CLASSIFICATION OF 2-MANIFOLDS WITH SINGULAR POINTS

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1. **Introduction.** By a *closed 2-manifold*, or simply a 2-manifold, we mean here a two-dimensional connected finite simplicial complex every point of which has a neighborhood homeomorphic to a circular disk, that is, the interior of a circle. If there is a point not having the latter property, we call it a *singular point* of it.

In this paper, we shall give a complete classification (§2) and some properties (§3) of 2-manifolds with a single singular point. Obviously one may get one such geometrical figure by identifying certain points of a 2-manifold or several ones together. Conversely, we shall show that every such figure may be obtained in such a manner (see (2.3)).

In §4, we generalize these results to 2-manifolds with any number of singularities.

2. **The classification.** The classification lies in the investigation of the nature of the neighborhoods of the singular point. Let[1] $\cdot\mathfrak{M}^2$ have its singular point at 0. We first establish the following lemma.

LEMMA (2.1). *Any neighborhood of* 0 *is homeomorphic to a finite number, say* p, *of circular disks with all their centers identified. We call it a* p*-bundle and call* 0 *its center*; *and the boundaries of these* p *disks are simply said to be the boundary of the* p*-bundle.*

PROOF. Consider a simplicial subdivision $\mathfrak{K}^2$ of $\cdot\mathfrak{M}^2$. We first note that 0 must be a vertex of $\mathfrak{K}^2$. For, if 0 were an inner point of a 2-simplex, then 0 could not belong to any other simplex and hence would be an ordinary point; and if it were an inner point of a 1-simplex, then all the points of this 1-simplex would be singular points for the same reason. It is also evident that 0 cannot be a vertex of a 1-simplex unless it is a vertex of a 2-simplex.

Let 0 be a vertex of a 2-simplex $\mathfrak{N}^2$. Then there must be many 2-simplexes including $\mathfrak{N}^2$ forming a circular disk surrounding 0, as otherwise there would be two edges of singularities. Besides, 0 must be a vertex of another 2-simplex, say $'\mathfrak{N}^2$, by noting that 0 is a singular point. Hence we get another circular disk consisting of

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[1] We use the notation $\cdot\mathfrak{M}$ instead of a dot directly over $\mathfrak{M}$ for typographical convenience.

2-simplexes surrounding 0. In such a way, finally, we obtain p such circular disks (since $\cdot\mathfrak{M}^2$ is finite), and the lemma is proved.

We say $\cdot\mathfrak{M}^2$ has a singularity of *order* p at 0. $\cdot\mathfrak{M}^2-(0)$ in general is not connected and consists of n components $\mathfrak{L}_i^2$ $(i=1, \cdots, n)$. We name $\mathfrak{L}_i^2+(0)$ a *sheet* of $\cdot\mathfrak{M}^2$. Then $\cdot\mathfrak{M}^2$ is the sum of these sheets with the identification of 0. Moreover, each circular disk of 0 belongs wholly to one and only one sheet. Let us denote these sheets by $\cdot\mathfrak{M}_i^2=\mathfrak{L}_i^2+(0)$ $(i=1, \cdots, n)$, then

$$\cdot\mathfrak{M}^2 = \sum_{i=1}^{n} \cdot\mathfrak{M}_i^2, \tag{1}$$

$$p = \sum_{i=1}^{n} p_i, \tag{2}$$

where p_i is the order of 0 in $\cdot\mathfrak{M}_i^2$ ($p_i=1$ in case $\cdot\mathfrak{M}_i^2$ itself is a 2-manifold).

