a Banach space which are possessed by transformations of the form $I+U$ where I is the identity and U is completely continuous. These are the principal properties given by Riesz, Acta Math. vol. 41 (1918) and Schauder, Studia Math. vol. 2 (1930) as well as certain related properties. The classes of transformations possessing these properties are characterized. (Received July 20, 1951.)

504*t*. C. Y. Yu: *On some functions holomorphic in an infinite region.*

S. Mandelbrojt indicated the following proposition: If a function is holomorphic and bounded in a half-strip of the z-plane containing the half-axis ox as a part of its central line and if this function and a certain infinite sequence of its derivatives vanish at the origin, then it is identically zero. In the present paper, one considers a function $F(z)$ holomorphic in a region Δ of the z-plane defined by $x \geqq d$, $|y| \leqq g(x)$, where $-\infty < d < 0$ and where $g(x)$ is a certain positive continuous function tending to zero with $1/x$. In this case, if in Δ, $F(z)$ tends to zero rapidly enough and uniformly with respect to y as x tends to infinity, and if $F(z)$ and a certain infinite sequence of its derivatives vanish at the origin, then $F(z)$ is identically zero. In order to establish this proposition, one proves at first a lemma on some entire functions of zero order by means of a theorem of G. Valiron (Annales de la Facultĕ des Sciences de l'Universite de Toulouse vol. 27 (1913) pp. 117–257, §32) and then applies a result of Mandelbrojt on generalized quasi-analyticity (*Sĕries adhĕrentes, rĕgularisations descuites, applications*, Paris, Gauthier-Villars, in press; for a particular case, see Ann. École Norm. vol. 63 (1946) p. 372). (Received May 7, 1951.)

Applied Mathematics

505. W. F. Bauer: *Modified Sturm-Liouville problems.*

The eigenvalue problem $y''+(\lambda+p)y=0$, $a_1y(0)+a_2y'(0)+a_3y''(0)=0$, $b_1y(1)+b_2y'(1)+b_3y''(1)=0$ has the following orthogonality relation which can be conveniently expressed as an inner product: $[y_i, y_j] \equiv \int_0^1 y_iy_jdx-(a_3/a_2)y_i(0)y_j(0)+(b_3/b_2)y_i(1)y_j(1)=0$. Under the conditions $a_2a_3 \leqq 0$ and $b_2b_3 \geqq 0$ the eigenvalues are shown to be real and simple. With the aid of the inner product notation and the Green's function it is shown that under further conditions on the coefficients a_i and b_i and on $p(x)$ the problem has an infinite number of real, positive eigenvalues and a minimal property is established for them. An expansion theorem is then obtained for functions of class C^2 which satisfy the boundary conditions by establishing the

Parseval's relation and the uniform convergence of the series. The eigenvalue problem arises in the study of the transient temperatures of a solid which is radiating heat from its faces while one of its ends is in contact with a well-stirred liquid whose container is in itself radiating heat into the surrounding medium. Properties of the eigenvalues and an expansion theorem can also be obtained by the application of the Laplace transform to the boundary value problem. (Received July 16, 1951.)

506. Abraham Charnes and H. J. Greenberg: *Plastic collapse and linear programming.* Preliminary report.

The critical value of the loading parameter at which plastic collapse occurs in structures or continua for certain types of loading has been characterized by means of a maximum and a minimum principle by Greenberg, Prager, and Drucker. They proved also that maximizing (resp. minimizing) solutions automatically satisfy compatibility, hence correspond to collapse states. It is shown here, for structures, that the maximum problem can be reduced in various ways to an equivalent form—maximization of a linear functional subject to linear inequalities. Application of linear programming techniques yields: (1) a new minimum principle for each reduction to a programming problem, (2) a criterion distinguishing collapse states from noncollapse states, (3) systematic algebraic procedures for computation of collapse states. (Received July 23, 1951.)

507. C. L. Dolph: *On the transmission and reflection problem for electro-magnetic waves.*

If S is a closed surface separating two homogeneous media with different dielectric, conductivity, and permeability constants upon which an electro-magnetic wave is incident from the exterior, the total electric field is shown to be a system of three linear Fredholm-Stieltjes integral equations. This representation is obtained by means of the fundamental tensor solution of the vector wave equation (cf. for example, Schwinger and Levine, Comm. Pure and App. Math. Dec. 1950) and the technique is a generalization of that of Sternberg (Compositio Mathematica vol. 3 (1936)). If the permeabilities of the two media agree, the device of Hilbert (*Gründzuge einer allgemeinen Theorie der linear Integralgleichungen*, Teubner, 1912, p. 206) can be used to reduce the system to a single Fredholm integral equation of the second kind. Thus this problem is conceptually no more difficult than the corresponding problem for the scalar wave equation. (Received July 12, 1951.)

508. G. E. Forsythe, M. R. Hestenes, and J. B. Rosser: *Iterative methods for solving linear equations.*

Several iterative methods are given for solving the equations $Ax=b$, where A is a given matrix and b is a vector. These methods appear to be particularly adapted to high speed computing machines. They have the property that if there were no round-off error the solution would be obtained in at most n steps where n is the rank of A. In the singular case the least square solution is obtained. At each iteration the problem is started anew. Accordingly there is no accumulation of errors. In the hermitian case the method is based on the following result. If A, $B>0$ are hermitian matrices which commute then the system $b, Ab, \cdots, A^n b$ may be replaced by a set of B-orthogonal vectors by the algorithm $z_0=b$, $z_1=z_0-a_0Az_0$, $z_{i+1}=b_iz_i-a_iAz_i+c_{i-1}z_{i-1}$. (Received July 23, 1951.)

509*t*. G. E. Forsythe and T. S. Motzkin: *An extension of Gauss's transformation for improving the condition of systems of linear equations.*

Let the real n-by-n matrix A, symmetric or not, have a non-negative semi-definite quadratic form of rank $n-d$ $(0 \leqq d < n)$. Let s be any n-rowed column-vector. Let $B = B(s)$ be the $(n+1)$-by-$(n+1)$ matrix formed by bordering A with an $(n+1)$th column As, an $(n+1)$th row $s^T A$, and an element $b_{n+1,n+1} = s^T As$. (T denotes transposition.) Let $\{\lambda_i\}$, $\{\mu_i\}$ be the eigenvalues of A, B, respectively. In a lemma it is proved that, for all s, $0 = \mu_0 = \lambda_0 = \mu_1 = \lambda_1 = \cdots = \lambda_d = \mu_d < \lambda_{d+1} \leqq \mu_{d+1} \leqq \lambda_{d+2} \leqq \mu_{d+2} \leqq \cdots \leqq \lambda_n \leqq \mu_n < \infty$, and conversely, given any $\{\mu_i\}$ separated by the $\{\lambda_i\}$ as above, that one can find an s for which the $\{\mu_i\}$ are the eigenvalues of B. A familiar condition number $P(A)$ is extended to these singular matrices by the definition $P(A) = \lambda_n / \lambda_{d+1}$. It is related to the speed of convergence of certain iterative processes for finding one solution of consistent systems of linear equations $Ax = b$. By the lemma we find that, by suitably choosing s, one can make $P(B)$ assume any value in the interval $\lambda_n/\lambda_{d+2} \leqq P(B) < \infty$, and no other value. Gauss (Letter to Gerling, 26 December 1823, *Werke*, vol. 9, pp. 278–281) proposed the special value $s = (1, 1, \cdots, 1)^T$ for speeding the relaxation solution of certain normal equations arising in surveying. (Received July 20, 1951.)

