L-S-HOMOTOPY CLASSES ON THE TOPOLOGICAL IMAGE OF A PROJECTIVE PLANE

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1. **Introduction.** Models for the L-S-(locally simple) homotopy classes of closed p-curves (p = parameterized) on any 2-manifold S have been announced in Morse [1].[1] Proofs have been given only for the case in which S is orientable. The present paper will treat the case in which S is the top. (topological) image of a projective plane. The proofs in the case of a general non-orientable surface can be given by an appropriate modification of methods of Morse [1] and of the present paper.

Recall that one writes $f \approx 0$ when f is a closed p-curve homotop. to zero. Deferring technical definitions until later sections, we can state the principal theorem as follows.

THEOREM 1.1. *Let h be a simple closed p-curve on the top. image S of a projective plane with h not ≈ 0 on S. Let $h^{(n)}$ $(n>0)$ be a closed p-curve on S which traces h n times. Any L-S-closed p-curve f on S is in the L-S-homotopy class of $h^{(1)}$ or $h^{(3)}$ if h not ≈ 0, and of $h^{(2)}$ or $h^{(4)}$ if $h \approx 0$. No two of the p-curves $h^{(1)}, h^{(2)}, h^{(3)}, h^{(4)}$ are in the same L-S-homotopy class.*

For theorems on regular closed curves in the plane see Whitney, and H. Hopf. For L-S-closed curves in the plane see Morse [2] and Morse and Heins [1]. For a use of L-S-curves in studying deformation classes of meromorphic functions see Morse and Heins [2].

2. **L-S-curves and deformations.** Let C represent the unit circle on which $|z| = 1$ in the plane of the complex variable $z = u+iv$. With $z = e^{i\theta}$ on C, we assign C the *sense* of increasing θ. Let S be an arbitrary 2-manifold. A *closed p-curve* on S is a continuous mapping f of C into S such that the image of z in C is a point $f(z)$ in S. Two p-curves f_1 and f_2 are regarded as the *same* if and only if

$$f_1(z) = f_2(z)$$

for every z in C. The union of the points $f(z)$ in S as z ranges over C is called the *carrier* of f. The simplest case arises when the points $f(z)$ are in 1–1 correspondence with their antecedents z in C, and in this case f is termed *simple.*

Let f be a continuous mapping of C into S. Let λ be any sense

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[1] Numbers in brackets refer to the references cited at the end of the paper.

preserving top. (that is, 1–1 and continuous) mapping of C onto C with $\lambda(z)$ the image in S of z in C. Then f and $f\lambda$ are termed *equivalent* mappings of C into S or equivalent p-curves on S. Here $f\lambda$ symbolizes the function whose value at z is $f[\lambda(z)]$. In earlier papers the class of mappings equivalent to a given mapping has been termed a *curve*, as distinguished from a p-curve. We shall here find it simpler to rely on p-curves and make use of Lemma 2.1, according to which any two equivalent closed p-curves are in the same L-S-homotopy class.

A closed p-curve f on S will be termed L-S if there is a positive constant e so small that the mapping under f of any arc of C with length less than e is top. Hence there exists a constant $e_1>0$ so small that any subarc of f whose carrier has a diameter less than e_1 is simple. Such a constant e_1 is called a *norm of local simplicity* of f.

The L-S-homotopy class $[f]$. Let J be the interval $0\leqq t\leqq 1$, t the "time." A *deformation* of f on S is a continuous mapping D of $C\times J$ into S such that the image of a point (z, t) in $C\times J$ is a point $D(z, t)$ in S, with

$$D(z, 0) = f(z) \qquad (z \in C)$$

initially (that is, when $t=0$). For t fixed in J, D defines a mapping[2] $D(\cdot, t)$ of C into S, and thus a p-curve f^t termed the *deform* of f at the time t. We say that f is deformed through the family f^t into f^1. If the deforms f^t, $0\leqq t\leqq 1$, possess a common norm of local simplicity, D will be said to be L-S. The class of p-curves f^1 into which f can be L-S-deformed on S is termed the L-S-homotopy class $[f]$ of f on S. By virtue of a proof similar to that of Lemma 28.1 of Morse [2] we can affirm the following.

LEMMA 2.1. *Any two equivalent closed p-curves on S are in the same L-S-homotopy class on S.*

A first objective of this paper is the proof of Theorem 1.1. Models for the L-S-homotopy class $[f]$ on the top. image S of a projective plane will be determined as indicated. In case the given p-curve $f\approx 0$, the M-order of f, as defined in Morse [1, §4], is 1, or 2 mod 2, according as $[f]=[h^{(2)}]$ or $[f]=[h^{(4)}]$. Here M is the top. sphere covering S.