Hence it is sufficient for us to consider $\cdot\mathfrak{M}_i^2$ separately. But the structure of $\cdot\mathfrak{M}_i^2$ is quite clear; for, if we take away the p_i circular disks surrounding 0, the rest is a bounded 2-manifold with p_i holes, the classification of which is already well known.[2] Therefore we get:

THEOREM (2.2). *Any 2-manifold with one singularity may be decomposed into the form* (1), *where* $\cdot\mathfrak{M}_i^2$ *are sheets, that is, bounded 2-manifolds with* p_i *holes adjoined with a* p_i*-bundle having its boundary identified with the boundaries of these holes; and all the centers of these bundles are to be identified.*

We may, however, consider the p-bundle separately as p circular disks and identify each of their circumferences with each of the boundaries of the holes. Thus we get 2-manifolds, and then identify the p centers. Hence we obtain:

THEOREM (2.3). *Every 2-manifold with one singular point is the sum of a finite number of 2-manifolds each with some points identified all together.*

The preceding two theorems lead us to obtain a 2-manifold with a single singular point from bounded and closed 2-manifolds respectively. In practice the latter is much more useful than the former.

3. Simple properties. The most evident property of 2-manifolds with one singularity is contained in the following theorem.

THEOREM (3.1). $\cdot\mathfrak{M}^2$ *and* $\cdot\mathfrak{M}^{*2}$ *are homeomorphic if and only if they*

[2] Cf., for example, Seifert-Threlfall, *Lehrbuch der Topologie*, 1934, §40.

have the same structures, hence necessarily, after suitably arranging their sheets,

$$(p_i) = (p_i^*) \tag{3}$$

(consequently $n=n^*$, $p=p^*$).

But we should notice that (3) is not a sufficient condition for $\cdot\mathfrak{M}^2$ and $\cdot\mathfrak{M}^{*2}$ to be homeomorphic, since the corresponding 2-manifolds constituting them may not be homeomorphic. We also note that the property of orientability is preserved:

THEOREM (3.2). *A 2-manifold with one singularity is orientable if and only if all the 2-manifolds constituting it are orientable.*

Now we come to prove the important theorem:

THEOREM (3.3). *The (integral) homology group $\mathfrak{h}$ of any dimension of a 2-manifold with one singularity is the direct sum of those of its sheets. The homology group of a sheet $\cdot\mathfrak{M}_i^2$ is the same as that of the 2-manifold $\mathfrak{M}_i^2$ except for dimension 1, where $\mathfrak{M}_i^2$ is the 2-manifold which is the same as $\cdot\mathfrak{M}_i^2$ but without those p_i points to be identified in $\cdot\mathfrak{M}_i^2$ being identified in $\mathfrak{M}_i^2$. For dimension 1, if $\cdot\mathfrak{M}_i^2$ and $\mathfrak{M}_i^2$ have the homology groups $\cdot\mathfrak{h}_i^1$ and $\mathfrak{h}_i^1$ respectively, then*

$$\cdot\mathfrak{h}_i^1 = \mathfrak{h}_i^1 + (p_i - 1)\mathfrak{g} \tag{4}$$

where $k\mathfrak{g}$ represents the direct sum of k free cyclic groups $\mathfrak{g}$.

PROOF. Let us consider the last statement only as the others are obvious. In case $p_i=1$, (4) is trivial.

Suppose $0_1, \cdots, 0_{p_i}$ are the points of $\mathfrak{M}_i^2$ which are to be identified in $\cdot\mathfrak{M}_i^2$. Take a sufficiently small simplicial subdivision of $\mathfrak{M}_i^2$ such that all these points are vertices, and it *induces* a simplicial subdivision on $\cdot\mathfrak{M}_i^2$. Then any 1-cycle Z^1 on $\cdot\mathfrak{M}_i^2$ is either a 1-cycle on $\mathfrak{M}_i^2$ or a broken line joining two 0's, say 0_j and 0_k. In the latter case Z^1 is neither homologous nor division homologous to zero. For if $mZ^1\sim 0$ $(m\neq 0)$, then there would exist a 2-chain C^2 whose boundary $\partial C^2=mZ^1$ and thence

$$\partial\partial C^2 = m\partial Z^1 = m(\pm 0_j \pm 0_k) \neq 0,$$

which is a contradiction. Hence Z^1 as an element of $\cdot\mathfrak{h}_i^1$ generates a free cyclic group.

We then join 0_1 to the other 0's and get p_i-1 broken lines, each of which generates a free cyclic group since no two of them are homologous or division homologous to zero by the same reason.