510*t*. Abolghassem Ghaffari: *On one-dimensional Brownian motion.*

In a problem of one-dimensional Brownian motion it is required to solve the functional equation (1) $f(x, s; y, t) = \int_V f(x, s; z, u) f(z, u; y, t) dz$, $s < u < t$, where f is the displacement-distribution function. By applying Fréchet's method (Proc. London Math. Soc. vol. 39 (1935) pp. 515–540) one is led to take for the solution of (1) the series (2) $f(x, s; y, t) = \sum_{n=0}^{\infty} A_n(x, s) B_n(y, t)$, where A_n and B_n form a complete biorthonormal set of functions over V. Using this method and taking for the region V the interval $(-\infty, +\infty)$ one finds that (3) $f(x, s; y, t) = \sum_{n=0}^{\infty} \phi_n(x)\phi_n(y)\theta^n(s, t)$, where the functions $\phi_n(x) = H_n(x) \exp(-x^2)/(2^n n! \pi^{1/2})^{-1/2}$ form a complete orthonormal set of functions over $(-\infty, +\infty)$, $\theta(s, t) = a(s)/a(t)$ where $a(s)$ is a positive increasing function $\neq 0$, and $H_n(x)$ is the nth Hermite polynomial. It is shown that, for s, t fixed such that $s < t$ and x, y varying arbitrarily, the series (3) is absolutely and uniformly convergent and also that the solution (3) is square summable over the whole plane. It is further shown that the solution (3) satisfies the equation $f_{xx} + [2a(s)/a'(s)] f_s + (1 - x^2) f = 0$. It is proved that for $s = t$ and $x \neq y$, $f(x, s; y, s) = 0$ and also for $s = t$ and $x = y$, $f(x, s; x, s) = \infty$. The author has shown that the solution (3) is not among those obtained by A. Kolmogoroff (Math. Ann. vol. 104 (1931) pp. 415–458). (Received July 11, 1951.)

511*t*. P. G. Hodge, Jr.: *The Brownian motion of coupled systems in neutral equilibrium.*

It is well known that colloidal particles suspended in a liquid will engage in random movements, known as *Brownian motion.* More recently, Barnes and Silverman [Review of Modern Physics vol. 6 (1934) pp. 162–192] considered the random deflections of measuring instruments from a position of stable equilibrium, due to the random impacts of the surrounding fluid molecules. In the present paper, the Brownian motion of systems in neutral equilibrium is considered. The kinetic energy of the system is assumed to be expressible in generalized coordinates x_i, in the form $K = (1/2) \sum a_{ij} \dot{x}_i \dot{x}_j + b_i \dot{x}_i + c$, where a_{ij}, b_i, and c may be functions of the coordinates

x_i, but not of the velocities $\dot{x}_i$. If the functions a_{ij}, b_i, and c are subject to certain restrictions, and if the generalized coordinates x_i are initially orthogonal, the mean-square deviations can be computed as functions of time, even in the case of neutral equilibrium. The method used is a generalization of that of Infeld [*On the theory of Brownian motion*, Applied Mathematics Series, no. 4, University of Toronto Press, Toronto, 1940]. The effect of certain approximations which were not discussed by Infeld is here considered in some detail. (Received June 15, 1951.)

512*t*. Andrew Sobczyk: *Resistances of spheres, and of packages of spheres.*

The resistance R between contacts at ends of a diameter of a conducting sphere, corresponding to a central plane angle 2α, is shown to satisfy the inequality $(1/\alpha) \leqq (2\pi aR/K) \leqq (1.442/\alpha)$, where a is the radius, K the resistivity of the material of the sphere. This result is obtained by use of the device of inserting appropriate insulating and conducting sheets, of integration with spherical shells as elements, of the known solutions of Laplace's equation for the point and hemispherical contact cases, and of methods in Smythe, *Static and dynamic electricity.* It is pointed out that there is an error of approximation in a recent paper by A. Gemant on resistance of soils. With view to obtaining a theory of the conductivity of a bar of powdered material, the resistances of packages of spheres, for the various possible packings of spheres in space described in Hilbert and Cohn-Vossen, *Anschauliche Geometrie*, are determined as functions of the contact angle α. (Received July 26, 1951.)

513. M. L. Stein: *Gradient methods in the solution of systems of linear equations.*

Let $Ax=b$ be a system of linear equations. Let a star (*) denote "conjugate transpose." Let H be a positive hermitian matrix. The minimum of $f(x) = (b-Ax)^*H(b-Ax)$ yields a solution of $Ax=b$ should one exist and otherwise a best fit of b by Ax in the metric H. The following algorithm for minimizing $f(x)$ has been tested experimentally. Let x_i be known, $\eta_i = A^*H(b-Ax_i)$, $\gamma_i = \eta_i^*\eta_i/\eta_i^*A^*HA\eta_i$. Then $x_{i+1} = x_i + \beta_i\gamma_i\eta_i$ where β_i is a scalar such that $1/\beta_i + 1/\bar{\beta}_i \geqq 1+\delta$, $|\beta_i| \geqq \delta$, $0<\delta<1$. For $\beta_i \equiv 1$ this is the optimum gradient method. Tests show that $\beta_i \equiv \beta < 1$ gives faster convergence than $\beta_i \equiv \beta \geqq 1$. In sharp contrast to methods using $\beta \geqq 1$ those using $\beta<1$ show an instability which yields large accelerations in convergence. This self-acceleration property is significant for the use of the method on computors having limited storage capacity. Byproducts of the method are estimates of bounds for eigenvalues of A. Tests in which the problem $Ax=b$ was transformed into an equivalent eigenvalue problem are also discussed. Modifications of the Hestenes-Karush optimum-α method (M. R. Hestenes and W. Karush, *A method of gradients for the calculation of the characteristic roots and vectors of a real symmetric matrix*, to appear in NBS Journal of Research) are compared. (Received July 20, 1951.)

514. H. F. Weinberger: *An error estimate for the method of Weinstein.*

The method of Weinstein (Mémorial des Sciences Mathématiques, no. 88, 1937) gives upper bounds for the eigenvalues λ_i of the projection of a completely continuous positive definite operator L into a linear subspace $\mathfrak{L}$ of the Hilbert space $\mathcal{H}$ in which L operates. The eigenvalues κ_i and eigenfunctions u_i of L in $\mathcal{H}$ are assumed to be known. These upper bounds are the corresponding eigenvalues of the projection of L

into $\mathfrak{H}\ominus\{p_1, \cdots, p_n\}$ where $p_1, \cdots, p_n$ are any vectors (constraints) in $\mathfrak{H}\ominus\mathfrak{Q}$. It is shown that a uniform error estimate which can be made arbitrarily small is obtained by taking p_i as the projection into $\mathfrak{H}\ominus\mathfrak{Q}$ of the eigenvector u_i. A fundamental inequality due to Aronszajn (Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) p. 476, Corollary I′) shows that the desired eigenvalues λ_i are related to the eigenvalues κ_i^n in $\mathfrak{H}\ominus\{p_1, \cdots, p_n\}$ by the inequality $\kappa_i^n \geqq \lambda_i \geqq \kappa_i^n - \kappa_{n+1}$. Because L is completely continuous, the error estimate $\kappa_{n+1} \to 0$ as $n \to \infty$. Since both upper and lower bounds are obtained, it is no longer necessary to use the Rayleigh-Ritz method or its generalizations. (Received June 25, 1951.)

