Coefficient regions for schlicht functions. By A. C. Schaeffer and D. C. Spencer. With a chapter on *The region of values of the derivative of a schlicht function* by Arthur Grad. (Amer. Math. Soc. Colloquium Publications, vol. 35.) New York, 1950. 14+311 pp., 2 plates. $6.00.

In the theory of schlicht functions there is considerable difference in depth between the elementary results exemplified by Koebe's distortion theorem and the advanced theory exemplified by Löwner's proof of the inequality $|a_3| \leqq 3$. Through Löwner's paper variational techniques were introduced in the coefficient problem, and since that time they have provided a more and more efficient tool. Löwner's result was at first quite isolated. To obtain similar results of a more general nature Schiffer had to overcome specific difficulties which seemed rather formidable. Later on the techniques were perfected by Schiffer himself, by the authors of the book under discussion, and by many others. The time was ripe for a systematic presentation of such methods and results, and the team of Schaeffer and Spencer has obliged us with an impressive monograph on the subject.

The historical introduction is brief and leads almost directly to the formulation of the main problem. One considers the class $\mathfrak{S}$ of normalized functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ which are regular and schlicht in $|z| < 1$. The problem is to characterize the sequences $\{a_n\}$ which define such functions; this problem will be solved if one can determine the region V_n in $(2n-2)$-dimensional space to which the point $(a_2, a_3, \cdots, a_n)$ is confined. The most likely way to success is by determination of the boundary of V_n through the extremal properties of the corresponding functions.

This problem is not the most general and not even necessarily the most natural; it is difficult to agree that the series development has special merits in comparison with other problems of interpolation. However, the coefficient problem is certainly typical in the sense that more general problems can very probably be treated by analogous methods, and the amount of interest it derives from the fact that Bieberbach's conjecture $|a_n| \leqq n$ has neither been proved nor disproved is not negligible.

Chapter II introduces the authors' specific variational method on which most of the book is based. A rough description of the method follows. Let C be the unit circle $|z| < 1$, S the Riemann sphere over the w-plane, and $w = f(z)$ a schlicht mapping. Draw a simple analytic arc γ in C, and introduce an analytic correspondence, different from the identity, between the two edges of a slit along γ. Through identification of corresponding points a new abstractly defined Riemann surface C^* is obtained which is conformally equivalent with C. A similar identification of the corresponding points on the image $f(\gamma)$ produces a Riemann surface S^* conformally equivalent with S. Map C conformally onto C^*, let this be followed by the mapping f of C^* into S^*, and finally by a conformal mapping of S^* onto S. This series of mappings produces a mapping of C into S which, properly normalized, can serve as a variation of $f(z)$.

It turns out that the mappings can be expressed explicitly by means of line integrals, in a first approximation, and the authors derive the existence of a normalized schlicht function given by

$$\begin{aligned} f^*(z) &= f(z) \\ (1) \qquad &+ \frac{\epsilon}{2\pi i}\int_\gamma \frac{p(u)}{2u^2}\left[\left(\frac{uf'(u)}{f(u)}\right)^2 \frac{2f(z)^2}{f(u)-f(z)} - zf'(z)\frac{u+z}{u-z}\right. \\ &\left. + f(z)\right] du \ + \frac{\bar\epsilon}{2\pi i}\int_\gamma \frac{\overline{p(u)}}{2\bar u^2}\left[zf'(z)\frac{1+\bar u z}{1-\bar u z} - f(z)\right] d\bar u + o(\epsilon), \end{aligned}$$

$p(u)$ being a fairly arbitrary weight function.

In the reviewer's opinion the author's choice is a somewhat artificial compromise. It is conceptually simpler to define C^* by the introduction of a new metric, defined for instance by $ds = |dz+h(z)d\bar z|$. The computations are fairly trivial and lead to (1) with a double integral in the place of the line integral. On the other hand, the results are not stronger, and the authors' method has the advantage that it need not be based on the general uniformization theorem.

By means of (1) it is proved that every function which maximizes a real-valued differentiable function of the coefficients must satisfy a differential equation of the form

$$(2) \qquad \left(\frac{z}{w}\frac{dw}{dz}\right)^2 P(w) = Q(z)$$

with

$$P(w) = \sum_{\nu=1}^{n-1} \frac{A_\nu}{w^\nu}, \qquad Q(z) = \sum_{\nu=-(n-1)}^{n-1} \frac{B_\nu}{z^\nu}.$$

In addition, $Q(z) \geqq 0$ on $|z| = 1$ with at least one zero there.

