*Nonlinear vibrations in mechanical and electrical systems.* By J. J. Stoker. New York, Interscience, 1950. 20+273 pp. $5.00.

Professor Stoker's book is an important landmark in the first decade of the studies of nonlinear problems in the United States and should be recommended to those who are interested in a systematic study of this subject. The presentation renders it particularly attractive to physicists and engineers owing to a considerable number of examples illustrating the various mathematical concepts such as singularities, limit cycles, and so on, which thus acquire an almost intuitive aspect. On the other hand, once this intuitive approach has been sufficiently prepared, the author proceeds with the necessary rigor to prove the existence theorems for these concepts introduced more or less on physical grounds.

As the author states in the introduction, it was necessary to restrict the subject to a somewhat limited number of selected topics in order to keep within the limits of the book. In fact, the Russian literature alone between 1929 and 1937 covers over 2000 pages of the various publications to say nothing of a considerable amount of the earlier material accumulated by Poincaré, Liapounoff, Bendixson, Birkhoff, and other forerunners in these studies. Nevertheless, Stoker's treatment of these selected topics is sufficiently broad to enable the reader to acquire the grasp of a much wider situation.

Perhaps the best way of abstracting the book is to proceed chapter by chapter as he does himself in his introduction. Chapter I is a short summary of the theory of linear vibrations for a system with one degree of freedom; it serves both as a reference and contrast for the nonlinear theory. Chapter II introduces the reader into the simplest case of nonlinear problems associated with the behavior of nonlinear conservative systems. The phase plane representation of integral curves is introduced in this chapter and is illustrated by a number of examples. Chapter III is also of a rather elementary character and deals with the differential equations of a more general type, namely, those which, in addition to the nonlinear restoring force, possess also a nonlinear damping. In this chapter the author gives an outline of the Liénard graphical method and closes the theoretical part by a summary of the theory and classification of simple singularities of the differential equations in real domain. The end of the chapter contains a number of physical examples illustrating the concept of singular points as various types of equilibria of dynamical systems. The last example concerning the behavior of the synchronous motor covers nearly 14 pages of the text and is treated in great detail.

Chapter IV is of a less elementary character and deals with the so-called Duffing equation with forcing term. This subject is of interest mostly in mechanical problems and is treated by two different methods, the iteration and the perturbation methods. §10 of this chapter touches a far more advanced point which has formed the subject of many discussions in the past, namely, the difficulty of the so-called "small divisors" which was encountered by Poincaré, Gylden, Lindstedt, and other mathematicians of that time in connection with astronomical problems. A more detailed treatment of this subject is given in Appendix II.

Chapter V is perhaps the most interesting and important of the whole book and is split into two parts. Part A deals with the self-excited systems. In this chapter the author introduces the fundamental concept of limit cycles on physical grounds, but a fuller mathematical treatment of this subject is given in Appendices III and VI. The second half of Part A is devoted to the so-called relaxation oscillations and contains a considerable amount of original work done by Stoker who has contributed much to recent studies of this difficult subject. Particularly interesting are asymptotic expansions appearing as approximations of higher orders in this case.

Part B deals with the forced oscillations in self-sustained systems. After a brief review of the earlier work of van der Pol, the author gives a very interesting and detailed presentation of the Andronow-Witt method, appearing as a further extension of the van der Pol theory. The method consists of investigating the distribution of singularities in a special parameter plane having "detuning" as one of the variables. Following this line of attack, the author treats the problem of stability of harmonic as well as combination oscillations in seven sections occupying nearly thirty pages of the text and extending considerably the original method of Andronow and Witt on the basis of "contact curves" of Poincaré. This is perhaps the most interesting part of the book. The only point to be regretted is that the connection between this theory and the well known phenomenon of synchronization (or "entrainment") is hardly mentioned at all. For a mathematical reader, this may not be so important, but for a physicist or an engineer more familiar with the phenomenon itself there is a risk of overlooking this important connection.

The last chapter is in some respects an anti-climax to the really remarkable Chapter V. The beginning of Chapter VI is devoted to a somewhat long review of the theory of linear differential equations with periodic coefficients (the Floquet theory), covering nearly 24 pages of the text. The reason for doing this, as Stoker explains, is that "the problem of the infinitesimal stability of the periodic solutions of nonlinear systems always leads to a Hill equation." Following this thought, the author reduces the problem of stability to the analysis of the local behavior of the variational equations (of the Hill type) in the various parts of the response curve and applies this procedure to the Duffing equation. It turns out, however, that, on this basis, the free oscillations are unstable. Having ascertained this seemingly paradoxical result, the author ascribes it to the fact that the criterion of the "infinitesimal stability" (that is, the stability in the sense of the variational equations) ought to be replaced by that of the orbital stability. In fact, after a somewhat delicate argument, in §7, the author proves this point. In spite of this, the reader, particularly the beginner, must inevitably feel somewhat confused as to when to use one criterion and when to use the other. This question does not seem to find a definite answer in the text, probably because the author, as he said in his introduction, had to curtail considerably the theory of stability owing to lack of space. It seems, however, sufficiently simple to show that if the differential equations are referred to the "amplitude-phase" plane $(a, \phi)$ instead of the usual $(x, \dot{x})$ phase plane (namely, $da/dt=f_1(a, \phi)$, $d\phi/dt=f_2(a, \phi)$) the singular point $f_1(a_0, \phi_0)=f_2(a_0, \phi_0)=0$ in this case represents the stationary periodic motion (if $a_0\neq 0$) and the variational equations ("the infinitesimal stability") give precisely the orbital stability in such a case, without any necessity of applying the theory of characteristic exponents of Poincaré. Reduction to this form is always possible if the differential equations do not contain time explicitly.

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*Differential algebra.* By Joseph Fels Ritt. (American Mathematical Society Colloquium Publications, vol. 33.) New York, American Mathematical Society, 1950. 8+181 pp. $4.40.

It was a gigantic task that J. F. Ritt undertook twenty years ago: to give the classical theory of nonlinear differential equations a rigorous algebraic foundation. Emmy Noether and her school had done the same thing for the theory of algebraic equations and algebraic varieties, but differential equations are much more difficult than algebraic equations. Luckily, Ritt has gathered around himself a whole school of able collaborators: Raudenbusch, Strodt, Kolchin, Howard Levi, Gourin, R. M. Cohn.

The present book is not just a revised and enlarged edition of the author's *Differential equations from the algebraic standpoint* (Colloquium Publications, vol. 14). It is written from a much higher point