BOOK REVIEWS

Operational calculus. By B. van der Pol and H. Bremmer. Cambridge University Press, 1950. 14+415 pp. $10.00.

Any new book on the Heaviside calculus or on the Laplace transformation must prove its worth in competition with a large and established literature containing such favorites as Titchmarsh, Doetsch, and Widder for the mathematician, or McLachlan and Churchill for the engineer and scientist. Even in this heavy competition the book under review is likely to succeed, for it offers some points of novelty of considerable interest, its choice of material and style of exposition is "best possible" for a certain class of readers, and it is very well written.

Some novel features of the book will be noted later, but it must be mentioned already here that the style of the exposition is likely to become the greatest asset of this book, and that it is somewhat of a novelty—at any rate as far as literature in English is concerned. Every mathematical book written primarily for engineers encounters this peculiar difficulty that a purely formal presentation is admittedly inadequate even from the engineer's point of view, and yet a mathematically sound and rigorous presentation under sufficiently general conditions is far beyond the scope of such a book; and insofar as a "sound" presentation remains unintelligible to many of its readers, it defeats its purpose. The reaction of the best authors to this situation varies. Some (for instance McLachlan in his last book on operational calculus) develop the mathematical matter that is necessary for the understanding of the finer points of the exposition. Others assume a certain standard of mathematical education, and give a sound presentation under suitably simplified conditions. For instance Churchill, in his *Modern operational mathematics in engineering*, develops the theory of the Laplace transforms of sectionally continuous functions, although he points out that more general types, for instance functions with infinite discontinuities, appear in very many applications. Van der Pol and Bremmer adopt a different attitude. They make a point of stating the results in a form sufficiently general for all applications which they envisage. Instead of sectionally continuous functions, they talk of functions of bounded variation, admit integrable infinite discontinuities, mention Stieltjes integrals, and in the inversion formula consider Cauchy principal values and integrals which are Cesàro summable rather than convergent. Of course they cannot prove their results in this general form. Instead of attempting a proof under more restrictive conditions, they altogether dispense with proofs of the basic theorems: they give a clear statement accompanied by a careful expanation of the terms, and reference to standard mathematical works (chiefly the books by Doetsch and Widder). The space saved by the omission of the proofs, and far more than that space, is devoted to a lavish discussion of the theorems. Illustrative examples are used to show the meaning of the various conditions, the role they play, and the practical use of the theorems. To the reviewer's mind this is an excellent plan—when it is carried out with as much understanding of the mathematical background and with as wide a knowledge of the applications as in the book under review. The technologist is put in a position to use the theorem under rather general conditions with comparative safety, and gets a much more intelligent grasp of the situation than an abstract proof would give him. The mathematical reader, on the other hand, is spared long routine proofs, and the references put him in the position to read up on the theorems with which he is not familiar. It must be admitted, though, that somehow this exposition lacks the last touch of precision. At no stage is it made quite clear just on what class of functions the authors defined the Laplace transform, and while the trained mathematician will have no difficulty in recognizing the conditions under which each result is valid, the engineer may be uncertain at times.

The authors deviate from the usual practice by basing the operational calculus, in chapter I, on the two-sided Laplace transform

$$f(p) = p\int_{-\infty}^{\infty} e^{-pt}h(t)dt \tag{1}$$

rather than on the one-sided transform (with 0 and ∞ as the limits of integration). They hold that the new operational calculus embraces a larger class of functions, simplifies the so-called rules, and leads to a more rigorous treatment. Actually, every complete discussion should (and the present book like many others does) include both the one-sided and the two-sided transformation, and there is little evidence to show that there is much to choose between the two as a starting point. Personal taste seems to be decisive.

In chapter II the Fourier integral in the complex domain is discussed, and tentative results on the domain of convergence of a Laplace integral are obtained. The notation used for the relation (1) is $f(p) \risingdotseq h(t)$ or $h(t) \fallingdotseq f(p)$. The function $h(t)$ is called the *original*, and $f(p)$ the *image*. One-sided Laplace transforms may be written as $f(p) \risingdotseq U(t)h(t)$ where Heaviside's unit function $U(t)$ is defined to be 1 for positive t, 1/2 for $t=0$, and 0 for negative t. A number of useful images are computed in chapter III, and the rules of the operational calculus are explained in chapter IV. At this stage the derivation is more or less formal, and although explanatory remarks are added, the conditions of validity of the various rules are not defined very precisely. Take for instance the differentiation rule. From $h(t) \doteqdot f(p)$ it follows that $h'(t) \doteqdot pf(p)$. It is pointed out that this formula is based on an interchange of two limiting processes, and must be handled with caution. Well-chosen examples illustrate the various possibilities, but no precise conditions of validity are given.

Chapter V is an excellent chapter on the unit function and its derivatives. Beside the definition given above, the authors give several families of continuous functions which approximate to the unit function. The impulse function $\delta(t)$ is defined as the derivative of the unit function, and approximations for it are obtained. The history of this function is traced through the writings of Cauchy, Poisson, Hermite, Kirchhoff, and more recent authors. The principal property of this function can be expressed by the "sifting integral"

$$\int_{-\infty}^{\infty} h(\tau)\delta(t - \tau)d\tau = h(t), \tag{2}$$

and the authors are careful in pointing out that this integral, and also the image of the delta function, can be given a mathematical meaning as a Stieltjes integral. They discuss several functions approximating to the delta function, and point out that not every function which has the sifting property (that is, not every singular kernel) can be regarded as an approximation for the delta function.

