THE JUNE MEETING IN SEATTLE

The four hundred sixtieth meeting of the American Mathematical Society was held at the University of Washington, Seattle, Washington, on Friday and Saturday, June 16–17, 1950, in conjunction with the Pacific Northwest Section of the Mathematical Association of America. Some 80 persons were in attendance, including the following 55 members of the Society:

T. M. Apostol, B. H. Arnold, S. P. Avann, R. W. Ball, Constance Ballantine, J. P. Ballantine, R. A. Beaumont, Z. W. Birnbaum, J. L. Brenner, D. G. Chapman, Harold Chatland, Paul Civin, C. M. Cramlet, D. B. Dekker, Worthie Doyle, F. E. Ehlers, Howard Eves, K. S. Ghent, F. L. Griffin, S. G. Hacker, Mary Haller, Edwin Hewitt, J. L. Hildebrand, J. W. Hurst, R. D. James, S. A. Jennings, L. G. Jones, J. M. Kingston, M. S. Knebelman, M. Z. Krzywoblocki, H. B. Mann, Rhoda Manning, L. H. McFarlan, A. S. Merrill, M. R. Moore, A. F. Moursund, D. C. Murdoch, Ivan Niven, Gloria Olive, D. B. Owen, T. S. Peterson, A. R. Poole, John Raymond, Louise Rosenbaum, R. A. Rosenbaum, E. C. Schlesinger, W. M. Stone, N. Y. Tang, F. H. Tingey, G. E. Uhrich, D. V. V. Wend, L. B. Williams, R. M. Winger, Fumio Yagi, H. S. Zuckerman.

On Friday evening there was a joint dinner with the Association at the banquet room of the University Commons. Following the dinner the visitors were guests of the Department of Mathematics at an informal evening at the Faculty Club.

On Saturday morning there was a general session for the invited address by Professor Edwin Hewitt of the University of Washington, entitled *Ideal theory in algebras of linear functionals*. This was followed by a session on Algebra for contributed papers. Professor M. S. Knebelman presided at both sessions. In the afternoon there was a session on analysis and geometry, Professor R. D. James presiding.

During the morning session, the members present voted unanimously to request the Council of the Society to schedule a meeting for Saturday, June 16, 1951, to be held in conjunction with a meeting of the Pacific Northwest Section of the Mathematical Association of America at the State College of Washington, Pullman, Washington, and expressed the hope that these joint meetings might become annual events. At the afternoon meeting the members voted a resolution of appreciation to the University of Washington, the Department of Mathematics, and the committees on arrangements and entertainment.

In the absence of the Associate Secretary, Professor R. M. Winger acted as secretary to the meeting.

Abstracts of papers presented at the meeting follow. Those abstracts whose numbers are followed by the letter "*t*" were presented by title. Paper number 419 was read by Professor Beaumont.

Algebra and Theory of Numbers

408. B. H. Arnold: *Birkhoff's Problem* 20.

Birkhoff (*Lattice theory*, 1948, p. 57, Problem 20) gives a relation which he suggests may generate a polarity between the classes of closed sets and the classes of valid statements of convergence in various topologizations of a set. In the present paper, an example is given to show that Birkhoff's relation does not give the desired polarity. Using a slightly different relation, the polarity is obtained. Abstract conditions are given which are necessary and sufficient for (Birkhoff's notation) $\Delta=(\Delta^*)^+$ and $T=(T^+)^*$. (Received March 31, 1950.)

409. J. L. Brenner: *Polynomial parametrizations.*

Let x, y, t be indeterminates, R a domain of integrity; $f=f(x, y)$ a homogeneous polynomial, $f\in R[x, y]$. If nonconstant polynomials $P_i=\sum a_{ij}t^j$ $(i=1, 2)$ exist, $a_{ij}\in R$, such that $G(t)=f(P_1, P_2)$ is a nonzero constant K, then $f(x, y)=c(a_{2n}x-ya_{1n})^m$ for some c, n; $c\in R$. Thus if R is also a unique factorization domain, then all polynomial parameterizations, if there are any, of $f=K$, $K\neq 0$, are given by $x=a_{1n}S(t)+d$, $y=a_{2n}S(t)$, $d\neq 0$. As an application, the 2×2 unitary matrixes over $R[t]$ are found. (Received April 7, 1950.)

410*t*. J. L. Brenner: *Matrices of quaternions.* Preliminary report.

Let A be an $n\times n$ matrix with quaternion elements. A nonsingular matrix T exists such that $TAT^{-1}=(b_{ij})$, where $b_{ij}=0$ whenever $i>j+1$. The elements of T are rational functions of the elements of A. Using this last semi-canonical form, it is proved that *every matrix of quaternions has a characteristic root* λ, such that $Ax=x\lambda$ for some vector x. The methods used appeared in Ann. of Math. vol. 39 (1938) p. 476; also used is the result that every polynomial of degree n and type 1 has a root (Eilenberg-Niven, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 246–248). (Received May 31, 1950.)

