Alternatives (1) and (2) represent extremes of optimism and pessimim respectively concerning the statistician's attitude toward Nature. Most statisticians will probably find themselves somewhere between the two extremes, and faced with the problem of how to utilize their rather vague feelings about the frequency with which various possible $F$'s can be expected to occur. For this reason, this reviewer suspects that minimax solutions as such are likely to be of little interest in statistics. For example, on p. 142 Wald cites the minimax point estimate of the mean $\theta$ of a binomial variate; the corresponding risk function is a constant $r_0 = [2(1+N^{1/2})]^{-2}$. The traditional (nonminimax) estimate has risk function $\theta(1-\theta)/N$. For large $N$ the ratio of this to $r_0$ is near zero except in a small interval about $\theta = 1/2$, where it is slightly greater than 1. It is very hard to believe in the superiority of the minimax estimate in this case, which is by no means unusual in its nature.

To those who are indifferent to minimax solutions the principal interest of the book will lie in the main theorem that, under very general conditions, the class $\mathcal{B}$ of all Bayes solutions is *essentially complete* in the following sense: for any decision function $\delta$ there exists a $\delta^*$ in $\mathcal{B}$ such that $r(F, \delta^*) \leqq r(F, \delta)$ for all $F$ in $\Omega$. (Mathematically, this theorem represents a highly nontrivial extension of the method of Lagrange multipliers in the calculus of variations.) There is obviously no loss involved in restricting the choice of a decision function to any essentially complete class, in particular to $\mathcal{B}$. But even a minimal essentially complete class will usually be so large that further reduction is necessary before the statistician can turn the problem of selecting a decision rule over to the experimenter. One criterion for reduction, the minimax principle, has already been mentioned. Other criteria exist (unbiasedness, invariance, and so on) but are not dealt with in the present volume.

The book makes effective use of the modern theory of measure and integration, and operates at a high level of rigor and abstraction. For this reason few statisticians will be prepared to read it, yet its ultimate liberating effect on statistical theory will be great. It is to be hoped that so rich and stimulating a book as this will reach an audience among mathematicians.

HERBERT ROBBINS

*Introduction to the theory of algebraic functions of one variable.* By C. Chevalley. (Mathematical Surveys, no. 6.) New York, American Mathematical Society, 1951. 12+188 pp. $4.00.

Here is algebra with a vengeance; algebraic austerity could go no further. "We have not tried to hide (says the author) our partiality to the algebraic attitude . . . "; he has not indeed; and, if it were not for a few hints in the introduction and one casual remark at the end of Chapter IV, one might never suspect him of having ever heard of algebraic curves or of taking any interest in them. Fields and only fields are the object of his study. A field is given, or rather two fields: one, the function-field $R$; the other, the field $K$ of constants; $K$ is algebraically closed in $R$; and $R$ is finitely generated and of degree of transcendency 1 over $K$. Everything must be "intrinsic," i.e. must be born from these by some standard operations. Later on the family circle is enlarged by the appearance of another function-field $S$ containing $R$, with a field of constants $L$ containing $K$, and a large portion of the book is devoted to the mutual relations between $R$ and $S$; but nowhere except in one or two lemmas is any element allowed to appear unless it is contained in those fields or canonically generated from them.

The contents of the book are as follows. Valuations are introduced and the basic existence theorem on valuations is proved in the standard manner (th. 1, p. 6), by the use of Zorn's lemma: this is the theorem according to which every "specialization" of a subring $\mathfrak{o}$ of a field $R$ (i.e. every homomorphic mapping of $\mathfrak{o}$ into a field) can be extended to a "valuation" of $R$ (i.e. a specialization of a subring $\mathfrak{O}$ of $R$ such that $R=\mathfrak{O}\cup\mathfrak{O}^{-1}$); the theorem, however, is not stated in its full generality. One might observe here that, in a function-field of dimension 1, every valuation-ring is finitely generated over the field of constants, and therefore, if a slightly different arrangement had been adopted, the use of Zorn's lemma (or of Zermelo's axiom) could have been avoided altogether; since Theorem 1 is formulated only for such fields, this treatment would have been more consistent, and the distinctive features of dimension 1 would have appeared more clearly.