515. Alexander Weinstein: *On cracks in shafts under torsion and seven-dimensional electrostatics.*

Let x and ρ be the coordinates in the meridian plane of a shaft of revolution subjected to torsion. The problem of a flat circular crack perpendicular to the x-axis is solved by an explicit determination of two functions ϕ and ψ satisfying the equations $\rho^3\phi_x = \psi_\rho$ and $\rho^3\phi_\rho = -\psi_x$. By putting $\psi = y^4\Phi(x, \rho)$, the problem is reduced to the determination of the electrostatic potential Φ of a disc in a space of seven dimensions in terms of ellipsoidal coordinates. The functions ϕ, ψ, and Φ can also be expressed in terms of Bessel functions by integrals of the Sonine-Schafheitlein type. (Received May 14, 1951.)

516t. L. A. Zadeh: *On the theory of filtration of signals.*

Let U be the signal space (generally a Hilbert space). A network N is called an ideal filter if it is idempotent, that is, if it is equivalent to a tandem combination of two networks each of which is identical with N. Thus, operation on the signals with an ideal filter corresponds to the projection of U on some linear manifold $\mathfrak{M}_P$ along a complementary manifold $\mathfrak{M}_S$. $\mathfrak{M}_P$ and $\mathfrak{M}_S$ constitute the "pass-band" and "stop-band" of the ideal filter. Two ideal filters N and N' are called complementary if the pass-band of N is the stop-band of N' and vice-versa. In general, an ideal filter is not physically realizable inasmuch as its impulsive response $W(t, \xi)$ does not vanish for $t<\xi$. However, in most cases one can approximate to $W(t-\beta, \xi)$, where β is a sufficiently large constant, by a physical filter. Two (or more) sets of signals $\{u_1(t)\}$ and $\{u_2(t)\}$ are called disjoint if the manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$ spanned by $\{u_1(t)\}$ and $\{u_2(t)\}$, respectively, are disjoint. Such signals can be transmitted simultaneously (multiplexed) and separated at the receiving end by two complementary ideal filters N_1 and N_1' of which N_1 has $\mathfrak{M}_1$ for the pass-band and $\mathfrak{M}_2$ for the stop-band. (Received July 18, 1951.)

Geometry

517t. D. O. Ellis: *Notes on abstract distance geometry.* III. *On self-congruences of metroids.*

This note deals with the relation between certain algebraic properties of a metroid and the self-congruences of the metroid. For the terminology see David Ellis, *Geometry in abstract distance spaces,* Publicationes Mathematicae Debrecen vol. 2 (1951) no. 1. The principal results are: 1. *A metroid with left cancellation* (*right cancellation*) *and a left zeroid* (*right zeroid*) *is metrically irreducible*; 2. *In a metroid with skew cancellation* ($x^*a = a^*y$ implies $x = y$) *every self-congruence is involutory*; 3. *If a metroid has a simply transitive group of motions every element of the form* x^*x *is idempotent*; 4. *In an associative metroid with unit the following are pairwise equivalent*: (a) *the group of motions is simply transitive,* (b) *the metroid has the property of free mobility,* (c) *all* *elements are nilpotent,* (d) *the metroid is (isomorphic to) a subgroup of the additive group of a Boolean ring.* (Received July 18, 1951.)

518. P. C. Hammer and Andrew Sobczyk: *Planar line families. II. Covering properties of outwardly simple line families.*

A family of lines in the plane is said to be outwardly simple if it simply covers all points in a region exterior to some circle, including points at infinity. The existence of instantaneous centers of rotation of such a family F is fully discussed. The accumulated positive area swept out in a bounded region by lines in the family is defined and various inequalities noted. Letting S_k be the set of points in the plane each of which lies on exactly k lines in an outwardly simple family F of lines, then we prove that the planar measure, $m(S_k)$, is zero if k is even or infinite. (Received July 6, 1951.)

519*t*. P. C. Hammer and Andrew Sobczyk: *Planar line families. III. Outwardly n-fold families.* Preliminary report.

A family of lines will be said to *cover the plane n-fold* if every point in the finite plane and every point on the line at infinity lies on precisely n lines. The line at infinity will not be included as a member of the line family. The following results are obtained. (1) No family of lines covers the plane simply (1-fold). (2) There exists a family of lines which covers the plane n-fold for every $n \geqq 2$—assuming the continuum hypothesis and the well ordering axiom. (3) There exists no family of lines which covers the plane continuously n-fold—for $n = 1, 2, \cdots$. (4). For every $n \geqq 1$ there exist families of lines covering continuously n-fold the exterior of any circle. (Points at infinity are assumed to be outside any bounded region.) Such a line family is called an *outwardly n-fold family of lines.* (Received July 6, 1951.)

520*t*. W. R. Hutcherson. *Fifth order neighborhood of an involution of period thirteen.*

An imperfect point (W. R. Hutcherson, *Point non parfait et courbes invariables,* Bulletin de la Société Royale des Sciences de Liège no. 11 (1950) pp. 485–489; *A cyclic involution of period eleven,* Canadian Journal of Mathematics vol. 3 (1951) no. 2) on a certain surface is found to possess perfect points in the neighborhood of orders three and four for I_{11}, whereas for I_{13} it is discovered to be five and three respectively for the two invariant directions. (Received June 27, 1951.)

521*t*. I. J. Schoenberg: *A generalization of the classical isoperimetric inequality to closed curves in higher even-dimensional spaces.*

Let Γ: $x_i = x_i(t)$ $(i = 1, \cdots, 2n)$ be a closed curve in E_{2n} where $x_i(t)$ are continuous functions of period 2π. We restrict ourselves to the class Ω of curves Γ which have the following property: There is no hyperplane in E_{2n} which is crossed more than $2n$ times by Γ. Curves Γ with this property are called *convex in* E_{2n}; for $n = 1$ we obtain the ordinary plane convex curves. Convex curves in E_{2n} are known to be rectifiable. Let $\Gamma \in \Omega$ and let L be its length. Let $K = K(\Gamma)$ denote the least convex extension of Γ and let $V = V(K)$ be the $2n$-dimensional volume of $K(\Gamma)$. Then the following inequality holds: $L^{2n} \geqq (2\pi n)^n n!(2n)!V$, where we have the equality sign if and only if Γ is obtained by a similitude, rotation and perhaps a reflexion from the curve Γ_0 defined by the equations $x_1 = \cos t$, $x_2 = \sin t$, $x_3 = (1/2) \cos 2t$, $x_4 = (1/2) \sin 2t, \cdots$, $x_{2n-1} = (1/n) \cos nt$, $x_{2n} = (1/n) \sin nt$. These are the curves which maximize $V(K)$ for a fixed value of their length L within the class of convex curves. (Received July 20, 1951.)