The main part of the mathematical theory of the Laplace transformation is contained in chapters VI and VII. In chapter VI, the region of convergence is investigated. The formulas for the abscissae of convergence, absolute convergence, and uniform convergence are stated without proof, but with reference to books where proofs can be found, and a number of examples serve to familiarize the reader with the use of the formulas. After a digression on summable series and integrals, the behavior of the Laplace integral on the boundary of the region of convergence, and the (complex) inversion formula are discussed. It is pointed out that the same image in two different strips may arise from two different originals, and operational relations having a line of convergence (rather than a strip of convergence) are introduced. One of the advantages of the bilateral transformation is that it exhibits a symmetry between the definition integral and the inversion integral.

The operational treatment of infinite series, and of asymptotic expansions, is taken up in chapter VII. The basic Abelian and Tauberian theorems are explained and applied to derive operational equalities. Asymptotic series are introduced and the usual asymptotic expansions are proved. This part of the book is not only more thorough, but also more detailed than that of any other book written for engineers—a very valuable feature since asymptotic expansions occur so frequently in applications of the operational calculus. The operational interpretation of (ascending or descending) power series, and Heaviside's expansion theorem are also discussed in this chapter. The various possibilities are outlined, but no very precise and satisfactory conditions of validity are given. This chapter concludes with Widder's inversion formula.

A little more than half of the book is devoted to the applications of operational calculus. There is no attempt at a general theory in these chapters, but the technique is explained and illustrated by a number of well-chosen (worked) examples. There are examples taken from electrical engineering problems, there are others illustrating the application of operational calculus to the investigation of special functions of mathematical physics, and yet other applications to the functions of number theory. At least the first two of these groups occur in other books on the subject, but the reviewer knows no book which gives such an extensive and well-balanced training in all three. Partial differential equations and integral equations are not illustrated to the same extent, but in view of Carslaw and Jaeger's book, and of the chapter on integral equations in Titchmarsh's book, this was not really necessary.

Chapter VIII is on linear differential equations with constant coefficients, including differential equations with one-point boundary conditions. Most examples here are taken from electrical network theory. The same is true of the examples of chapter IX on systems of linear differential equations with constant coefficients. Linear differential equations with variable coefficients are discussed in chapter X. Laguerre and Hermite polynomials, Bessel, Legendre, and hypergeometric functions are discussed in great detail, and as a matter of fact this chapter in conjunction with many of the examples in other chapters forms an excellent introduction to the special functions of mathematical physics. Of general theory there is none, and it appears that so far no book on the Laplace transformation ventured on a presentation of the Birkhoff-Horn theory of integration of linear differential equations by Laplace integrals.

Samples of the problems discussed in chapter XI are: the relation between the originals belonging to $f(p)$ and to $p^k f(1/p)$, the connection between the images of $h(t)$ and $h(g(t))$ where $g(t)$ is a standard function such as t^{-1}, t^2, e^t, sinh t, and so on, interesting applications to hypergeometric series, the Parseval relation, again with interesting applications to Bessel function identities. Chapter XII introduces step- and discontinuous functions. This is the chapter in which the functions of number theory play such a conspicuous and welcome part. While the work is an excellent practice in operational technique, the results are also relevant for the theory of numbers. The discussion of difference equations in chapter XIII is another opportunity for deriving many relations between special functions.

Chapter XIV explains the technique of solving integral equations by means of the operational calculus. Chapter XV, on partial differential equations, is somewhat disappointing. It makes stimulating reading, the technique is explained carefully, and is illustrated by excellent examples; yet the work is more or less formal. The peculiar difficulties involved in verifying the solution are not even mentioned.

In chapter XVI the operational calculus is extended to several variables, and it is shown how this simultaneous operational calculus can be utilized for the solution of partial differential equations, and other problems.

Chapter XVII contains ten pages of general formulas and rules of the operational calculus, and is aptly labelled *Grammar*. Chapter XVIII, *Dictionary*, contains 27 pages of well-classified operational transform pairs.

The book developed from lectures given by van der Pol in 1938 and 1940, the original Dutch manuscript was prepared during the late war, and the English translation was edited by Dr. C. J. Bouwkamp. The translation deserves special praise for preserving the freshness and flavor of the original. It goes without saying that it is always clear what the authors mean, even where they do not express themselves idiomatically. Passages which may lead to misunderstandings, as for instance on p. 135 where the authors say "the exponents of x may increase" when they presumably mean "the exponents of x are assumed to increase" are very rare. A very special praise is also due to the printing of this book, one of the most beautifully printed of the recent mathematical books.

A. ERDÉLYI

Existence theorems in partial differential equations. By Dorothy L. Bernstein. (Annals of Mathematics Studies, no. 23.) Princeton University Press, 1950. 10+228 pp. $2.50.

An enormous number of papers have been written that deal with