Places are defined as being in one-to-one correspondence with the non-trivial valuation-rings of $R$, i.e. with those proper subrings $\mathfrak{o}$ of $R$ which contain the field of constants $K$ of $R$ and satisfy $R=\mathfrak{o}\cup\mathfrak{o}^{-1}$. In Zariski's terminology, on the other hand, a place is a homomorphic mapping of a valuation-ring $\mathfrak{o}$ into a field; in consequence, if $\mathfrak{o}$ is a non-trivial valuation-ring of $R$, and $\mathfrak{p}$ the ideal of non-units in $\mathfrak{o}$ (the "place" in Chevalley's sense), there will be as many "places" belonging to $\mathfrak{o}$ and $\mathfrak{p}$, with values in a given "universal domain" $\Omega$, as there are isomorphisms of the "residue-field" $\Sigma=\mathfrak{o}/\mathfrak{p}$ into $\Omega$; their number is equal to the degree $d_s$ over $K$ of the maximal separable extension $\Sigma_s$ of $K$ contained in $\Sigma$. According to Chevalley's selfimposed taboos, however, only the field $\mathfrak{o}/\mathfrak{p}$ is allowed to exist, and the "place" determined by $\mathfrak{o}$ and $\mathfrak{p}$ (or by either of them) must be unique. This has far-reaching consequences: while otherwise sums (e.g. the sum of the residues of a differential) could be extended over all the $d_s$ places belonging to $\mathfrak{o}$ and $\mathfrak{p}$, here the $d_s$ terms belonging to such a sum can never be separated from each other. It is true that traces (of elements of $\Sigma_s$ over $K$) are adequate substitutes for such sums; but it may well be doubted whether the constant use of traces is not an unnecessary complication, and whether it helps a beginner to understand the subject.

Chapter I then brings, as usual, proofs for the existence of a uniformizing variable at a place (i.e. a $t \in \mathfrak{p}$ such that $\mathfrak{p} = t\mathfrak{o}$), for the independence of valuations (or of "places"), and for the existence of the divisor of a function; it ends up with the theorem that the degree of the divisor of zeros of $x \in R$ is equal to $[R:K(x)]$. Chapter II follows, with the definition of differentials and the proof of the Riemann-Roch theorem due to A. Weil. The genus is defined by means of Riemann's theorem. A "repartition" is defined as a function assigning to each place $\mathfrak{p}$ an element $x(\mathfrak{p})$ of $R$ (or, later, an element $x(\mathfrak{p})$ of the $\mathfrak{p}$-adic completion $\overline{R}_{\mathfrak{p}}$ of $R$ at $\mathfrak{p}$), so that those places $\mathfrak{p}$ for which $x(\mathfrak{p})$ has a pole at $\mathfrak{p}$ are in finite number; then a differential is a linear function on the space of repartitions, continuous in a suitable sense, which vanishes on the subspace of "principal" repartitions (those for which $x(\mathfrak{p}) = x$ for all $\mathfrak{p}$, with $x \in R$). This rather abstract concept of differential is of course what makes possible such a brief proof of the Riemann-Roch theorem; while this is very convenient for many purposes, one should not forget that eventually (in the case where $R$ is separably generated over $K$) differentials have to be identified with the expressions $ydx$, or, what amounts roughly to the same thing, it must be shown that the sum of the residues of $ydx$ is 0; for this, in the present volume, one has to wait until p. 117.

Chapter III introduces the local or $\mathfrak{p}$-adic completions of the function-field $K$ by means of the usual definitions and of Hensel's lemma; $\Sigma_s$ being defined as before, it is shown that $\Sigma_s$ can be canonically identified with a subfield of the completion $\overline{R}$ of $R$ at $\mathfrak{p}$, and that, if $\Sigma = \Sigma_s$, this completion is essentially the ring $\Sigma((t))$ of power-series in $t$ with coefficients in $\Sigma$, where $t$ is any uniformizing variable at $\mathfrak{p}$; the structure of $\overline{R}$ when $\Sigma$ is not separable over $K$ is not further discussed. The last § of Chapter III brings the concept of residue of a differential, in terms of the values of the differential at certain "repartitions"; it then becomes trivial that the sum, not of the residues, but of the traces of the residues of a differential is 0.

So far only one function-field has been considered. Now another one, $S$, is introduced, with the field of constants $L$, such that $S \supset R$ and $K = L \cap R$; the next three chapters (nearly half the book) are devoted to the simultaneous study of the two fields $R$, $S$, under various assumptions. Some of the questions raised here by the author had never been treated before; unfortunately, as he treads new ground, his footsteps become more uncertain, and to follow in them is at times no easy task. In the language of algebraic geometry, the passage from $R$ to $S$ consists partly in enlarging the field of constants of a given curve, partly in considering the mutual relations of two curves in a $(1, m)$-correspondence with each other; the author tries to treat both problems by the same methods; however tempting this idea may appear to the algebraist, it is not altogether successful, and may well have caused some blurring of the picture.

Chapter IV is chiefly devoted to the case where $S$ is of finite degree over $R$, and to the behavior of the places of $R$ and $S$ with respect to one another; these questions are fairly familiar, at least in the parallel case of number-fields, and no surprises are to be expected here. Because of the too special formulation which has been given of the theorem on the extension of specializations in Chapter I, the existence of a place of $S$ lying over a given place $\mathfrak{p}$ of $R$ is made to depend, strangely enough, upon Riemann's theorem. The ramification indexes $e_\lambda$ and the relative degrees $d_\lambda$ of the places $\mathfrak{P}_\lambda$ of $S$ lying over $\mathfrak{p}$ are defined, and it is proved that $\sum_\lambda d_\lambda e_\lambda = [S:R]$; this might well have been postponed until it is shown that $d_\lambda e_\lambda$ is the degree of the $\mathfrak{P}_\lambda$-adic completion $\overline{S}_\lambda$ of $S$ over the $\mathfrak{p}$-adic completion $\overline{R}$ of $R$, and a basis is explicitly given for the former over the latter (Theorems 4 and 5, p. 60–61); in between those results are inserted some remarks on the case of normal extensions, the proof for the existence of a base of $S$, integral at $\mathfrak{p}$, and auxiliary definitions and results on the Kronecker product of fields or commutative algebras, the latter being necessary in order to show that the direct sum of the $\overline{S}_\lambda$ is no other than the algebra over $\overline{R}$ obtained by considering $S$ as an algebra over $R$ and extending its ground-field to $\overline{R}$.