Chapters III–VI are devoted to a study of the differential equation (2) and especially those solutions which belong to the class $\mathfrak{S}$. Although the variables can be separated, the multiple-valuedness of the integrals make this study far from trivial. The aim is of course to characterize those solutions which can correspond to boundary points of the coefficient region V_n. Such investigations tend to be rather laborious, the main chore being to refute certain undesirable possibilities. In places the presentation is not very lucid, although the painstaking reader will ultimately become convinced that the argument is correct. On the whole the authors seem to prefer blunt frontal attacks to simpler but more devious reasonings. A typical case is the proof of Lemma XI which requires almost three pages and a concentrated effort on the part of the reader. In the reviewer's opinion it would take much less effort to relate the angles, the Euler characteristic, and the signatures of the singularities by a general formula.

In the next chapter the authors use Teichmüller's method to prove the converse, namely that the solutions of (2) belong to the boundary of V_n. There is thus one-to-one correspondence between boundary points and solutions, but it may happen that a solution belongs to several differential equations of the type (2). This complication is due to the fact that V_n will be bounded by several hypersurfaces which intersect in manifolds of lower dimension.

By this time the solution of the coefficient problem begins to take rather definite shape. The extremal functions map the unit circle onto the complement of a "tree" whose branches are characterized by the condition Im $P(w)^{1/2}dw/w=0$. It is ultimately possible to characterize these trees by their topological nature and by the requisite number of numerical parameters. In this sense the solution is complete, but it leads to practically no explicit statements concerning the coefficients.

An interesting chapter deals exhaustively with the lowest nontrivial case, the inequalities between a_2 and a_3. Since a_2 may be supposed real the problem is to determine a three-dimensional region. In this case all functions can be expressed in terms of elementary functions; numerical tables are given, and two plates give excellent visual impressions. The authors fail, however, to give the geometric interpretation which in the eyes of the reviewer is the most revealing. Remove the points $\eta \leqq 0$, $|\xi| \leqq 1$ from the $\zeta=(\xi+i\eta)$-plane, and identify boundary points which are symmetric with respect to the imaginary axis; the closed surface obtained in this manner may be considered as a model of the whole complex plane. In this model we introduce a rectilinear slit with one end point at the origin, and consider the slit surface as a model of the unit circle. If we let $\zeta=\infty$ correspond to $z=0$, $w=0$ and $\zeta=0$ to $w=\infty$ the identical mapping can be considered as a conformal mapping $w=f(z)$ of $|z|<1$ into the w-plane; it is the coefficients a_2, a_3 of these mappings which form the boundary of V_3. Two cases are possible: the slit may consist of a single segment of length ρ forming an angle $\phi \neq 0$, π with the real axis, or the slit may lie on the real axis with the pieces of absolute value $\leqq 1$ identified with each other, in which case the slit appears as a fork with end points α, β. Accordingly, the boundary of V_3 consists of one surface corresponding to the parameters ρ, ϕ, and another surface with the parameters α, β.

The last chapter, written by Arthur Grad, gives an explicit determination of the region of values taken by $f'(z)$ at a fixed point z. It is an excellent illustration of the method in a case different from but of the same degree of difficulty as the determination of V_3. The reader who is anxious to learn the technique from the point of view of actual application will find this chapter most rewarding.

The authors can be congratulated on the accurate work they have accomplished. Great professional skill and painstaking detailed analysis are dominating features throughout the book. Books of this sort are never easy to read, and they offer little to the impatient skimmer. This book is definitely written by conscientious authors for conscientious readers.

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BRIEF MENTION

Funzioni quasi-periodiche. By S. Cinquini. (Scuola Normale Superiore, Pisa, Quaderni Matematici, no. 4.) Pisa, Tacchi, 1950. 132+7 pp.

The class of Bohr almost periodic functions includes in particular the subclass of exponential polynomials $\sum a_n e^{i\lambda_n x}$, and Bohr's uniqueness theorem states that the whole class arises from the subclass on closing the latter by the norm

$$\|f\| = \sup_{-\infty<x<\infty} |f(x)|. \tag{1}$$

In the periodic case this corresponds to the C-functions, and the so-called generalizations of almost periodic functions are various modes for introducing closure norms that would give the analogues to all L_p-functions as well. The narrowest such generalization known is due to Stepanoff, and it uses the norm

$$\|f\| = \sup_{-\infty<x<\infty} \left(\int_0^l |f(x+\xi)|^p d\xi \right)^{1/p} \tag{2}$$

for some (and hence any) finite length l; the purpose of the present tract is to give an account of the theory that would feature the Stepanoff functions in the main.