522. Andrew Sobczyk and P. C. Hammer: *Symmetrical types of convex regions.*

If a bounded convex region C has involutory symmetry in each of a set of complementary linear subspaces $L_1, \cdots, L_k$ of n-dimensional linear space L, then C is symmetric with respect to the origin (that is, it may be taken as the unit sphere for a Minkowski or Banach norm on L). Possible regions which have specified sections $C_1, \cdots, C_k$ in $L_1, \cdots, L_k$ are discussed. Convex regions are classified according to kinds of symmetry which they may possess; rotational and discrete symmetry are special cases. The notion of *convex transformation* is defined and studied. Classes of convex transformations, including affine transformations as subclass, are introduced; the existence and properties of completely unsymmetrical convex regions (group of symmetries in a corresponding class consists of only the identity) are investigated. (Received July 6, 1951.)

Logic and Foundations

523*t*. S. C. Kleene: *Permutability of inferences in Gentzen's calculi LK and LJ.*

After establishing the possible interchanges of logical inferences in Gentzen's calculi *LK* and *LJ*, the following theorem is proved. In a proof in *LK* or *LJ* without cut, one may assign the logical symbol occurrences L in the endsequent to classes $C_1, \cdots, C_k$, and then order the inferences belonging to those symbol occurrences (call them also L) so that those by symbols of C_1 are uppermost, those by C_2 next, and so on, provided the classification meets two restrictions: (1) If $L_1 \in C_i$, $L_2 \in C_j$, and L_2 contains L_1 within its scope, then $i \leqq j$. (2) If $L_1 \in C_i$, $L_2 \in C_j$, and L_1/L_2 is one of the following exceptional pairs of inferences, then $i \leqq j$. For *LK*: $(\forall\rightarrow$ or $\rightarrow\exists)/(\rightarrow\forall$ or $\exists\rightarrow)$. For *LJ*: $\forall\rightarrow/\rightarrow\forall$, $(\forall\rightarrow$ or $\rightarrow\exists)/\exists\rightarrow$, $(\supset\rightarrow$ or $\neg\rightarrow)/(\rightarrow\supset$ or $\rightarrow\neg)$, $(\supset\rightarrow$, $\rightarrow\vee$, $\neg\rightarrow$ or $\rightarrow\exists)/\vee\rightarrow$. Counterexamples are given to the theorem with (1), or any one of the exceptions under (2), omitted. (Received June 11, 1951.)

524*t*. J. C. C. McKinsey, A. C. Sugar, and Patrick Suppes: *A set of axioms for classical particle mechanics.*

A system of axioms is given for classical particle mechanics, based on five primitive notions: (1) P is the set of *particles*; (2) T is a set of real numbers measuring elapsed *times*; (3) for $p \in P$, $m(p)$ is a positive real number, measuring the *mass* of p; (4) for $p \in P$ and $t \in T$, $s(p, t)$ is a vector giving the *position* of p at time t; (5) for $p \in P$, $t \in T$, and n a positive integer, $f(p, t, n)$ is a vector giving the nth *force* acting on p at time t. The axioms, which are eight in number, include a form of Newton's Second Law, and otherwise consist chiefly of so-called "closure axioms," and axioms giving continuity properties. Theorems are established which provide grounds for considering this an adequate axiomatization of the usual particle mechanics; in particular, it is proved that every model can be embedded in a model which in addition satisfies Newton's third law (and the first law, as would be expected from less formal developments, is of course a theorem). It is also shown that the axioms are independent; and—some received opinions to the contrary notwithstanding—that m, s, and f are not definable in terms of the other notions, even if Newton's third law is added as an axiom. (Received July 9, 1951.)

Statistics and Probability

525t. Abolghassem Ghaffari: *On a functional equation arising in the the theory of chain probabilities.*

The author considers one-dimensional Chapman's functional equation (C) $f(x, s; y, t) = \int_V f(x, s; z, u) f(z, u; y, t) dz$, $s<u<t$, where f denotes the probability density which is subjected to specific conditions such as: $f(x, s; y, t) \geqq 0$; $\int_V f(x, s; y, t) dy = 1$; $f(x, s; y, s) = 0$; $f(x, s; x, s) = \infty$. Taking for the region V the interval $(-\infty, +\infty)$ and trying for the solution of (C) a preassigned exponential expression $f(x, s; y, t) = A(s, t) \exp[-(a(s, t)x^2 + 2b(s, t)xy + c(s, t)y^2)]$, one finds that (1) $f(x, s; y, t) = g^{1/2}(t)[\pi(H(t)-H(s))]^{-1/2} \exp (xg^{1/2}(s) - yg^{1/2}(t))^2(H(t)-H(s))^{-1}$, where $g(s)$ is an arbitrarily chosen positive continuous function and $H(s)$ is an increasing function. The solution (1), satisfying the equation (C) and the supplementary conditions indicated above, is then the most general such solution and moreover contains the solution of L. Bachelier as a special case. (Received July 11, 1951.)

526t. H. P. Mulholland: *On the distribution of a convex even function of several independent rounding-off errors.*

The following theorem, in which $\phi(w)$ may be any continuous convex even function of the real vector $w = (w_1, w_2, \cdots, w_n)$ with $\phi(0) = 0$, generalizes Z. W. Birnbaum's results (*On random variables with comparable peakedness*, Ann. Math. Statist. vol. 19 (1948) pp. 76–81) for the case $\phi(w) \equiv |w_1 + w_2 + \cdots + w_n|$. Let u_i $(i = 1, 2, \cdots, n)$ be independent random variables with unimodal continuous distributions symmetrical about 0 and cumulative distribution functions (c.d.f.'s) $P_i(x)$, and similarly for v_i with c.d.f.'s $Q_i(x)$: let the distribution of u_i be more peaked about 0 than that of v_i, that is, $P_i(x) \geqq Q_i(x)$ $(x \geqq 0;\ i = 1, 2, \cdots, n)$. Then the distribution of $\phi(u)$ is more peaked about 0 than that of $\phi(v)$. The theorem is applicable when u_i is a (composite) rounding-off error, that is, $P_i(x)$ is the c.d.f. of the l_i-fold convolution of the rectangular distribution on $(-1/2, 1/2)$, and $Q_i(x)$ is the normal c.d.f. with mean 0 and variance σ^2, if we take $\sigma \geqq \rho_l (l/12)^{1/2}$ where $l = \max l_i$, $\rho_1 = (6/\pi)^{1/2}$, and $1 < \rho_l < \rho_2 < 1.0707$ $(l > 2)$. This result is simpler and stronger than H. H. Goldstine and J. von Neumann's fundamental lemma (*Numerical inverting of matrices of high order*. II, Proceedings of the American Mathematical Society vol. 2 (1951) pp. 188–202). (Received July 2, 1951.)