Any broken line between 0_j and 0_k may be replaced by an algebraic sum of two broken lines starting from 0_1 and ending in 0_j, 0_k respectively, and a 1-cycle through these three points on $\mathfrak{M}_i^2$ will be discussed below.

A 1-cycle on $\mathfrak{M}_i^2$ not passing through any 0 is not influenced in constructing $\cdot\mathfrak{M}_i^2$, while one passing any 0, say 0_k, may be modified by omitting the two edges through it and adding the third edge of the 2-simplex that is incident with 0_k as well as the two edges through 0_k (see (2.1)). Hence the homology classes made by the 1-cycles on $\mathfrak{M}_i^2$ are unchanged on $\cdot\mathfrak{M}_i^2$. Thus (3.3) is established.

COROLLARY (3.4).

$$\cdot\mathfrak{h}^1 = \sum_{i=1}^{n} \cdot\mathfrak{h}_i^1 + (p - n)\mathfrak{g}.$$

This shows us a method for constructing a 2-complex with any preassigned Betti number whatever.

Analogously, we have the following theorem.

THEOREM (3.5). *The fundamental group of a 2-manifold with one singularity is the free product of those of its sheets, and the fundamental group* $\cdot\mathfrak{f}_i$ *of a sheet* $\cdot\mathfrak{M}_i^2$ *is the free product of the fundamental group* $\mathfrak{f}_i$ *of* $\mathfrak{M}_i^2$ *and* $p_i - 1$ *free cyclic groups.*[3]

4. **Generalizations.** By the finiteness of a 2-manifold it is evident that the number of singular points on it, if any, is finite. Let ${}'0, {}''0, \cdots, {}^{(m)}0$ be the only singular points on a 2-manifold $\cdot\mathfrak{M}^2$.

Lemma (2.1) is valid for each ${}^{(j)}0$ $(j=1, \cdots, m)$ and we may speak of the *order* at ${}^{(j)}0$, say ${}^{(j)}p$. The generalized Theorems (2.2) and (2.3) have their natural forms, the latter of which we state as follows.

THEOREM (4.1). $\cdot\mathfrak{M}^2$ *is the sum of a finite number of 2-manifolds* $\cdot\mathfrak{M}_i^2$ $(i=1, \cdots, n)$ *on which* ${}^{(j)}p_i$ *points are identified to the point* ${}^{(j)}0$ $(j=1, \cdots, m)$.

Moreover,

$$(5) \qquad p_i = \sum_{j=1}^{m} {}^{(j)}p_i, \qquad {}^{(j)}p = \sum_{i=1}^{n} {}^{(j)}p_i, \qquad p = \sum_{i=1}^{n} p_i = \sum_{j=1}^{m} {}^{(j)}p,$$

where p_i and p are defined as the *orders* of $\cdot\mathfrak{M}_i^2$ and $\cdot\mathfrak{M}^2$ respectively.

[3] We may first prove (3.5) and so (3.3) follows immediately by a relation between the homology group and the fundamental group, cf. ibid. p. 173.

Theorem (3.1) now takes the form:

THEOREM (4.2). *$\cdot\mathfrak{M}^2$ and $\cdot\mathfrak{M}^{*2}$ are homeomorphic if and only if they have the same structures, hence necessarily, after suitably arranging their sheets and the order of their singular points,*

$$(6) \qquad ({}^{(i)}p_i) = ({}^{(i)}p_i^*)$$

(consequently $m = m^*$, $n = n^*$, $p = p^*$).

Theorem (3.2) is true in its original form. We establish now the following theorem.

THEOREM (4.3). *The homology group $\cdot\mathfrak{h}_i^1$ of $\cdot\mathfrak{M}_i^2$ is given by*

$$(7) \qquad \cdot\mathfrak{h}_i^1 = \mathfrak{h}_i^1 + (p_i - m)\mathfrak{g},$$

where $\mathfrak{h}_i^1$ is the homology group of $\mathfrak{M}_i^2$, the 2-manifold which is the same as $\cdot\mathfrak{M}_i^2$ but without any points being identified.