Then norms and conorms, traces and cotraces are defined for divisors and repartitions in $S$ and in $R$; norms and traces are defined as usual; the conorm and cotrace are the dual operations, i.e. consist in "lifting" divisors, repartitions, etc., from $R$ to $S$; the consistent use of these terms (rather than the more usual identification of divisors, etc., in $R$ with the corresponding ones in $S$) is perhaps cumbersome, but is very helpful in keeping apart essentially distinct concepts while their main properties are being developed. The different is then defined, and its basic properties are given.

Chapter V discusses the extension of the field of constants; in the absence of a universal domain, such extensions have to be generated by the clumsy device of the tensor-product of fields; in the inseparable case, one has then to face the disagreeable appearance of radicals, whose rather arbitrary dismissal follows at once, not without the intervention of minimal ideals. More than half of the chapter is spent in such awkward discussions, beginning with the definition of separable (not necessarily algebraic) extensions by the non-existence of nilpotent elements in certain tensor-products and leading up to Theorem 3 (p. 92) which expresses, again in terms of such products, the effect on a place of $R$ of an extension of the field of constants. There is hardly any connection between the foregoing and the basic Theorem 4; according to the latter, if the field of constants $K$ of $R$ is extended to a field $L$, separable over $K$, and if $\mathfrak{a}$ is a divisor of $R$, every element $y$ of the extended function-field which is a multiple of $\mathfrak{a}$ (more accurately, of the "conorm" of $\mathfrak{a}$) is a linear combination, with coefficients in $L$, of elements of $R$ which are multiples of $\mathfrak{a}$ (cf. A. Weil's *Foundations*, Chapter VIII, th. 10); from this it is deduced that the genus is not altered by the extension of the field of constants from $K$ to $L$ if $L$ is separable over $K$, and that it can only decrease by an arbitrary extension.

Chapter VI takes up the behavior of differentials under an extension of the function-field; one of its main objectives is to identify the differentials in $R$ with the symbols $ydx$, provided $R$ is separable (i.e., separably generated) over $K$. This is done by means of a general theory for the "lifting" of a differential from a field $R$ to a field $S$ under suitable conditions; this operation is called the "cotrace." An explicit definition being given for a certain differential, called $dx$, in a purely transcendental extension $K(x)$ of the field of constants, $dx$ is then "lifted" from $K(x)$ to $R$ for every non-constant $x$ in $R$. A drawback of this method is, of course, that, as $dx$ and $dy$ are lifted from different fields, there is no obvious connection between them, and $d(x+y)=dx+dy$ becomes a deep theorem; perhaps a more satisfactory arrangement would have been provided by a definition similar to that adopted for meromorphic differentials in Chapter VII. However that may be, after a preliminary discussion of the field $K(x)$, the fields $K$, $R$, $L$, $S$ are again considered; the trace of a differential of $S$ is defined, in the case where $S$ is of finite degree over $R$; and the cotrace of a differential of $R$ is defined, but merely for the case $K=L$; the behavior of residues under the operations of trace and cotrace, and other elementary properties, are established. The different (which had disappeared during the whole of Chapter V) turns up again, and it is shown that, when a differential is lifted from $R$ to $S$, its divisor is multiplied by the different of $S$ over $R$; it is a pity that this § is separated from the § on the different, as both could just as easily have been put together, either here or, even better, in Chapter IV. One then comes back to $dx$, which can now be lifted from $K(x)$ to $R$; among other results, its divisor is calculated; and it is shown that, if $x$ is a uniformizing variable at a place $\mathfrak{p}$ of degree 1, the residue of $ydx$ at $\mathfrak{p}$ is the coefficient of $x^{-1}$ in the power-series expressing $y$ in terms of $x$ in the completion of $R$ at $\mathfrak{p}$. The investigation is again interrupted, this time in order to introduce the general concept of derivation in fields, algebraic function-fields and power-series fields; it is resumed for the proof of the decisive Theorem 9, according to which, for a given $x$, $dy/dx$ is the derivation $D_xy$ of $R$ which vanishes on $K$ and has the value 1 at $x$; this is ingeniously proved by showing that the differential $dy-(D_xy)dx$ has infinitely many zeros.