527. Murray Rosenblatt: *On a question of J. L. Doob.*

Let $X(t)$ be a time-homogeneous Markov process with a denumerable number of states. J. L. Doob has asked whether there are Markov chains $X(t)$ whose sample functions have discontinuities that are not well-ordered with probability one (Trans. Amer. Math. Soc. vol. 58 (1945) pp. 455–473). An example of such a Markov chain is given. The transition probabilities of the given process satisfy the "second system of differential equations" (forward equations) but do not satisfy the "first system of differential equations" (backward equations) (A. Kolmogorov, Math. Ann. vol. 104 (1931) pp. 415–458). (Received July 16, 1951.)

528. J. L. Snell: *Applications of martingale system theorems.*

A sequence of random variables $\{x_n\}$ with $E\{|x_n|\} < \infty$ is called a martingale if $E\{x_n | x_1 \cdots x_{n-1}\} = x_{n-1}$ with probability one for all n, a semimartingale if $E\{x_n | x_1 \cdots x_{n-1}\} \geqq x_{n-1}$ with probability one for all n. If $\{x_n\}$ is a martingale (semimartingale) the interpretation of x_n as a gambler's fortune playing a fair (favorable) game has suggested theorems, called system theorems. Such theorems have been proven by Doob and Halmos. In this paper these system theorems are generalized and their interpretation in terms of von Neuman's theory of games is given. The theorems are then applied to the study of convergence properties of the random variables of a martingale and semimartingale. It is proven that lim $x_n(\omega)$ exists and is finite for almost all ω in the set $\{\omega | \inf_k \sup_{n \geq k} E\{|x_n| | x_1 \cdots x_2\} < \infty\}$ if $\{x_n\}$ is a semimartingale. Finally, the system theorems are applied to the problem of the existence of Bayes solutions in the theory of statistical decision functions. Results of Arrow Blackwell and Girshick are obtained from martingale theory. (Received June 29, 1951.)

Topology

529t. Richard Arens: *Extension of mappings.*

Let A be a closed subset of a normal space X. Let f be a continuous mapping into a Banach space L. Then f can not always be continuously extended to X (even if L is a Hilbert space), although it always can when $f(A)$ is separable. Two separate necessary and sufficient conditions that f can be extended to X are (1) every neighborhood finite covering of A has a neighborhood finite refinement which can be extended neighborhood finitely to X, (2) every pseudo-metric on A has a refinement which can be extended to all of X. If an extension of f is possible it may be so chosen that the new values lie in the convex hull of the closure of $f(A)$, but when moreover A is a G_δ (or compact) the new values can be forced to lie in the convex hull of $f(A)$. However, in the absence of the G_δ condition it is sometimes necessary for mappings into the plane (but never, for mappings into the line) to have extended values outside of the convex hull of $f(A)$. (Received July 27, 1951.)

530t. R. H. Bing: *Convexifying continuous curves so as to obtain many unique segments.*

It is shown that each continuous curve M has a dense subset R and a convex metric $D(x, y)$ such that if p and q are two points of R there is only one arc in M from p to q whose length is $D(p, q)$. The method used is to get a decreasing sequence of core partitionings of M and define the convex metric $D(x, y)$ in terms of the number of elements of the various partitionings it takes to link x and y. (Received July 23, 1951.)

531t. C. E. Burgess: *Concerning continua and their complementary domains in the plane.*

A continuum K in the plane is said to be compactly connected if every pair of its points is contained in a compact subcontinuum of K. The following theorems are proved: (1) If the continuum M is the boundary of the sum of a finite number of its complementary domains and there is a sequence of mutually exclusive compactly connected continua in M converging to M, then either M is indecomposable or there is only one pair of indecomposable proper subcontinua of M whose sum is M. (2) If M is a continuum containing no domain and some domain intersects M and only a finite number of complementary domains of M, and there is a sequence of mutually exclusive continuous curves in M converging to M, then either M is indecomposable or it is the sum of two of its proper subcontinua of which one is indecomposable. (Received July 11, 1951.)

532. C. E. Burgess: *Concerning continua in the plane.* Preliminary report.

The following theorems are proved: (1) In order that for each three domains R_1, R_2, and R_3 intersecting the compact continuum M there should exist three complementary domains of M each intersecting each of the domains R_1, R_2, and R_3, it is necessary and sufficient that either M be the boundary of each of three of its complementary domains or there exist a sequence of distinct complementary domains of M converging to M. (2) If M is a compact decomposable continuum and for each two domains R_1 and R_2 intersecting M there exist three complementary domains of M each intersecting each of the domains R_1 and R_2, then there is a collection G consisting of less than five indecomposable continua such that M is the sum of the continua of G and such that, furthermore, if X and Y are two continua of G, then $X+Y$ is a continuum which is irreducible between some two points, and either there is a complementary domain of M having $X+Y$ as its boundary or there is a sequence α of distinct complementary domains of M converging to $X+Y$ such that each domain of α is a complementary domain of $X+Y$. (Received July 11, 1951.)

533. Eugenio Calabi and Beno Eckmann: *A class of simply connected, compact complex manifolds which are not algebraic.*

It is demonstrated that the product of any two odd-dimensional spheres $S^{2p+1} \times S^{2q+1} = M$ can be given a complex analytic structure. The second Betti number of M vanishes (except in the trivial case $p=q=0$), so that M does not admit any Kähler metric, hence M can not be an algebraic manifold (theorem of Lefschetz-Hodge). In the special case where $p=0$ a complex structure on $S^1 \times S^{2q+1}$ was first exhibited by Hopf (*Courant anniversary volume*, 1948, p. 168) and the variety is uniformizable, while in the general case p, $q \geqq 1$, the manifold M, being simply connected, can not be uniformized. The manifold M can be considered as a complex analytic fibre space over the product of two complex projective spaces of dimension p, q respectively, the fibre being a 2-torus with any specified conformal structure. (Received July 23, 1951.)

534. K. T. Chen: *Isotopy invariants of links.*

This is an improved result of a previous paper [Bull. Amer. Math. Soc. vol. 57 (1951) p. 143] of the author. By a δ-approximation L' of an arbitrary link L in the euclidean 3-space E, we mean a polygonal link L' in E such that there is a homeomorphism ϕ: $L \rightarrow L'$ with $|\phi(x)-x| < \delta$ for any $x \in L$. It is proved that, for each $d \geqq 1$, there is a $\delta > 0$ such that any two δ-approximations L' and L'' of L have $G'/G'_d \cong G''/G''_d$, where G'_d and G''_d are the dth lower central commutator subgroups of the fundamental groups $G' = \pi(E-L')$ and $G'' = \pi(E-L'')$ respectively. Set $H_d = G'/G'_d$. Thus, for each d, H_d is well defined for L up to isomorphism. Furthermore, this group H_d is invariant under any isotopy of the link L, that is, a continuous family of mappings h_t: $L \rightarrow E$, $0 \leqq t \leqq 1$, each of which maps L topologically into E with $h_0(x) = x$, $x \in L$. If L is polygonal, then $H_d \cong G/G_d$, where $G = \pi(E-L)$. Through the groups H_d, one finds some numerical invariants which distinguish the members of a certain infinite series of links, each of which consists of only two curves with vanishing linking number. (Received July 9, 1951.)