411*t*. Frank Harary: *A generalization of ring duality.*

Several results of A. L. Foster's papers *The idempotent elements of a commutative ring form a Boolean algebra*; *ring duality and transformation theory* and *Maximal idempotent sets in a ring with unit* are generalized to the class of all rings having a nonzero idempotent element. In particular, (i) an analogous duality theory is found, and (ii) if an idem-set is a subset of a ring in which each element is idempotent and any two elements commute, then any maximal idem-set is a generalized Boolean algebra. (Received April 6, 1950.)

412*t*. G. P. Hochschild: *Automorphisms of simple algebras.*

A theory of automorphisms of a simple algebra, A, is established by combining the Galois theory for fields with the theory of commutators in simple algebras. The results concern the usual relationships between groups of automorphisms, subjected to appropriate regularity and completeness conditions, and simple subrings B of A, such that the center of A is finite over its intersection with B, and B generates a simple algebra over the center of A. (Received May 3, 1950.)

413. D. C. Murdoch: *Isolated component ideals in a noncommutative ring.*

Let R be a noncommutative ring. N. H. McCoy (Amer. J. Math. vol. 71 (1949) pp. 823–833) has defined an m-system as a set M of elements such that if $a \in M$ and $b \in M$ then $axb \in M$ for some x in R. The complement in R of a prime ideal is an m-system. Corresponding to a fixed m-system M, a right M-n-system N is defined as a set of elements containing M such that $m \in M$ and $n \in N$ imply $nxm \in N$ for some x in R. If $\mathfrak{a}$ is an ideal and M an m-system, the right upper M-component of $\mathfrak{a}$ is an ideal $\mathfrak{u}(\mathfrak{a}, M)$ consisting of all elements x such that every right M-n-system containing x meets $\mathfrak{a}$. The right lower M-component $\mathfrak{l}(\mathfrak{a}, M)$ is the ideal consisting of all x such that $xRm \subseteq \mathfrak{a}$ for some m in M. Both $\mathfrak{u}(\mathfrak{a}, M)$ and $\mathfrak{l}(\mathfrak{a}, M)$ are generalizations of the isolated component ideals defined by Krull in the commutative case. Properties of $\mathfrak{u}(\mathfrak{a}, M)$ and $\mathfrak{l}(\mathfrak{a}, M)$ and relations between them are investigated. In certain cases principal components of $\mathfrak{a}$, in the sense of Krull, can be defined and $\mathfrak{a}$ is then the crosscut of its right lower principal components. (Received April 3, 1950.)

414t. D. C. Murdoch: *Isolated primary components of an ideal in a noncommutative ring.*

An ideal $\mathfrak{q}$ in a noncommutative ring R is defined to be right primary if, when a is not in $\mathfrak{q}$, $aRb \subseteq \mathfrak{q}$ implies $b \in \mathfrak{r}(\mathfrak{q})$ where $\mathfrak{r}(\mathfrak{q})$ is the radical of $\mathfrak{q}$ in the sense of McCoy (Amer. J. Math. vol. 71 (1949) pp. 823–833). If either the ascending or descending chain condition holds in all quotient rings $R/\mathfrak{a}$, with $\mathfrak{a}$ not prime, then the radical of a right primary ideal is prime. In rings with ascending chain condition not every ideal is representable as the crosscut of a finite number of right primary ideals. However if an ideal is so representable, then each upper M-component $\mathfrak{u}(\mathfrak{a}, M)$ (see preceding abstract), where M is the complement of a minimal prime divisor of $\mathfrak{a}$, is right primary. These are called the isolated primary components of $\mathfrak{a}$ and they must all occur among the right primary components in any minimal representation of $\mathfrak{a}$ as a finite crosscut of right primary ideals. (Received March 6, 1950.)

415. Ivan Niven: *A binary operation on sets of positive integers.*

Let A and B denote the sets of increasing positive integers $\{a_i\}$ and $\{b_i\}$. Define the "product" AB to be the set $\{b_{a_i}\}$. Under this operation, sets of positive integers form an associative, non-commutative groupoid. Basic properties are determined, including the relations of the densities of A, B, and AB, for both limit and number theoretic densities. (Received May 5, 1950.)

416t. R. M. Robinson: *Arithmetical definability of field elements.*

It is shown that an element of an algebraic field is arithmetically definable if and only if it is fixed for all automorphisms of the field. (Received April 27, 1950.)

417t. A. R. Schweitzer: *Grassmann's extensive algebra and modern number theory.* II.

The author considers Grassmann's algebra in relation to investigations of Cauchy and Kronecker. In C. R. Acad. Sci. Paris (1847) p. 1120 Cauchy obtained a "real" interpretation of the complex number $a+bi$ ($i^2=-1$) by specializing a polynomial generalization of Gauss's relation of congruence. Subsequently (C. R. Acad. Sci. Paris (1853) p. 75) Cauchy construed the units of the complex number $a+bi$ (and also the units of Hamilton's quaternion) as instances of his "algebraic keys" (clefs algébriques) and showed that his theory of the latter is applicable to his method of congruence (C. R. Acad. Sci. Paris (1853) p. 168). Grassmann noted (C. R. Acad. Sci. Paris (1854)) that Cauchy's algebraic keys correspond to his "extensive magnitudes." The two standpoints are essentially distinct, as Grassmann has suggested (*Gesammelte Werke*, vol. 1, part 2, pp. 9, 10, 399). Kronecker extended, in effect, Cauchy's method of congruence to representations of negative numbers and rationals (J. Reine Angew. Math. vol. 101 (1887) pp. 345, 346) as a contribution to a plan of "arithmetization" of mathematical disciplines (ibid. pp. 338, 339, 355) which included replacement of algebraic numbers (ibid. p. 347) with the aid of modular systems (J. Reine Angew. Math. vol. 92 (1882) pp. 1–122). (Received May 3, 1950.)