The concept of cotrace is then extended to the case $K\neq L$, provided $R$ is separable; since in that case the differentials of $R$ can be written as $ydx$, with $x$, $y$ in $R$, these same expressions can be used to lift them into $S$; in particular, under an extension of the field of constants of $R$, it is shown that the residues of a differential remain the same, that the divisor of a differential is unchanged if the genus is unchanged, that otherwise it is divided by an integral divisor. A further section (the purpose of which remains unexplained) discusses the effect on differentials and their residues of a derivation of the field of constants; and the chapter ends up, rather disappointingly, with a theory of differentials of the second kind confined to characteristic 0, in which case it is an easy application of the Riemann-Roch theorem.

Maybe the appearance of characteristic 0 at the end of Chapter VI was meant as a transition to the extensive Chapter VII (more than 50 pages), which treats the "classical" case, i.e. the case where the field of constants is the complex number-field, with its topology; this is almost a different book. It is hard to say what knowledge is assumed of the reader in this chapter; while it is tediously proved that meromorphic functions in an open set form a field, and one full page is devoted to the calculation of $\int dx/x$ on a circle surrounding the origin in the complex plane (the value being found to be $\pm 2\pi(-1)^{1/2}$), Schwarz's lemma suddenly turns up from nowhere (p. 152) in order to prove that holomorphic mappings preserve the orientation, a fact for which, fortunately, a more reasonable justification is given later (p. 181). The reader is further required to take for granted the validity of all the "axioms" of Eilenberg and Steenrod for the singular homology theory in arbitrary topological spaces, a statement of which is given in §3; thanks to this, says the author, "we have avoided the cumbersome decomposition of the Riemann surface into triangles." The truth is that this triangulation is a quite trivial matter; and, while the reduction of a triangulation to standard form (as done e.g. in Seifert-Threlfall) is a somewhat clumsy process, the canonical dissection of the Riemann surface which is so obtained has immense advantages over a purely homological theory; it shows that all Riemann surfaces of a given genus are homeomorphic; it gives the structure of the fundamental group, which, even to the pure algebraist, is of prime importance in determining the nature of the non-abelian extensions of the given function-field; such advantages seem to be more than enough to outweigh those of the more algebraic (and "intrinsic") procedure adopted by the author.

The chapter begins with the definition of the Riemann surface, i.e. of the set of places of the given field, as a topological space; unfortunately, its definition as an analytic manifold is given only much later, so that orientation is defined twice, and various special cases of Stokes' formula have to be proved separately. Meromorphic functions and differentials on open subsets of the Riemann surface are defined; it is shown in the usual manner that the meromorphic functions on the Riemann surface are the elements of the function-field. Periods of differentials are defined, essentially by analytic continuation (not by integration, since the 1-chains are not assumed to be differentiable), so that their definition virtually depends upon the concept of fundamental group, which however is carefully avoided.

We come now to one of the most interesting and original features of the whole book. With the author, let us denote by $S$ the Riemann surface, by $P$ and $Q$ two mutually disjoint finite subsets of $S$. Then $H_1(S-P, Q)$ is the "relative" homology group of the open set $S-P$ modulo $Q$; in other words, it is the group of classes of 1-chains lying in $S-P$, with boundary in $Q$ (the "relative cycles" in $S-P$ mod. $Q$), such a chain being homologous to 0 if it bounds in $S-P$. If $\gamma$ is such a relative cycle, and $\gamma'$ is a relative cycle in $S-Q$ mod. $P$, then, as the boundary of each cycle is disjoint from the other, the intersection-number or Kronecker index $I(\gamma, \gamma')$ is defined; it depends only upon the homology classes of $\gamma, \gamma'$; and it determines a duality between $H_1(S-P, Q)$ and $H_1(S-Q, P)$, in the sense that $I(\gamma, \gamma')$ cannot be 0 for all $\gamma$ unless $\gamma'$ is homologous to 0, and that there is a $\gamma'$ such that $I(\gamma, \gamma')$ is equal to an arbitrarily given integral-valued linear function on $H_1(S-P, Q)$. These groups, and the duality between them, can now be translated into algebraic terms by means of the following concepts. Let $E(P, Q)$ be the set of differentials on $S$ with no poles at the points of $Q$ and no residue $\neq 0$ at any point outside $P$. Take a canonical dissection of $S$ by means of curves, not going through the points of $P$ and $Q$; $S$ is then represented as a canonical polygon of $4g$ sides (where $g$ is the genus, which we assume to be $\neq 0$, the case $g=0$ being similar but simpler), all the vertices corresponding to one and the same point of $S$, and the sides occuring in the order $a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}$; join the origin $O$ of $a_1$ (the extremity of $b_g^{-1}$) to the points $P_\mu$ of $P$, and to the points $Q_\nu$ of $Q$, by mutually disjoint simple arcs $p_\mu$ resp. $q_\nu$, interior (except for their common origin $O$) to the fundamental polygon. In the polygon, cut along the arcs $q_\nu$, an element $\omega$ of $E(Q, P)$ is the differential $\omega = d\phi$ of a one-valued function $\phi$; similarly, in the polygon, cut along the arcs $p_\mu$, an element $\omega'$ of $E(P, Q)$ is the differential $\omega' = d\phi'$ of a one-valued function $\phi'$; we may assume that, at the vertex $O$, $\phi = \phi' = 0$. The integral of $\phi d\phi'$, or that of $-\phi' d\phi$, along the contour of the canonical polygon is equal to