535. Mary E. Estill: *Concerning a problem of Souslin.*

In order that there exist a linear nonseparable space not containing uncountably many mutually exclusive domains it is necessary and sufficient that there exist a locally connected nonseparable space not containing uncountably many mutually exclusive domains which satisfies the first three parts of R. L. Moore's axiom 1 such that, if A and B are two points, there exists a countable point set separating A and B. It is also a necessary and sufficient condition for the existance of a locally connected, nonseparable space not containing uncountably many mutually exclusive domains which satisfies the first three parts of R. L. Moore's axiom 1 such that, if A and B are two points, there exists a separable point set separating A and B. (Received July 23, 1951.)

536*t*. E. E. Floyd: *On periodic maps and the Euler characteristics of associated spaces.*

Let p be a fixed prime. If A is a locally compact Hausdorff space, denote by $H_i(A)$ the Čech homology group of compact i-cycles of A modulo cycles bounding on compact subsets of A, with the integers mod p as coefficients. Denote by dH_i the minimum number of generators of $H_i(A)$, and by $\chi(A)$ the sum $\sum(-1)^i dH_i$, in case these numbers are defined and finite. Let now T be a periodic map of period p on a locally compact finite-dimensional Hausdorff space X. Denote by L the fixed point set of T, and by Y the orbit decomposition space of T. Then if $\chi(X)$ exists, we prove that $\chi(X)+(p-1)\chi(L)=p\chi(Y)$, where all terms are well-defined. Various consequences are obtained. An immediate one is that $\chi(L)=\chi(X)$ mod p. (Received July 23, 1951.)

537. E. E. Floyd: *On related periodic maps.*

Consider two periodic maps S and T of period p on a compact n-manifold X. The problem studied is as follows. Suppose S and T are related in some simple topological fashion. What, then, are the implied relationships between the fixed point sets of S and T? Between their orbit decompositions spaces? For example, if S and T are homotopic then, when properly interpreted and p is prime, the Euler characteristics of the fixed point sets are equal. Results of another type concern the case in which S is a fixed map of period q^a, q prime. Then there exists $\epsilon>0$ such that if T is of the same period as S, and $\rho(S, T)<\epsilon$, then their fixed point sets have isomorphic mod q homology groups. (Received July 18, 1951.)

538*t*. V. G. Gorciu: *Continuous-convergence in a non-metric uniform space.* Preliminary report.

A Hausdorff, nonmetric, uniform space is defined by assigning to each point an indexed set of neighborhoods. The fundamental set of indices $\Lambda(\lambda, \mu, \cdots; >)$ is directed, noncountable and does not possess a last element. *Definition.* The directed set of functions $\{f_\lambda(x)\}$ is said to be continuous-convergent at a point $a\in\overline{A}$ if for every directed set of points $\{x_\lambda\}\rightarrow a$, the directed set $\{f_\lambda(x_\lambda)\}$ converges; $x_\lambda\in A$. If $\Lambda_1(\lambda_1, \mu_1, \cdots; >)$ and $\Lambda_2[\lambda_2, \mu_2, \cdots; >]$ are two sets order-isomorphic with Λ, a new directed set $\Lambda_3[\Lambda_1\cup\lambda_2; >]$ is defined by introducing following partial ordering: $\lambda_2>\lambda_1$, $\mu_2>\mu_1, \cdots; \lambda>\mu\leftrightarrow\lambda_1>\mu_1$, $\lambda_2>\mu_2$, $\lambda_1>\mu_2$, $\lambda_2>\mu_1$. In this new set, both Λ_1 and Λ_2 are cofinal subsets. By using this new set, it is easily proved that, in the case of continuous convergence, the limit of $\{f_\lambda(x_\lambda)\}$ does not depend on the directed set $\{x_\lambda\}\rightarrow a$ that is chosen. A necessary and sufficient condition that $\{f_\lambda(x_\lambda)\}$ be continuous-convergent, at a point $a\in\overline{A}$, is established. (Received July 18, 1951.)

539. O. G. Harrold: *Almost locally polyhedral spheres.*

Recently examples of topological spheres in space have been given illustrating the fact that a set need be only slightly pathological for the classical Schoenflies relation to fail. In particular, if a sphere is sufficiently twisted in the vicinity of two points, the fundamental group of a complementary domain may be nontrivial. A supplementary result is derived. Set $K\subset R^n$, Euclidean space, is called locally polyhedral at $p\in R^n$ if and only if for some subdivision of R^n into convex cells: (1) only a finite number of cells meet any spherical domain; (2) there is a neighborhood U of p in R^n such that $U\cap K$ is a union of cells of the subdivision or their lower dimensional faces. If K contains no points where this property fails, call K locally polyhedral. If a topological sphere in R^3 is locally polyhedral save for one (two) point(s), then both complementary domains are (at least one complementary domain is) simply connected. (Received July 23, 1951.)

540*t*. S. T. Hu: *Cohomology theory in topological groups.*

For a given topological group Q and a given topological abelian group G, three kinds of cohomology groups of Q over G are defined in this paper, namely, the pth cohomology group $H^p(Q, G)$, the pth cohomology group $H^p(Q, G)$ with empty supports, and the reduced cohomology group $H^p(Q, G)$ for any integer $p\geqq 0$. They form in a natural way an exact sequence. Among the various applications, the following decomposition theorem has been proved. If E is a compact connected topological group with a center C which is a Lie group, then E is locally isomorphic with the direct product $C\times Q$ of the center C and the quotient group $Q=E/C$. If E is a compact Lie group, then this is a classical result of E. Cartan. (Received July 23, 1951.)

541*t*. S. T. Hu: *On imbedding a local group in a topological group.*

For any given local group L, there is constructed in this paper a topological group Q_L and a local homomorphism h_L of L into Q_L such that, for every local homomorphism f of L into a topological group G, there exist a homomorphism F of Q_L into G and a neighborhood U of the identity element in L such that $f(x)=Fh_L(x)$ for all $x\in U$. Then it is proved that L is locally isomorphic with a local subgroup of some topological group if and only if there exists a neighborhood V of the identity element in L such that, for any given element $v\in V$, there is a local homomorphism f of L into a topological group G such that $f(xy)=f(x)f(y)$ for all $x, y\in V$ and that $f(v)$ is different from the identity element of G. (Received August 20, 1951.)

542*t*. J. R. Jackson: *A note on function spaces.*

Let X and Y be T-spaces, $\{X_\alpha, Y_\alpha\}$ be a collection of pairs of subsets $X_\alpha\subset X$ and $Y_\alpha\subset Y$, and denote by $Y^X\{X_\alpha, Y_\alpha\}$ the compact-open topologized space of continuous functions on X into Y which carry each X_α into the corresponding Y_α. Suppose $X_0=\bigcap X_\alpha$ and $Y_0=\bigcap Y_\alpha$ are nonempty. A topological imbedding of Y_0 in $Y^X\{X_\alpha, Y_\alpha\}$ is obtained by identifying each point $y_0\in Y$ with the constantly y_0-valued function. If $x_0\in X_0$, then $\rho: Y^X\{X_\alpha, Y_\alpha\}\to Y_0$, defined by $\rho(f)=f(x_0)$, is a retraction: *Y_0 may be imbedded as a retract of $Y^X\{X_\alpha, Y_\alpha\}$.* It follows that if property P of topological spaces is preserved by retractions, then a necessary condition that $Y^X\{X_\alpha, Y_\alpha\}$ have property P is that Y_0 have property P. Another consequence is that the homotopy group $\Pi_m(Y^X\{X_\alpha, Y_\alpha\}, y_0)$ is a split extension of some subgroup by $\Pi_m(Y_0, y_0)$. [For definitions and theorems used above, see S. T. Hu, *Homotopy theory*, vol. 1, Department of Mathematics, Tulane University, 1950.] (Received July 2, 1951.)