418*t*. A. R. Schweitzer: *On the place of the algebraic equation in Grassman's extensive algebra.*

The author discusses the following problem: Given the points $x_1, x_2, \cdots, x_{n+1}, y_1, y_2, \cdots, y_n, t_1, t_2, \cdots, t_n$ in Grassmann space, to find the point y_{n+1} such that $f(u_1, u_2, \cdots, u_{n+1})=f(v_1, v_2, \cdots, v_{n+1})$ where $u_i=f(x_i, t_1, t_2, \cdots, t_n)$, $v_i=f(y_i, t_1, t_2, \cdots, t_n)$ $(i=1, 2, \cdots, n+1)$; $f(x_1, x_2, \cdots, x_{n+1})$ is a linear function of the x's with numerical coefficients a_i different from zero and belonging to a field F; $f(u_1, u_2, \cdots, u_{n+1})=f(\lambda_1 x_2, \lambda_2 x_3, \cdots, \lambda_{n+1}x_1)$ where the numerical coefficients λ_i belong to F. For $n=1$ the preceding generalizes a problem equivalent to a problem considered by Grassmann, *Gesammelte Werke*, vol. 1, part 1, pp. 158, 159. The author shows that the solution of the preceding problem requires that $a_1^n(x_1-y_1)+\lambda_1 a_1^{n-1}(x_2-y_2)+\cdots+(\lambda_1\lambda_2\cdots\lambda_n)(x_{n+1}-y_{n+1})=0$, where the λ's are different from zero but otherwise arbitrary; the sum of the coefficients of the differences x_i-y_i is an algebraic equation in a_1 and of the nth degree. Also λ_{n+1} must satisfy a vanishing determinant whose rows are respectively: $1, \lambda_1, \lambda_1\lambda_2, \cdots, (\lambda_1\lambda_2\cdots\lambda_n)$; $1, \lambda_2, \lambda_2\lambda_3, \cdots, (\lambda_2\lambda_3\cdots\lambda_{n+1})$; $\cdots 1, \lambda_{n+1}, \lambda_{n+1}\lambda_1, \cdots, (\lambda_{n+1}\lambda_1\cdots\lambda_{n-1})$. Reference is made to Grassmann, ibid. pp. 165, 373. (Received May 3, 1950.)

419. H. S. Zuckerman and R. A. Beaumont: *A characterization of the subgroups of the additive rationals.*

All of the subgroups of the additive group of rational numbers are determined by means of a simple characterization. Many of their properties are readily obtained. For example, all isomorphisms between the subgroups are determined. Also, a complete survey of the rings which have these subgroups as additive groups is given. (Received April 28, 1950.)

ANALYSIS

420. T. M. Apostol: *Asymptotic series related to the partition function.*

Let $G_p(x)$ be the Lambert series $\sum_{n=1}^{\infty} n^{-p}x^n/(1-x^n)$ and let $f_p(x)=\exp(G_p(x))$. If $f_p(x)$ is developed into a power series $f_p(x)=1+\sum_{n=1}^{\infty} a_p(n)x^n$, then for $p=1$ the coefficients reduce to $p(n)$, the number of unrestricted partitions of n. For $p>1$, the coefficients are no longer integers but they still have some properties resembling those of $p(n)$. For odd $p>1$ the author obtains an asymptotic formula for $a_p(n)$ analogous to Rademacher's famous convergent series for $p(n)$. The series for $a_p(n)$ is obtained with bounded error term. The method used is the classic Farey-dissection method and use is made of transformation formulae of the author (see Bull. Amer. Math. Soc. vol. 55 (1949) p. 715) which give the behaviour of $f_p(x)$ near the singularities on the unit circle. It is possible that the error term can be replaced by an error which tends to zero, thus giving a convergent series for $a_p(n)$, but this would require improving the estimate on a complicated type of exponential sum. (Received April 25, 1950.)

421. D. G. Chapman: *A note on quasi-analytic functions.*

Let M_n be a sequence of positive numbers and C_{m_n} be the class of infinitely differentiable functions over the closed interval (0, 1) such that to each function f there exists a real number $k>0$ such that $|f^n(x)|\leqq k^n M_n$ for $0\leqq x\leqq 1$, $n\geqq 1$. If for any f in C_{m_n}, $f^n(x_0)=0$ for some x_0 in (0, 1) and all n implies $f(x)=0$, then C_{m_n} is quasi-analytic. A simple new proof is here given for the theorem that a necessary condition that C_{m_n} be quasi-analytic is that $\sum_{n=1}^{\infty}1/M_n^*$ diverge where $M_n^*=\inf_{r\geqq n} M_r^{1/r}$ (this condition is also known to be sufficient). Some extensions to abstract spaces are considered. (Received May 5, 1950.)