$$\begin{aligned} I(\omega, \omega') &= \int \phi d\phi' = -\int \phi' d\phi \\ &= \sum_{\lambda=1}^{g} \left( \int_{a_\lambda} \omega \int_{b_\lambda} \omega' - \int_{b_\lambda} \omega \int_{a_\lambda} \omega' \right). \end{aligned} \tag{1}$$

Apply now Cauchy's theorem, either to the differential $\phi\omega' = \phi d\phi'$ and to the polygon cut along the arcs $q_\nu$, or to the differential $-\phi'\omega = -\phi' d\phi$ and to the polygon cut along the arcs $p_\mu$. We get

$$\begin{aligned} \frac{1}{2\pi i} I(\omega, \omega') + \sum_\nu \phi'(Q_\nu) \cdot \mathrm{Res}_{Q_\nu}\, \omega + \sum_\nu \mathrm{Res}_{Q_\nu} \left\{ [(\phi' - \phi'(Q_\nu)]\omega \right\} \\ = \sum_\mu \mathrm{Res}_{P_\mu}\, (\phi\omega') + \sum_\rho \mathrm{Res}_{R_\rho}\, (\phi\omega') \end{aligned}$$

where the $R_\rho$ are all the poles of $\omega$ or of $\omega'$, other than the $P_\mu$ and $Q_\nu$, and therefore:

$$\begin{aligned} j(\omega, \omega') &= \sum_\mu \mathrm{Res}_{P_\mu} \left\{ [\phi - \phi(P_\mu)]\omega' \right\} - \sum_\nu \mathrm{Res}_{Q_\nu} \left\{ [\phi' - \phi'(Q_\nu)]\omega \right\} \\ &\quad + \sum_\rho \mathrm{Res}_{R_\rho}\, (\phi\omega') \\ &= \frac{1}{2\pi i} I(\omega, \omega') + \sum_\nu \phi'(Q_\nu)\, \mathrm{Res}_{Q_\nu}\, \omega - \sum_\mu \phi(P_\mu)\, \mathrm{Res}_{P_\mu}\, \omega'. \end{aligned} \tag{2}$$

Here $j(\omega, \omega')$ is an alternating bilinear form, defined for $\omega \in E(Q, P)$,

$\omega' \in E(P, Q)$. Of the two expressions for it given by (2), the first one depends only upon the power-series expansions of $\omega$, $\omega'$ at the points $P_\mu$, $Q_\nu$, $R_\rho$; in fact, $\phi - \phi(P_\mu)$ is the function vanishing at $P_\mu$ with the differential $\omega$; $\phi' - \phi'(Q_\nu)$ is the function vanishing at $Q_\nu$ with the differential $\omega'$; and at a point $R_\rho$ we may, in calculating $\mathrm{Res}_{R_\rho}(\phi\omega')$, take for $\phi$ any function with the differential $\omega$ (this being meromorphic there since $\mathrm{Res}_{R_\rho}\,\omega = 0$), the residue of $\phi\omega'$ being independent of the choice of the additive constant in $\phi$ since $\mathrm{Res}_{R_\rho}\,\omega'$ is 0; and we have $\mathrm{Res}_{R_\rho}(\phi\omega') = -\mathrm{Res}_{R_\rho}(\phi'\omega)$. Put $j(\omega, \omega') = j_\omega(\omega')$; then $\omega \to j_\omega$ maps $E(Q, P)$ into the space of linear functions on $E(P, Q)$; the second expression for $j(\omega, \omega')$ in (2) shows at once that the kernel of this mapping consists of the differentials $\omega = df$ of the meromorphic functions $f$ on $S$ which are 0 at the $P_\mu$; if $F(P)$ is this kernel, then that same expression shows that $E(Q, P)/F(P)$, $E(P, Q)/F(Q)$ are two vector-spaces of finite dimension equal to $2g + (p-1)^+ + (q-1)^+$ (where $p$, $q$ are the numbers of points in $P$, $Q$ respectively, and $a^+ = \max(a, 0)$), and that $j(\omega, \omega')$ establishes a duality between them.

Now take $\omega' \in E(P, Q)$, and $c \in H_1(S-P, Q)$; let $\gamma$ be a relative cycle in $S-P$ mod. $Q$ belonging to the homology class $c$ and disjoint from the poles of $\omega'$; it is easy to see that $\int_\gamma \omega'$ depends only upon $c$ and upon the class of $\omega'$ mod. $F(Q)$; therefore there is an $\omega_c \in E(Q, P)$ such that $\int_\gamma \omega' = j(\omega_c, \omega')$ for all $\omega'$, and the class of $\omega_c$ mod. $F(P)$ depends only upon $c$. Similarly, if $c' \in H_1(S-Q, P)$, one will attach to it an element $\omega'_{c'}$, of $E(P, Q)$, well defined mod. $F(Q)$.