PROOF. For each j we consider the ${}^{(j)}p_i$ points to be identified to ${}^{(j)}0$ as in the proof of (3.3), that is, join broken lines from one of them to all the others. They are 1-cycles on $\cdot\mathfrak{M}_i^2$, each of which generates a free cyclic group in $\cdot\mathfrak{h}_i^1$ and any two of which are neither homologous nor division homologous to zero on $\cdot\mathfrak{M}_i^2$. Any 1-cycle on $\cdot\mathfrak{M}_i^2$ may be replaced by an algebraic sum of these broken lines and a 1-cycle on $\mathfrak{M}_i^2$. Hence by the same reason as in (3.3), from the first equation of (5) we have (7), and thus the theorem is proved.

In order to get the homology group $\cdot\mathfrak{h}^1$ of $\cdot\mathfrak{M}^2$, we again introduce a lemma which may be readily proved.

LEMMA (4.4). *If $\mathfrak{K}$ is a connected simplicial complex of any dimension and $P_1, \cdots, P_k$ are k arbitrary distinct points on it, and $\mathfrak{K}^*$ is the complex made by $\mathfrak{K}$ in addition with the 1-simplexes (P_1P_2), (P_1P_3), $\cdots$ (P_1P_k) (not in $\mathfrak{K}$); then*

$$(8) \qquad \mathfrak{h}^{*1} = \mathfrak{h}^1 + (k - 1)\mathfrak{g},$$

or

$$(8)' \qquad \mathfrak{h}^1 = \mathfrak{h}^{*1} - (k - 1)\mathfrak{g},$$

where $\mathfrak{h}^1$ and $\mathfrak{h}^{*1}$ are homology groups of $\mathfrak{K}$ and $\mathfrak{K}^*$ respectively and the minus sign indicates a difference group.

Eventually, we have the following theorem.

THEOREM (4.5). *The homology group $\cdot\mathfrak{h}^1$ of $\cdot\mathfrak{M}^2$ may be written as*

$$(9) \qquad \cdot\mathfrak{h}^1 = \sum_{i=1}^{n} \mathfrak{h}_i^1 + (p - n - m + 1)\mathfrak{g}.$$

PROOF. In constructing 1-simplexes $('0''0), \cdots, ('0^{(m)}0)$ (not belonging to $\cdot\mathfrak{M}^2$), we get $\cdot\mathfrak{M}^{*2}, \cdot\mathfrak{M}_1^{*2}, \cdots$ as in the lemma. By (7) and (8), we have

$$(10) \qquad \cdot\mathfrak{h}_i^{*1} = \mathfrak{h}_i^1 + (p_i - 1)\mathfrak{g},$$

where $\cdot\mathfrak{h}_i^{*1}$ is the homology group of $\cdot\mathfrak{M}_i^{*2}$. The newly constructed simplexes form a connected 1-complex whose 1-dimensional homology group contains the identity only. Therefore from a famous theorem (cf. Seifert-Threfall, p. 179), by (5) we get

$$(11) \qquad \cdot\mathfrak{h}^{*1} = \textstyle\sum_{i=1}^{n} \mathfrak{h}_i^1 + (p - n)\mathfrak{g},$$

where $\cdot\mathfrak{h}^{*1}$ is the homology group of $\cdot\mathfrak{M}^{*2}$. Therefore (9) is finally established in virtue of (11) and (8)'.

Theorem (3.5) may be extended analogously.

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A NOTE ON EQUICONTINUITY

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During a recent seminar discussion of his paper *Transitivity and equicontinuity* [1],[1] W. H. Gottschalk proposed the following question:

"Is the center of every algebraically transitive group of homeomorphisms on a compact metric space equicontinuous?"

An affirmative answer to the above question is given in this note.

1. **Definitions.** We let X and Y be compact metric spaces and let d be the metric for Y.

A set F of functions on X into X is *algebraically transitive* if corresponding to each pair p and q of points in X there exists $f \in F$ such that $f(p) = q$.

A sequence $[g_n]$ of functions on X into Y converges to a function

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[1] Numbers in brackets refer to the bibliography at the end of the paper.