422. Paul Civin: *Approximation to conjugate functions.*

Suppose the trigonometric polynomials $P_n(x)$ and a function $f(x)$ satisfy the relation $f(x)-P_n(x)=O(n^{-\alpha})$, $0<\alpha<\infty$, uniformly in x. The conjugate trigonometric polynomials and the conjugate function then satisfy the relation $\bar{f}(x)-\bar{P}_n(x)=O(n^{-\alpha}\log n)$ uniformly in x, and this order is the best possible for arbitrary $P_n(x)$. For the Fourier partial sums Salem and Zygmund [Trans. Amer. Math. Soc. vol. 59 (1946) pp. 11–22] showed that the order of the conjugate approximation was $n^{-\alpha}$. (Received April 26, 1950.)

423*t*. V. L. Klee: *Some remarks concerning reflexive spaces and adjoint spaces.*

With the aid of a characterization of weak compactness due to Šmulian, there is established a characterization of reflexivity which has the following corollary: (I) *A normed linear space is reflexive if and only if in each of its isomorphs, every continuous linear functional attains its maximum on the unit sphere.* For spaces having a basis, this result was announced by R. C. James (Bull. Amer. Math. Soc. Abstract 56-1-80). Also proved is (II) *The space* (m) *has an isomorph which is not equivalent to any adjoint space.* This answers a question of J. Dixmier (Duke Math. J. vol. 15 (1948) p. 1070). (Received May 5, 1950.)

424*t*. B. O. Koopman: *A probabilistic generalization of matric Banach algebras.*

Let X be any nonempty abstract set and S any σ-algebra of subsets of X. Let $\mathfrak{B}$ be the normed algebraic system whose elements are the complex-valued functions $\phi(x, E)$, defined and bounded over $(x, E)\in(X, S)$, measurable (S) in x (E fixed), and countably additive in E (x fixed). Linear combinations of elements of $\mathfrak{B}$ with complex coefficients are defined as usual; the (noncommutative) product $(\phi\psi)(x, E)$ is defined as the integral of $\psi(y, E)$ with respect to the measure $\phi(x, F_y)$, where x and E remain fixed while y, F_y are variables of integration. The norm $N(\phi)$ in $\mathfrak{B}$ is defined as the least upper bound over $x \in X$ of the total absolute variation over X of $\phi(x, E)$ regarded as a set function. It is shown that the system $\mathfrak{B}$ is a Banach algebra over the complex field, possessing the identity $\epsilon = \epsilon(x, E) = \chi_E(x)$ the characteristic function of E. This $\mathfrak{B}$ is evidently a generalization of matric algebra. It arises in the study of stochastic processes, where transition probabilities $P(x \rightarrow E) = \pi(x, E)$ are elements of B. (Received May 24, 1950.)

425t. E. R. Lorch: *Differentiable inequalities.* II.

The present report constitutes a continuation of Bull. Amer. Math. Soc., Abstract 55-7-356. The subject here dealt with is that of *approximation by differentiable surfaces*. A natural class $\mathfrak{C}$ of differentiable surfaces is one in which the surfaces and their adjoints have a common domain Ω and a common domain Ω^* respectively. Furthermore, $\mathfrak{C}$ is closed under positive scalar multiplication and addition (if $G(x) = c$ and $H(x) = c$ are in $\mathfrak{C}$, so is $\lambda G(x) + \mu H(x) = c$). For such a class a separation theorem may be established: If the surface $\mathfrak{S}_0$ separates $\mathfrak{S}_1$ from the origin, then $\mathfrak{S}_1^*$ separates $\mathfrak{S}_0^*$ from the origin. It is now possible to introduce a metric in a natural class. The distance between two surfaces is measured by the amount of magnification and contraction necessary to produce separation. This metric space is completed by means of convergent sequences. For every surface of the completed space there is a unique adjoint surface and these two give rise to a bilinear inequality $\sum x_i^* y_i \leqq \psi(x^*) \cdot \phi(y)$. The theory is applied to the natural class of closed convex surfaces. It is shown that any surface bounding a body convex in the classical sense may be approximated by surfaces convex in the sense of this paper. (Received April 25, 1950.)

426t. E. R. Lorch: *Differentiable inequalities.* III.

