Cardinal algebras. By A. Tarski. New York, Oxford University Press, 1949. 12+326 pp. $10.00.

Professor Tarski's latest book is built around an axiomatic study of cardinal numbers (including zero) under finite and countable addition. Besides the commutative and associative laws in their general (countably infinite) form, these operations satisfy a Refinement Postulate ($a+b=\sum c_i$ implies the existence of a_i, b_i such that $\sum a_i=a$, $\sum b_i=b$, and $a_i+b_i=c_i$), and a Remainder Postulate (if $a_1=b_1+b_2+b_3+\cdots+b_n+a_{n+1}$ for all finite n, then c exists such that $a_1=\sum_{i<\infty}b_i+c$).

A "cardinal algebra" is a system satisfying these postulates. Among cardinal algebras, we may include: Boolean σ-algebras; the non-negative elements of any complete l-group, if $+\infty$ is adjoined; and relation numbers under cardinal addition. On the other hand, since $3|2\cdot 2\cdot 2\cdots$ (countable multiplicands) without dividing any factor, cardinal numbers do *not* form a cardinal algebra under multiplication. The first part of the book is devoted to the formal properties of such cardinal algebras. After defining $a\leqq b$ to mean that $a+x=b$ has a solution, and $n\cdot a$ as $a+\cdots+a$ (n summands), the following typical results are proved: $a\leqq b$ and $b\leqq a$ imply $a=b$ (Schroeder-Bernstein Theorem); if $a+n\cdot c\leqq b+n\cdot c$, then $a+c\leqq b+c$—whence $n\cdot a=n\cdot b$ implies $a=b$; if $a_i\leqq b_j$ for all i, j of a finite or countable set, then c exists such that $a_i\leqq c\leqq b_j$ for all i, j. Again, if $a\cap b$ exists in the sense of lattice theory, then $a\cup b$ exists and $a+b=(a\cap b)+(a\cup b)$; under other existential hypotheses, $a\cap\sum_{i<\infty}b_i=\sum_{i<\infty}(a\cap b_i)$ and $a+\bigcap_{i<\infty}b_i=\bigcap_{i<\infty}(a+b_i)$. In fact, since $\sum_{i<\infty}a_i=\text{Sup}\ \{\sum_{i<n}a_i\}$, one can define countable sums in terms of binary sums; but the postulates in terms of binary sums alone would be more awkward.

Since the proof of the postulates for addition of cardinal numbers does not involve the Axiom of Choice for uncountable sets, deductions from the postulates are also independent of this Axiom, provided no further appeal to it is made. Hence the book is an important axiomatic contribution to the foundations of set theory.

On the other hand, some readers may find it difficult to accept the author's use of "the class of all sets" (p. 215). When it comes to "generalized cardinal algebras," in which closure under the operations is not assumed, there are further logical difficulties, noted in the addenda.

However, many noteworthy theorems about generalized cardinal algebras are proved. Thus every generalized cardinal algebra can be extended to a cardinal algebra; again the ideals of any cardinal algebra form, under set-union, a cardinal algebra in which $a+a=a$.

Another interesting feature of the book is the construction, given a cardinal algebra $\mathfrak{A}$ and a group G of "partial automorphisms" of $\mathfrak{A}$, of a "refinement algebra," very similar to a cardinal algebra, in which elements of $\mathfrak{A}$ equivalent under G (directly or by decomposition) are identified. In this direction, various abstract analogs of theorems relating to the existence of measure, and to the Banach-Tarski paradox, are proved. (Let $\mathfrak{A}$ consist of Borel sets, and let G be the group of isometries of space.) The algebra of cardinal numbers under addition is deduced from the algebra of sets by the same construction.

Another section deals with the relation between "cardinal algebras" and other types of algebraic systems—especially semigroups and distributive lattices. Finally, an appendix discusses "cardinal products of isomorphism types"—that is, with the direct factorization of general algebraic systems with a binary operation and a zero. The results here are related to the monograph *Direct decompositions of finite algebraic systems*, by the author and Bjarni Jónsson.

It seems certain, to the reviewer, that the postulates and ingenious deductions of Professor Tarski will permanently enrich modern algebra—at the same time that they show once more the value of considering infinitary operations. On the other hand, it seems less clear that the particular combinations of conditions labelled by the author "cardinal algebra," "generalized cardinal algebra," and "refinement algebra" will survive without modification. At all events, the author is to be congratulated for penetrating deep into new and heretofore uncharted mathematical territory; the book is a "must" for everyone seriously interested in modern algebra or set theory.

GARRETT BIRKHOFF

Theory of functions. By J. F. Ritt. Rev. ed. New York, Kings Crown Press, 1947. 10+181 pp. $3.00.

A student's introduction to the theory of functions has certain aspects in common with a youth's emergence into the adult world. Here he meets directness and subtlety, power and simplicity, beauty and rigor in quantity and proportion not previously experienced. A mathematician who undertakes to mediate this introduction by means of a book must be in control of these properties. Beyond that he needs the tact which prevents him from overwhelming the student with needless detail or deluding him with insufficient mathematical content. In the book under review these conditions have been met with ease. The teacher in search of a textbook will find, agreeably,