It is easy to determine, in terms of the canonical dissection, the structure of $H_1(S-P, Q)$; this is generated by the $a_\lambda$, $b_\lambda$, by small circles $\gamma_\mu$ surrounding the $P_\mu$ and positively oriented, and by the linear combinations $\sum_\nu m_\nu q_\nu$ of the arcs $q_\nu$, with integral coefficients $m_\nu$ satisfying $\sum_\nu m_\nu = 0$; the only relation between these generators is $\sum_\mu \gamma_\mu \backsim 0$; $H_1(S-P, Q)$ is therefore a free abelian group of rank $2g + (p-1)^+ + (q-1)^+$. Also the intersection-numbers of cycles in $H_1(S-P, Q)$ with cycles in $H_1(S-Q, P)$ are then obvious. On the other hand, if one uses the second expression in (2) for $j(\omega, \omega')$, one obtains the conditions which $\omega = \omega_c$ has to satisfy, for a given $c$, in terms of the periods of $\omega$ and of the $\phi(P_\mu)$; proceeding similarly for $\omega'_{c'}$, one finds that $j(\omega_c, \omega'_{c'})$ *is equal to the intersection-number of* $c$, $c'$ (i.e. of two cycles belonging to these homology classes).

While these are some of the main results of the author, he proceeds in an entirely different way. He first gives the algebraic definition of $j(\omega, \omega')$, and shows in a purely algebraic manner that this is a bilinear function on the spaces $E(Q, P)/F(P)$, $E(P, Q)/F(Q)$ and establishes a duality between them. He next defines $\omega_c$, $\omega'_{c'}$ as above; and he calls $j(\omega_c, \omega'_{c'})$ the intersection-number of $c$, $c'$! He then (without condescending to say so) goes on to show that this intersection-number has the properties which characterize it from the topological point of view, viz., that it is an integer, that $I(\gamma, \gamma')=0$ if $\gamma$, $\gamma'$ are disjoint, and that $I(\gamma, \gamma')=1$ if $\gamma$, $\gamma'$ are arcs inside a small circle, with extremities on that circle, the cyclic order of these extremities being suitably related to the orientation; the proof for the second one of these facts (pp. 158–161) is a singularly difficult and tortuous one, appeals to a theorem (attributed to Montel) on so-called "normal families," and also (without any reference) to the fact that a continuous function of two complex variables, separately holomorphic in each, is holomorphic in both. All this could easily have been shown by means of the canonical dissection, even if one did not want merely to verify it *a posteriori* in the manner sketched above. The author now proceeds to the determination of the homology groups, which is far from easy and requires all the combined resources of topology and of algebraic function-theory. As this gives him the technical equivalent of the tools ordinarily provided by the canonical polygon and integration along its contour, he can then prove Abel's theorem and Riemann's bilinear inequalities; even at that stage, he needs two pages to prove Stokes' formula for the complement of the union of finitely many small circles on $S$, and two more pages, involving the use of differentials of the second kind, for the proof of the bilinear inequalities for periods of differentials of the first kind. On the other hand, a very simple and direct proof is given for the fact that the group of divisor-classes of degree 0 is isomorphic to the torus-group of real dimension $2g$.

We have not yet mentioned some illustrative sections in this and the earlier chapters, on fields of genus 0 and 1, on fields of elliptic functions, and one (Chapter IV, §9) on hyperelliptic fields. Except for the latter which is a kind of *tour de force* (hyperelliptic fields over an arbitrary field of constants had probably never been discussed before), these are elementary and could well have been given in the form of exercises or series of exercises; and it is greatly to be regretted that the author has not added many more in that form, out of his rich stock of knowledge on such subjects; he could thus have greatly enhanced the usefulness of the book at the cost of a very moderate increase in size. As it is, one will not even find in it a calculation of the genus of a field $k(x, y)$ defined by $y^2=P(x)$, where $P(x)$ is a polynomial. The book is also without any bibliography beyond a small number of references in the brief introduction, a few lines of which comprise all that the reader is told about the history of the subject; the name of Riemann occurs only as a label for some theorems and in "Riemann surface." Chapters I–VI are without a single reference; it is true that they are almost entirely self-sufficient, but even at places where a reference could help the reader (as e.g. in the section on elliptic functions) none is given; of the four references in Chapter VII, two (those to Montel and to Bourbaki) are irrelevant to the author's main purposes. There is nothing to indicate to the reader which results should be considered important, which ones could be further extended. As to the style, it is that of the modern algebraic or formalistic school; the resources of the English vocabulary and syntax could not be cut down any further. Definitions are either not motivated, or else the patronizing tone in which this is done indicates that it is mere condescension to human weaknesses of which the author does not approve. At the same time, as one could expect of him, he achieves everywhere the utmost precision; there is never one vague word to mislead the unwary, perplex the novice, or let loose the fancy of the imaginative. The reader is not to look forward to a conducted tour through a picturesque countryside; he is on a bus which runs to a schedule. Why should he want to look out of the window?