This section gives an application of the theory of convexity outlined in two previous reports to the theory of volumes and mixed volumes. A large number of formulas are developed for these, including all well known classic equations. If $\phi(x)$ is positively homogeneous of degree 1, if $rG(x) = \phi^r(x)$, $r > 1$, and if $G(x) = c$ is a closed convex surface $\mathfrak{S}$, then the volume V^* bounded by the adjoint surface $\mathfrak{S}^*$ satisfies $V^* = \lim_{r \to 1} [n(r-1)]^{-1} \int_\Omega |G_{ij}(x)| d\omega$ where the integration is over the unit sphere. If Φ_{ij} is the cofactor ϕ_{ij} in $|\phi_{ij}|$, then $V^*(i, j) = n^{-1} \int_\Omega \phi \Phi_{ij} x_i x_j d\omega$ is called a partial volume. One has $\sum V^*(i, j) = V^*$. The theory of mixed volumes is developed. If $\mathfrak{K}_0$ is a convex body given by $rG(x) \leqq 1$ and $\mathfrak{K}_1$ by $rH(x) \leqq 1$, then $V(\lambda \mathfrak{K}_0^* + \mu \mathfrak{K}_1^*) = \lim_{r \to 1} [n(r-1)]^{-1} \int_\Omega |\lambda G_{ij} + \mu H_{ij}| d\omega$. The mixed volume of order ν, V_ν^*, may be represented as the integral of the trace of a matrix. Given a matrix $\|a_{ij}\|$, let $\|a_{ij}\|^{(\nu)}$ represent the νth compound matrix of $\|a_{ij}\|$. It is the matrix of the transformation induced by $\|a_{ij}\|$ and operating on the determinants defining the Grassmanian coordinates. Then $C_{n,\nu} V_\nu^* = \lim_{r \to 1} [n(r-1)]^{-1} \int_\Omega |G_{ij}| \cdot T(\|H_{ij}\| \, \|G_{ij}\|^{-1}) d\omega$ where $T\|a_{ij}\|$ denotes a trace. (Received April 25, 1950.)

427t. G. M. Wing: *Cesàro averages of the coefficients of Schlicht functions.*

Let $f(z) = \sum_{n=1}^\infty a_n z^n$ $(a_1 = 1)$ be analytic and schlicht in $|z| < 1$. Let $S_N(k) = \sum_{n=0}^{N-1} C_{n+k-1,\, k-1} a_{N-n}$ and $\sigma_N(k) = S_N(k)/C_{N+k+1,k+1}$. If the Bieberbach conjecture,

$|a_n| \leqq n$, holds, then $|\sigma_N(k)| \leqq 1$. By means of an integral representation of $\sigma_N(k)$ numbers $A(k)$ are determined such that $\lim \sup_{N\to\infty} |\sigma_N(k)| \leqq A(k)$, and it is proved that $\lim_{k\to\infty} A(k) = 1$. For this result, only the classical distortion theorem of Koebe is needed. Use is also made of recent results on the area of the map of $|z| \leqq r < 1$ by $f(z)$ (I. E. Basilevitch, Dokladii Akademii Nauk USSR. vol. 65 (1949) pp. 253–255) to show that $A(1) \leqq 1.31$, $A(2) \leqq 1.14$, $A(3) \leqq 1.11$. These compare favorably with the best known estimate of the coefficients themselves, $\lim \sup_{n\to\infty} |a_n|/n \leqq e/2$ (ibid.). (Received May 1, 1850.)

Applied Mathematics

428. M. Z. Krzywoblocki: *On the latest developments in the integral operator method in compressible flow.*

After a brief representation of the principal ideas of Bergman's integral operator method as applied to the solution of partial differential equations of a compressible fluid, the author discusses the main three types of flow equations: subsonic (elliptic), supersonic (hyperbolic), transonic (mixed). Next, the author represents the application of the method to the solution of problems in a three-dimensional space, and particularly to the axially symmetric flow patterns of a compressible fluid. (Received May 4, 1950.)

429. Howard Eves: *A note on Schick's theorem.*

This paper gives a new and simple proof, based upon a construction in the Gauss plane, of Schick's theorem to the effect that the pedal triangle of one of four points with respect to the other three is invariant in form under a direct circular transformation. Some known associated theorems are reestablished in a simple way, and then some probably new theorems are proved. The following may be mentioned as an example of the latter: *Let $A'B'C'$, $A''B''C''$ be the pedal triangles of a pair of isogonal conjugate points M' and M'' of a given triangle ABC. Then the pedal triangle of M' for triangle $A'B'C'$ and the pedal triangle of M'' for triangle $A''B''C''$ are, respectively, directly similar to triangles $A''B''C''$ and $A'B'C'$.* (Received April 12, 1950.)

430t. Howard Eves: *Some consequences of a fundamental construction in the Gauss plane.*

This paper gives a simple construction for finding the point D of the Gauss plane such that $(ABCD) = (A'B'C'D')$, where A, B, C, A', B', C', D' are any given points. As special cases simple constructions are given for finding the limit and double points of the homography $(ABCZ) = (A'B'C'Z')$. Known theorems concerning the situations where $(ABCD)$ is real, harmonic, equianharmonic, or orthocyclic are reestablished as corollaries to the above constructions. The geometry of the situation where $(ABCD)$ is pure imaginary is considered in detail. As an example of a new theorem the following is proved: *Let M be the Miquel point for triangle ABC of the triad A', B', C', and let O' be the circumcenter of triangle $A'B'C'$. Then M and O' are corresponding points under the direct circular transformation set up by A, A'; B, B'; C, C'.* H. F. Sandham has shown that under the same transformation M' and M are corresponding points, where M' is the isogonal conjugate of M for triangle ABC. (Received April 12, 1950.)