543*t*. Shizuo Kakutani: *Continuity of proper functions on minimal dynamical systems.*

Let $x'=\phi(x)$ be a homeomorphism of a compact metric space X onto itself. The pair (X, ϕ) is called a *dynamical system.* (X, ϕ) is called *minimal* if for any $x\in X$ the orbit $O(x)=\{\phi^n(x)\mid n=0, \pm 1, \pm 2, \cdots\}$ is dense in X. (X, ϕ) is called *equicontinuous* if the iterations $x'=\phi^n(x)$ of $x'=\phi(x)$ for $n=0, \pm 1, \pm 2, \cdots$ are equicontinuous on X. Given two dynamical systems (X, ϕ), (Y, ψ), a mapping $y=\sigma(x)$ of X into Y is called a *homomorphism* if $\psi(\sigma(x))=\sigma(\phi(x))$ for all $x\in X$. It is proved that, for any Baire homomorphism of a minimal dynamical system into an equicontinuous dynamical system, there exists a continuous homomorphism which differs from it only on a set of first category. From this follows that, for any Baire proper function on a minimal dynamical system, there exists a continuous proper function which differs from it only on a set of first category. This result makes it possible to develop a purely topological ergodic theory of minimal dynamical systems without using the notion of invariant measures or invariant integrals. For example, the notions of minimal dynamical systems with a pure point spectrum and minimal dynamical systems without point spectrum (that is, topologically weakly mixing dynamical systems) can be introduced and their properties can be discussed. (Received July 23, 1951.)

544*t*. W. S. Massey and G. W. Whitehead: *The Eilenberg-MacLane groups and homotopy groups of spheres.*

It is known that for any fixed value of p, the homotopy groups $\pi_{n+p}(S^n)$ are isomorphic for all sufficiently large values of n (H. Freudenthal, Compositio Math. vol. 5 (1937) pp. 299–314). Identify these isomorphic groups and call the resulting group G_p. If Π and G are any abelian groups, then $A_q(\Pi, G)$ will denote the q-dimensional homology group of the abstract complex, $A(\Pi)$ (see Eilenberg and MacLane, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 657–663) with coefficients in the group G. With this notation, the authors' main result, relating the groups $A_q(\Pi, G)$ and homotopy groups of spheres, may be stated as follows. For any abelian group Π, there exists a Leray-Koszul sequence $(E^n(\Pi), d_n)$, $n=2, 3, 4, \cdots$, of abelian groups with differential operators having the following four properties: (1) $E^n(\Pi)$ has a bigraded structure defined by a doubly-indexed family of subgroups, $E^n_{p,q}(\Pi)$. (2) The differential operator d_n is homogeneous of degree $(-n, n-1)$. (3) The group $E^2_{p\,q}(\Pi)$ is naturally isomorphic to $A_p(\Pi, G_q)$. (4) If $E^\infty(\Pi)$ denotes the limit group of the sequence, then $E^\infty_{1,0}(\Pi)=\Pi$, and $E^\infty_{p,q}(\Pi)=0$ for all other pairs (p, q). For the definition of a Leray-Koszul sequence, see H. Cartan, C. R. Acad Sci. Paris vol. 226 (1948) pp. 148 and 303; also, J. Leray, Journal de Mathématiques vol. 29 (1950) pp. 1–139. (Received July 3, 1951.)

545*t*. W. S. Massey and G. W. Whitehead: *The* $(n+3)$*-dimensional homotopy group of the n-sphere.*

Let Π be an infinite cyclic group. By making use of the calculation by Eilenberg and MacLane of the groups $A_1(\Pi), \cdots, A_5(\Pi)$ (see Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 657–663), and of the Leray-Koszul sequence associated with the group Π (see the preceding abstract), it is possible to prove that the homotopy groups $\pi_{n+3}(S^n)$ are all of order 12 or all of order 24 if $n>4$. It readily follows from the Freudenthal suspension theorems (Compositio Math. vol. 5 (1937) pp. 299–314) and the fact that the suspension $\pi_6(S^3)\rightarrow\pi_7(S^4)$ is an isomorphism into (see

Hurewicz and Steenrod, Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) pp. 60–64) that if $\pi_8(S^5)$ is a group of order m, then $\pi_6(S^3)$ is a group of order $m/2$; hence $\pi_6(S^3)$ must be of order 6 or 12. However, from the existence of an element in $\pi_6(S^3)$ of nonzero generalized Hopf invariant (see Blakers and Massey, Ann. of Math. vol. 53 (1951) p. 202), and from the fact that the suspension $\pi_5(S^2)\rightarrow\pi_6(S^3)$ is an isomorphism into (an unpublished result discovered independently by G. W. Whitehead and P. Hilton), one concludes that $\pi_6(S^3)$ cannot be a group of order 6. Therefore, $\pi_6(S^3)$ is a group of order 12, and $\pi_{n+3}(S^n)$ is a group of order 24 if $n>4$. The structure of these groups is not determined; however, $\pi_6(S^3)$ is cyclic or not according as the map $(x, y)\rightarrow x^6yx^{-6}y^{-1}$ of $S^3\times S^3$ into S^3 is essential or not. (Received July 3, 1951.)

546*t*. C. E. Miller: *The topology of rotation groups.*

A boundary formula is obtained for the cell complex of J. H. C. Whitehead (Proc. London Math. Soc. (1944) p. 243) on the Stiefel manifold $V(n, r)$ of r-frames in euclidean n-space. From this the homology and cohomology groups of $V(n, r)$ are computed. The cup products with rational and with modulo 2 coefficients are computed, by using appropriate mappings and spaces to reduce the problem to known or simple cases. The Steenrod squaring homomorphisms are computed similarly, and these are then used to show the non-existence of cross sections to certain of the bundles $V(n, r)\rightarrow S^{n-1}$. This gives a simple proof of the Steenrod-Whitehead Theorem (Proc. Nat. Acad. Sci. U.S.A. (1951) p. 58) asserting that 2^k vector fields do not exist on S^{n-1}, where 2^k is the largest power of 2 dividing n. Similar techniques are used to analyze the factor space $R(2n)/U(n)$, where $R(m)$ $(U(m))$ is the rotation (unitary) group in m real (complex) variables. $R(n)$ is homeomorphic to $V(n, n-1)$ and is a special case of $V(n, r)$. The Pontrjagin product is also computed for $R(n)$ and it is shown that the Pontrjagin and cohomology rings are not isomorphic when modulo 2 coefficients are used. (Received June 18, 1951.)