Enough has been said to indicate that, in spite of some shortcomings which it was our duty to point out, this is a valuable and useful book, and also a timely one. While it is not as attractively written as the classical paper of Dedekind and Weber, or as H. Weyl's *Idee der Riemannschen Fläche*, it covers far more ground than the former, and, even in its final chapter, has little in common with the latter. It was highly desirable that the principles of the theory of algebraic functions should be treated at least once in their full generality by purely algebraic methods; this is what the author has done as perhaps no one but he could do it, and for this he has a right to expect the gratitude of the mathematical community. His attitude towards his subject has been professedly one-sided; but his work should be of value, not only to those who will always prefer the algebraic methods for their own sake, but also to those who wish to ascertain both their scope and their limitations. Indeed some conclusions already seem to emerge from it, and will now briefly be set forth.

This branch of mathematics, says the author (p. v), has an "algebraico-arithmetical" and a geometric aspect; it is surprising that he should not even mention the function-theoretic method, which was that of Riemann, and is still in many ways the most powerful one of all; alone it supplies the proof for Riemann's existence theorems, and is therefore the only source of our present knowledge about the structure of the non-abelian extensions of a function-field; in higher-dimensional problems, it leads directly to harmonic integrals and the theory of Kähler manifolds, which has achieved such striking successes in the last 20 years. Even now its advantages are such that one who is chiefly interested in characteristic $p$ will frequently begin by investigating the "classical" case and will do so by using function-theory, topology and harmonic integrals. However, as the author points out, there are valid reasons for considering other fields of constants than the complex numbers; if the characteristic is 0, it is still possible, by "Lefschetz' principle," to apply to them many of the results obtained in the classical case by function-theory; but finite fields of constants are becoming increasingly important, both for their own sake and because of possible applications to analytic number-theory; in particular, the so-called "singular series" depend upon numbers of solutions of equations over finite fields. Therefore, if for no other reason, one must be able to deal at any rate with fairly general fields of constants; and while in substance the means for doing so must be of a purely algebraic nature, one has to choose here between the language and technique of algebraic geometry and another language, originating in the theory of fields of algebraic numbers, which takes the function-field as the primary object of its study. "Whichever method is adopted," says the author, "the main results to be established are the same"; his book is good evidence to the contrary. On the one hand, the algebraic method which he follows emphasizes the analogies with algebraic numbers; one of its main advantages, in fact, is that many questions concerning algebraic numbers and functions of one variable can be so treated simultaneously, as has been shown by Artin and Whaples, and more recently by Hasse in his *Zahlentheorie*; in the book we are reviewing, the greater part of Chapters I, III and IV applies with little change to number-fields, and it is perhaps unfortunate that this is not pointed out there. For the same reasons, class-field theory can best be treated, at least at present, from that point of view; it is true that a treatment of class-field theory over function-fields of dimension 1 by the methods of algebraic geometry is greatly to be desired, for its own sake and as a preparation for the same theory in higher dimensions; but this would not, at least for some time to come, deprive the algebraic method of the advantages which it now seems to possess.

It also appears that the algebraic method is as yet the only one to deal with certain phenomena connected with inseparability. The algebraic geometer, in order to study a function-field, must assume, in the case of dimension 1, that it possesses a model without multiple points (a curve whose points are all absolutely simple in the sense of Zariski). But perhaps not much is lost by that assumption. It is always fulfilled in the case of a finite ground-field, since such fields are perfect; and the curves which arise from the application of the Picard method to varieties of higher dimension are without singular points, provided one starts from a "normal" variety. The fields which have no non-singular model may therefore, at the present moment, be regarded as pathological beings; these are the fields whose genus decreases under a suitable extension of the ground-field, and they are necessarily inseparable. If the author had excluded them, he could have spared himself some of the complications inherent in Chapters IV, V and VI; and the reader would not be left with the uncomfortable feeling of being told (sometimes without proof) that certain unpleasant things can happen, but not how and when they may happen. Thus, by imposing upon himself the task of working with an entirely unrestricted field of constants, the author, perhaps unintentionally, has overemphasized certain more or less pathological features connected with characteristic $p$, because he had to devise procedures which do not exclude them; on the other hand, far more interesting facts on characteristic $p$, such as Witt's residue-theorem for "Witt's vectors" (Crelles J., 176 (1936), p. 140), are not included, perhaps because the author considers them as pertaining to class-field theory.