431. M. S. Knebelman: *Spaces of relative parallelism.*

Parallelism with respect to a continuous curve joining two, not necessarily distinct, points in an n-dimensional space is defined by means of a nonsingular matrix, the structure matrix of the space. The elements of this matrix are functionals from the space of directed continuous arcs to the space of real numbers. If $\lambda(P)$ is a contravariant vector, the vector obtained by parallel displacement with respect to the arc C_P^Q is given by $\lambda(Q)=M(C_P^Q)\lambda(P)$. For covariant vectors the relation is $\mu(Q)=\mu(P)\cdot M^{-1}(C_P^Q)$. The structure matrix is subject to a number of conditions, Fréchet differentiability being one and $M(C_P^R)=M(C_P^R)\cdot M(C_P^Q)$ another. From this one obtains the affine structure matrix $L(C(\alpha))=dM(C_{\alpha_0}^x)\cdot M^{-1}(C_{x_0}^x)$, L being independent of x_0 and thus defining the local properties of the structure. Covariant differentiation is defined in terms of L, and some fundamental invariants, such as scalar and vector torsions, of the curvature matrix are obtained. The projective structure is defined by the matrix $\mathcal{M}=M/(\det M)^{1/n}$ and from it one obtains the projective connection matrix $\Pi=d\mathcal{M}\cdot\mathcal{M}^{-1}$. Regarding $\mathcal{M}$ as a mapping of the tangent space at P on the tangent space at Q one obtains a generalization of Cartan's results dealing with the holonomy group of the space. (Received April 13, 1950.)

432t. Yeh Mo: *Foci of spherical conics.*

The intersection of a sphere with a quadric cone whose vertex lies at the center of the sphere is called a spherical conic. A real non-circular spherical conic consists of two real disjoint closed parts. Within each part there are two real foci. Using elementary analytic methods, the author proves the following theorem: The sum of the spherical distances of any point of a part of a real non-circular spherical conic from two real foci within this part is constant. The tangent to a real non-circular spherical conic at any point makes equal angles with the tangents to the focal arcs of that point with respect to two real foci within any one part. (Received April 27, 1950.)

433t. Yeh Mo: *Foci of plane curves.*

In this paper the author proves the following theorems: (1) The point F and the line D are respectively a focus and a corresponding directrix of the curve which is the locus of a point P such that $(FP^2)^m=eMP^n$ where MP is the perpendicular distance from P to the line D, and m, n are positive integers such that $n\geqq 2$, $n>m$, and $(n, m)=1$. (2) The two points F' and F are two foci of the curve which is the locus of a point P such that $\pm\lambda F'P\pm\mu FP=a$ where λ, μ are two nonzero constants with absolute values both different from $a/|F'F|$; and if $\lambda F'P+\mu FP=a$ is the real part of the curve, and if θ and ϕ are respectively the angles which the tangent line to this part at P makes with lines PF' and PF, then $\lambda\cos\theta+\mu\cos\phi=0$. (3) The conformal transform of a focus of a given curve is, in general, a focus of the conformal transform of the given curve. (4) If a curve has at least one axis of symmetry and at least one focus, then, with respect to each axis, there is at least one focus situated on it and one of the corresponding directrices is perpendicular to it; and any remaining foci are situated symmetrically with respect to this axis. (Received April 27, 1950.)

434t. A. R. Schweitzer: *On the derivation of the regressive product in Grassmann's geometrical calculus.*

In his *Ausdehnungslehre* of 1844 Grassmann derives the regressive product (eingewandtes Produkt) by means of his calculus of systems or spaces (compare Gebietslehre: *Gesammelte Werke*, vol. 1, part 2, pp. 16–23; see also R. Grassmann, *Die Ausdehnungslehre*, Stettin, 1891, pp. 13–25, 50) including concepts applicable to the algebra of logic such as "gemeinschaftliches System," "nächstumfassendes System," "Unterordnung," "Ueberordnung," and so on. In this paper regressive products associated with an n-simplex ($n=2, 3$) are derived with the aid of elementary properties of determinants. Reference is made to Grassmann, *Gesammelte Werke*, vol. 1, part 2, p. 43, paragraphs 62, 63, and p. 400; F. Caspary, J. Reine Angew. Math. vol. 92 (1882) pp. 123–144. (Received May 3, 1950.)

435*t*. A. R. Schweitzer: *A metric generalization of Grassmann's geometrical calculus.*