547. E. E. Moise: *Affine structures in 3-manifolds.* I. *Polyhedral approximations of solids.*

Theorem 1. Let S be a set in Euclidean 3-space which is homeomorphic to the Cartesian product of a compact connected 2-manifold and a closed linear interval. Then there is a polyhedral surface, homeomorphic to the given 2-manifold, which separates the two components of the boundary of S from each other. The conclusion of this theorem can be strengthened by requiring the polyhedral surface to satisfy a rather weak regularity condition. The methods of proof are for the most part elementary. Forthcoming in Ann. of Math. (Received July 18, 1951.)

548*t*. E. E. Moise: *Affine structures in 3-manifolds.* II. *Positional properties of 2-spheres.*

Theorem 1. Given polyhedral 2-spheres S and S', lying in the interior of a cube P in Euclidean 3-space E^3, there is a piecewise linear homeomorphism f of E^3 onto itself which is the identity outside P and maps S upon S'. This theorem evidently implies a strengthened form of a well known theorem of J. W. Alexander (Proc. Nat. Acad. Sci. U.S.A. vol. 10 (1924) pp. 6–8) to the effect that every polyhedral 2-sphere in E^3 is the boundary of a topological 3-cell. Theorem 1 has already been given, essentially, by W. Graeub (Sitzungsberichte der Heidelberger Akademie der Wissenschaften (1950) pp. 205–272). If $K\subset E^3$ is homeomorphic to a (finite) polyhedron, then K is *semi-locally tamely imbedded* if there is an open neighborhood U of K, and a homeomorphism f throwing U into E^3, such that $f(K)$ is a polyhedron. *Theorem* 2. The complement of a semi-locally tamely imbedded 2-sphere in E^3 is homeomorphic to the complement of the unit sphere $x^2+y^2+z^2=1$. Forthcoming in Ann. of Math. (Received July 18, 1951.)

549. O. M. Nikodým: *Two general criteria for continuity of linear functionals.*

Let L be an abstract real linear space on which an open-set-topology is given, satisfying the following conditions: (1) Any translation of an open set is open; (2) if $E\neq\emptyset$ is open and $I\neq 0$, then $I\cdot E$ is open; (3) if E is an open set containing the zero vector $0^{\rightarrow}$, then there exists an open set E_1 such that $0^{\rightarrow}\in E_1$, $E_1\subseteq E$, and if $x\neq 0^{\rightarrow}$, $x\in E_1$, then the closed segment $\langle 0^{\rightarrow}, x\rangle\subseteq E_1$. No separation axiom is supposed. L is more general than a linear topological space. For linear functionals $f(x)$ on L the following is proved: $f(x)$ is continuous if and only if there exists an open set $A\neq\emptyset$ and a real number μ such that $f(x)$ does not admit, on any point of A, the value μ. If the condition (3) is dropped, the theorem is not true, but in this case $f(x)$ is continuous if and only if there exists an open set $A\neq\emptyset$ on which $f(x)$ is bounded from above. These criteria contain, as particular cases, all criteria known until now. A generalization of the theorem by J. P. LaSalle on the existence of nonvanishing linear continuous functionals is given. (Received July 11, 1951.)

550*t*. M. J. Norris: *Pointwise periodic mappings of simple curves.*

It is shown that every pointwise periodic mapping of the real numbers other than the identity is a strictly decreasing involution. The nature of these involutions is determined in terms of the strictly increasing functions with the positive real numbers as domain and range. Finally the same problem is studied for pointwise periodic mappings of simple curves. (Received July 23, 1951.)

551*t*. M. J. Norris: *Semi-regularity and strict extension.*

The notions of semi-regularity and strict extension as defined by M. H. Stone (*Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. vol. 41 (1937) no. 3) are considered in this paper as pointwise properties of spaces. Stone has shown that a semi-regular space is a strict extension of every dense subset. In this paper some auxiliary conditions on the space are found which, with the condition that the space is a strict extension of every dense subset at the point p, assure semi-regularity at p. The most natural of these conditions is local semi-regularity at p. An example is given to show that a space can be a strict extension of every dense subset at p without being semi-regular at p. The question as to whether or not a space can be a strict extension of every dense subset at every point without being semi-regular is not answered. (Received July 23, 1951.)

552. Everett Pitcher: *Suspension of primary components of homotopy groups of spheres.*

The Freudenthal theorems on suspension may be extended in the following form. Let $w\geqq 3$ denote a prime and $C(G, w)$ the set of elements in the Abelian group G with order a power of w. Suppose $n\geqq 3$. Let E_w: $C(\pi_r(S^n), w)\rightarrow C(\pi_{r+1}(S^{n+1}), w)$ denote the syspension homomorphism with restricted domain. *Theorem* 1: *E_w is an isomorphism onto if $r\leqq r_0=\min\ (3n-3,\ 2n+2w-5)$ and a homomorphism onto if $r=r_0+1$. Theorem* 2: *If $w\leqq n-1$ and n is even then $\pi_{2n+2w-4}(S^n)$ contains an element of* *order w which is not a suspension of any element.* Theorems 1 and 2 depend on results of Serre (C. R. Acad. Sci. Paris vol. 232, pp. 142–144, Theorems 3, 4) and on the Freudenthal theorems in the strong form of G. W. Whitehead (Ann. of Math. vol. 51, pp. 192–237, (3.49)) and are proved through use of the model introduced by the writer (Bull. Amer. Math. Soc. Abstract 56-1-38). Theorem 1 improves on the remark of Serre (op. cit. p. 143). Theorem 2 shows that r_0 in Theorem 1 cannot be increased. (Received July 23, 1951.)

553. G. H. M. Thomas: *Simultaneous partitionings of two sets.* II.

The concept of simultaneous partitionings was introduced by the author in a previous paper (Bull. Amer. Math. Soc. Abstract 57-4-366). The following additional result is proved in the present paper. Theorem: Suppose M_1 is a compact partitionable set in a metric space, and M_2 is a closed partitionable subset. Then there exists a sequence $\{G_i\}$ of simultaneous partitionings of M_1 and M_2 such that all the elements of G_i have diameters less than $1/i$, G_{i+1} is a refinement of G_i, and if U_{ij}, $j=1, 2, 3, \cdots, m$, is any collection of elements of G_i then the interior of $\sum_{i=1}^{m} \overline{U_{ij}}$ and the interior of $\sum_{j=1}^{m} \overline{U_{ij}M_2}$ are each uniformly locally connected. Since a set in a metric space is partitionable if and only if it has property S, and every continuous curve has property S, the theorem holds in case M_1 and M_2 are continuous curves. (Received July 20, 1951.)

554*t*. W. R. Utz: *A note on unrestricted regular transformations.*

The action of unrestricted regular transformations (cf. W. A. Wilson, *On certain types of continuous transformations of metric spaces*, Amer. J. Math. vol. 57 (1935) pp. 62–68) on metric spaces is considered. A typical theorem is that if M is a bounded metric space and $T(M)=M$ is unrestricted regular and has equicontinuous powers, then T is an isometry. (Received July 2, 1951.)

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