The algebraic method begins to show its weakness when it comes to dealing with extensions of the field of constants. Here also a new language and new techniques had to be invented by the author, chiefly in order to show the invariance of the most important properties of a function-field under such extensions; in his introduction, he acknowledges the considerable effort which this has cost him, and, strangely enough, finds no better justification for it than a reference to a notably unsuccessful paper of Deuring on correspondence-theory, where the latter rediscovered rather clumsily a few of Severi's more elementary results on the same subject. Undoubtedly, whatever method is adopted, there are some crucial proofs (e.g. that of Theorem 4, Chapter V, p. 96) which cannot be avoided; but most readers of this book will feel the need for a language by which those properties and results which are invariant under an extension of the ground-field can be expressed and proved in a manner independent of that field; this is what algebra does not do, and what algebraic geometry does without any effort.

All this would not be decisive; as long as the geometer is exploring curves and nothing else than curves, the algebraist can keep pace with him; he will sometimes be in front, and at worst not far behind. What is decisive is that algebra stops short of higher-dimensional problems; and, even in the theory of curves, these cannot be avoided. To begin with, there are times when curves have to be embedded into projective spaces; even the author could not refrain at least once (at the end of Chapter IV) from interpreting in this manner a statement on the differentials of a non-hyperelliptic field. But the crucial test is supplied by the theory of correspondences, which is the theory of the product of two curves, and by that of the jacobian variety of a curve; there it would be impossible to take function-fields as the primary object, since one has to deal with properties of surfaces and varieties which depend upon the use of a particular model, and are not invariant under arbitrary birational transformations. It is therefore no accident that in the present book the group of divisor-classes of degree 0, which is nothing else than the jacobian, is discussed only in the "classical case" and by topological methods (v. Theorem 16 of Chapter VII, p. 176, and its corollary); and it is no accident that the algebraists who attacked those problems by their own methods failed to obtain any significant results.

Thus it appears that the author has somewhat overstated his claims, and has been too partial to the method dearest to his algebraic heart. Who would throw the first stone at him? It is rather with relief that one observes such signs of human frailty in this severely dehumanized book. And it would only remain for us to congratulate him on the service he has rendered to the mathematical public, if it were not necessary to devote some of our attention to typographical matters.

The book is generally well printed, and fairly free from misprints, much more so indeed than most previous publications of the same author; here are a few which might embarrass the reader: pp. 98–99, the references to Theorem 5 are really to Theorem 4; p. 111, l. 6–7, instead of "for the elements . . . to all be" read "for all the elements . . . to be"; p. 121, l. 6 from below, for "ramification index" read "differential exponent"; p. 123, l. 14, for "Theorem 5, V, §5" read "Theorem 4, V, §6"; p. 132, last line, for "such that" read "such that $\omega \equiv 0$ (mod. $\mathfrak{a}^{-2}$) and that"; p. 165, l. 22, for 2-chain, read 2-cycle. It is regrettable that the running title at the top of each page does not include the indication of the Chapter and §, as this makes references unnecessarily hard to find. But the point upon which we wish to draw attention is a far more serious one, and one which affects not merely this volume, but all modern mathematical printing in America. No typesetter would separate the word "and" into "a-" and "nd" at the end of a line. Yet on p. 169 of this book, $H_1(S-P, Q)$ is broken into $H_1(S-$ and $P, Q)$; on p. 149–150, a similar formula is similarly broken between one page and the next; on p. 121, the two factors of a product occur, one on line 22 and the other on line 23; several dozens of such instances could easily be given. It is difficult enough to follow such a text in detail without having constantly to reconstruct in one's mind what has been separated on paper; and, apart from all aesthetical considerations, such practices, which in this country are fast becoming the rule rather than the exception, may soon make many of our mathematical texts intolerably hard to read. It is high time that a reaction should set in against the tendency to cram as much text as possible into each page at the lowest possible cost, regardless of the effect on the reader; this will require a coordinated effort on the part of authors, editors and the printing-presses. The authors, who undoubtedly bear some responsibility for the present situation, should be more mindful of such matters in the preparation of their manuscripts; editors and editorial assistants should cooperate with them to a greater extent than sometimes happens now. As to the typesetters, who are doing an extraordinarily good job of setting the most complicated formulas, they could very easily be trained to avoid broken formulas, if their attention were drawn to it by the presses; they could well be trusted to use their judgment in displaying some long formulas, even in the absence of an indication from the author or editor; as to short formulas, all that is mostly required is some adjustment in the spacing of words; this might sometimes take more time than mechanically running along, but would still be far less expensive than later corrections which may affect a whole paragraph of type. Possibly, at least in the transitional period until typesetters acquire experience in such matters, the average cost of the printed page in mathematical texts would increase slightly; possibly the number of pages to be printed every year by mathematical journals would have to be somewhat cut down. Maybe the gain would be greater than the loss.

A. WEIL

*Projektive Differentialgeometrie.* Part I. By G. Bol. (Studia Mathematica, vol. 4.) Göttingen, Vandenhoeck and Ruprecht, 1950. 8+365 pp. 20 DM.

The present book is the first part of a treatise on projective differential geometry. This volume is divided into four sections: I. *Plane curves*; II. *Introduction to space geometry*; III. *Space curves*; IV. *Surface strips* (*Flächenstreifen*). The second (and last) volume will