The relation between n-simplexes, $\alpha_1\alpha_2 \cdots \alpha_{n+1}=k\cdot\beta_1\beta_2 \cdots \beta_{n+1}$ $(n=1, 2, 3, \cdots)$ fundamental for Grassmann's calculus (Amer. J. Math. vol. 35 (1913) pp. 37–56) is generalized by the author to the relation $\alpha_1\alpha_2 \cdots \alpha_{n+1}\equiv k\cdot\beta_1\beta_2 \cdots \beta_{n+1}$ (mod $\xi_1\xi_2 \cdots \xi_{n+1}$) with meaning: There exists the n-simplex $\xi_1\xi_2 \cdots \xi_{n+1}$ such that $\alpha_1\alpha_2 \cdots \alpha_{n+1}\equiv\xi_1\xi_2 \cdots \xi_{n+1}$ and $\xi_1\xi_2 \cdots \xi_{n+1}=k\cdot\beta_1\beta_2 \cdots \beta_{n+1}$. Then Grassmann's relation $\alpha_1\alpha_2 \cdots \alpha_{n+1}=k\cdot\beta_1\beta_2 \cdots \beta_{n+1}$ is replaced by taking $\alpha_1\alpha_2 \cdots \alpha_{n+1}$ for the modulus and the relation of geometric congruence $\alpha_1\alpha_2 \cdots \alpha_{n+1}\equiv\beta_1\beta_2 \cdots \beta_{n+1}$ is replaced by taking $k=1$ and $\beta_1\beta_2 \cdots \beta_{n+1}$ for the modulus. Metrical properties corresponding to Hilbert's theory in his *Grundlagen der Geometrie* (7th ed., Leipzig and Berlin, 1930, pp. 11–14) are developed postulationally in terms of an undefined relation of congruence between n-simplexes. Relatively to the author's system in terms of his relation K_n (Amer. J. Math. vol. 31 (1909) pp. 365–410) congruence between n-simplexes is expressed in terms of an undefined relation E_n. Reference is made to the author's note, Bull. Amer. Math. Soc. vol. 15 (1908) pp. 79–81. (Received May 3, 1950.)

436*t*. S. K. B. Stein: *Projection of 2 polygons.*

Let P and P' be two simple polygons in the plane. We prove the following theorem. It is possible to move P' rigidly onto P in such a manner that the area of intersection of the two interiors is greater than zero and that the directed projection in every direction of the portion of P' interior to P is zero. Indeed this can be done in an infinity of manners. The theorem holds for all dimensions and for almost all curves. (Received May 3, 1950.)

437*t*. Fred Supnick: *On the perspective deformation of polyhedra.* II. *Solution of the convexity problem.*

In the first part of this paper the question raised by S. S. Cairns whether every geodesic triangulation of the sphere is a central projection of a convex polyhedron is answered. (Geodesic triangulation means edges $<\pi$, and faces $<$hemisphere in area.) It is shown that certain geodesic triangulations of the sphere of octahedral type cannot be central projections of convex polyhedra. In the second part of this paper the problems of determining the necessary and sufficient conditions for any geodesic triangulation of the sphere to be a central projection of a convex polyhedron and of determining an algorithm for finding the solutions are solved. This is done by reducing the characterization problem to that of solving systems of linear inequalities, thus permitting the use of known results. (Received May 29, 1950.)

Logic and Foundations

438*t*. F. B. Jones: *An elementary two color problem.* II.

Consider a unit square region in the number plane with (0, 0) and (1, 1) as the ends of a diagonal. It is possible to color certain of the points of this region red and certain others green such that (1) no point on $y=x$ is colored, (2) all other points are colored either red or green, and (3) if H_x is any uncountable set of straight line arcs spanning the square parallel to the y-axis and H_y is the image of H_x with respect to $y=x$, then the intersection of H_x with H_y contains both a red and a green point. This settles the question raised in a previous paper (Bull. Amer. Math. Soc. Abstract 56-2-212) and may be used to establish a characterization of the first transfinite cardinal based on the decomposition of the pairs of a set into sets. (Received May 5, 1950.)

439*t*. Hao Wang: *The non-finitizability of impredicative principles.*

Let A be an ordinary second-order functional calculus with number theory as its theory of individuals and B be the system obtained from it by omitting classes which cannot be defined without using bound class variables. The principles of class formation in both A and B embody infinitely many special cases. We know that the principle of B can be replaced by a finite number. In this paper it is shown that this is impossible for that of A. An enumeration of all the classes required to exist by the axioms of B can actually be expressed in A. So that by the diagonal method, there exists a class of A not enumerated and therefore some case of the impredicative principle of A which is independent of the axioms of B. Similar things can be proved for any subsystem B' of A which contains finitely many axioms. Hence A is non-finitizable. It also follows that there exists a sequence of classes each containing its predecessors such that each of them can be proved to exist in B' but their union cannot. Similarly impredicative principles in functional calculi of higher orders up to ω are non-finitizable. (Received May 22, 1950.)

440*t*. Hao Wang: *Translatability and relative consistency.*

With an arithmetization of syntax we can express the consistency of a system S by an arithmetic proposition Con (S). Two systems S and S' are said to be of similar strength if Con (S) is derivable from Con (S') in number theory and vice versa. Therefore, for every system S, if Con (S') is provable in S and S contains number theory then S and S' are not of similar strength. S is said to be translatable into S' if there is a recursive function mapping propositions and theorems of S respectively into those of S'. It is proved that if S and S' are translatable into each other, then they are of similar strength. All consistent systems with decision procedures are of similar strength because their consistency can all be proved in number theory. However, the consistency of quantification theory can be proved in number theory too, although it is known to be an undecidable system. It is further shown by extending a result of Hilbert-Bernays that each system S is translatable into the system obtained from number theory by adding Con (S) as a new axiom. (Received May 22, 1950.)

441*t*. Hao Wang: *On the relative strength of certain ordinary systems.*

L is an ordinary set theory with finitely many axioms, being adequate for developing the second-order functional calculus founded on integers. L'' is related to L as an nth-order functional calculus is to an $(n-1)$th, and L' is like L'' except that variables of the highest type are not allowed in defining classes of lower types. $L\#$ is L plus the proposition Con (L) as a new axiom. The following things are proved. $L\#$ and L' are of similar strength. Although a Tarski truth set for $L\#$ is obtainable in L', Con $(L\#)$ is not provable in L'. If we use in L' and L the same definition for the class of natural numbers, then the proposition that every class of natural numbers has a least member is independent of the axioms of L' and Con (L) is an unprovable proposition of L' of the form $(k)\phi k$ for which we can prove $\phi 0$ and $\phi n \supset \phi n'$ in L'. By choosing suitable definitions, Con (L) becomes a theorem of L'. Con (L) and Con $(L\#)$ are theorems of L''. It follows that Con (L') is also a theorem of L''. (Received May 22, 1950.)

Statistics and Probability

442*t*. Murray Rosenblatt: *On distributions of certain Wiener functionals.*

Let $x_1(t)$, $x_2(t)$ be elements of two independent Wiener spaces. $V(x, y)$ is assumed to be non-negative and sufficiently regular except on a rectifiable curve of discontinuity C. The Wiener functional $\int_0^t V(x(\tau), y(\tau))d\tau$ is investigated. The principal result is the following: If $\sigma(\alpha; t)$ is the distribution of $\int_0^t V(x(\tau), y(\tau))d\tau$, then $\int\int_0^\infty \exp(-u\alpha - \delta t)d_\alpha\sigma(\alpha; t) = \int\int_{-\infty}^\infty \psi(x, y)dxdy$ where $\psi(x, y)$ is the fundamental solution of $2^{-1}\nabla^2\psi - (s+uV(x, y))\psi = 0$, $(x, y) \neq (0, 0)$, $\lim_{\epsilon\to 0}\int_0^{2\pi}\partial\psi(\epsilon\cos\theta, \epsilon\sin\theta)/\partial t d\theta = -2$. The methods used are similar to those of M. Kac (*On the distributions of certain Wiener functionals*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 1–13). A small discussion indicates that the results of the paper can be immediately extended to the product space of n Wiener spaces in like manner. (Received April 25, 1950.)

443*t*. Abraham Wald and Jacob Wolfowitz: *Characterization of the minimal complete class of decision functions when the number of distributions and decisions is finite.*

Let m be the number of cumulative distribution functions $\{f_i\}$ (of any kind whatever) and ξ the generic designation of an a priori probability distribution on $\{f_i\}$. A Bayes solution with respect to the sequence $\xi_1, \cdots, \xi_h$ is defined inductively as follows: When $h=1$ it is a Bayes solution with respect to ξ_1. For $h>1$ it is a Bayes solution with respect to ξ_h among all decision functions which are Bayes solutions with respect to the sequence $\xi_1, \cdots, \xi_{h-1}$. Let ξ_{ij} be the probability of f_j according to ξ_i. The authors prove the following: In order that a decision function be admissible it is necessary and sufficient that it be a Bayes solution with respect to a sequence of h $(\leqq m)$ a priori distribution functions $(\xi_1, \cdots, \xi_h)$, such that the matrix $\{\xi_{ij}\}$, $i=1, \cdots, h; j=1, \cdots, m$, has the following properties: (a) for any j there exists an i such that $\xi_{ij}>0$, (b) The matrix $\{\xi_{ij}\}$, $i=1, \cdots, (h-1); j=1, \cdots, m$, does not possess property (a). (Received April 24, 1950.)

Topology

444*t*. S. K. B. Stein: *Convex maps.*

The following theorem is proved: Let M be a regular map in the plane with the two additional properties that each country is simply connected and that the union of two touching countries is simply connected. Then it is possible to deform this map

M into a map in which all the countries are convex still preserving all incidence relations. The proof is inductive and tells how to construct the convex map. (Received May 3, 1950.)

445*t*. A. D. Wallace: *Cohomology groups of collections.*

Fix a space X and an abelian group G. Let $\mathfrak{X} \supset \mathfrak{A} \supset \mathfrak{B}$ be three collections of subsets of X. We define cohomology groups $H^p(\mathfrak{X}, \mathfrak{A}), \cdots$ with G as coefficient group. Each of these groups depends on X as well as the displayed arguments. The standard algebraic theorems hold as well as an exact sequence theorem for $\cdots \rightarrow H^p(\mathfrak{X}, \mathfrak{A}) \rightarrow H^p(\mathfrak{X}, \mathfrak{B}) \rightarrow H^p(\mathfrak{A}, \mathfrak{B}) \rightarrow \cdots$. It is possible also to define groups $H^p(X, \mathfrak{A}), \cdots$. For these there are comparable results with an exact sequence $\cdots \rightarrow H^p(X, \mathfrak{A}) \rightarrow H^p(X) \rightarrow H^p(\mathfrak{A}) \rightarrow \cdots$, the last group depending on X as well as on the displayed argument. This last case is interesting when the collection A is $\{f^{-1}(y) \mid y \in Y\}$, where $f: X \rightarrow Y$ is a map. (Received March 20, 1950.)

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