In case the given p-curve f not ≈ 0, a new *S-difference order* $d_S(f)$ is defined in §7 and $[f]=[h^{(1)}]$ or $[h^{(3)}]$ according as

$$d_S(f) = 1 \text{ or } 3 \qquad (\text{mod } 4).$$

The value of $d_S(f)$ will be shown to be independent of f in its L-S-

[2] For fixed t, $D(\cdot, t)$ symbolizes the function whose value at z in C is $D(z, t)$.

homotopy class, and to be a top. invariant in the following sense. If S' is a top. image of S with f' on S' the top. image of f on S, then

$$d_S(f) = d_{S'}(f') \qquad (\text{mod } 4).$$

An important special result in the case f not ≈ 0 is that $d_S(f) = 1 \bmod 4$ if and only if $[f]$ contains a simple closed p-curve.

3. **The 2-sphere M covering Π.** We shall begin with a special model Π of a projective plane obtained by identifying diametrically opposite points

$$x = (x_1, x_2, x_3), \quad - x = (- x_1, - x_2, - x_3)$$

of a 2-sphere

$$M: \ x_1^2 + x_2^2 + x_3^2 = 1. \tag{3.1}$$

A point in Π can accordingly be given by a pair $(x, -x)$ of diametrically opposite points x and $-x$ in M. We understand that the point $(x, -x)$ in Π equals the point $(-x, x)$ of Π. We say that x in M *covers* the point $(x, -x) = (-x, x)$ in Π, and denote this point in Π by $A(x)$. We say also that x on M *projects* into $A(x)$ on Π.

The mapping A of M onto Π *has two fundamental properties*: (1) *for x and y in M*

$$A(x) = A(y) \tag{3.2}$$

if and only if $x = \pm y$; (2) *the mapping A is locally top.*

Let ϕ be any mapping of an abstract set E into M. Then[3] $A\phi$ is a mapping of E into Π termed the *A-projection* of ϕ into Π.

A closed p-curve F mapping C into M will be termed R-invariant (reflection invariant) if for every z in C

$$F(- z) = - F(z). \tag{3.3}$$

The mapping μ of C onto the semi-circle C_1. We shall make frequent use of a mapping μ of C onto the semi-circle

$$C_1: \ z = e^{\theta i}, \qquad 0 \leqq \theta < \pi, \tag{3.4}$$

of C. Explicitly, with $z = e^{i\theta}$, set

$$\mu(z) = e^{i(\theta/2)} \qquad (0 \leqq \theta < 2\pi). \tag{3.5}$$

The mapping μ of C into C_1 is continuous, except at $z = 1$. Let C_2 be the residual semi-circle $C - C_1$ of C. When F is R-invariant the

[3] Here and elsewhere a functional operation such as A on ϕ is indicated by $A\phi$. The value of $A\phi$ at a point x of E will be denoted by $A\phi(x)$ and not by $A[\phi(x)]$.

mapping $F\mu$ of C into M determines the mapping F of C into M in accordance with the equations

$$(3.6)\qquad \begin{aligned} F\mu(z^2) &= F(z) & [z \in C_1],\\ F\mu(z^2) &= -F(z) & [z \in C_2]. \end{aligned}$$

By definition μ is single-valued; it is one branch of $z^{1/2}$ on C, and to avoid confusion is not to be continued into the other branch. The mapping $F\mu$ of C into M is discontinuous at the point $z=1$ of C. In fact,

$$(3.7)\qquad F\mu(1) = -F\mu(1^-)$$

where $F\mu(1^-)$ indicates the limit of $F\mu(z)$ as z tends to $z=1$ on C from the arc of C on which $\pi < \text{arc } z < 2\pi$.

When F is an R-invariant closed p-curve on M, the mapping

$$(3.8)\qquad f^F = AF\mu$$

of C into Π is a *closed* p-curve on Π by virtue of (3.7). The p-curve f^F is the A-projection of $F\mu$ on Π.

A *deformation* Δ of a closed p-curve F on M is a continuous mapping of $C\times J$ into M in which the image of a point (z, t) of $C\times J$ is a point $\Delta(z, t)$ in M, with

$$(3.9)\qquad \Delta(z, 0) = F(z) \qquad [z \in C]$$

initially, that is, for $t=0$. Such a deformation is termed *R-invariant* if

$$\Delta(-z, t) = -\Delta(z, t)$$

for each point (z, t) in $C\times J$. Suppose then that Δ is R-invariant. Then F is necessarily R-invariant. The deform F^t of F under Δ is the R-invariant mapping $\Delta(\cdot, t)$ of C into M. The A-projection $AF\mu$ of $F\mu$ is a closed p-curve f^F on Π, and $A\Delta\mu$ is a deformation of f^F on Π. We draw the following conclusion:

LEMMA 3.1. *If an R-invariant L-S-closed p-curve F on M admits an R-invariant L-S-deformation on M through p-curves F^t, then the A projection on* Π *of* $F\mu$ *admits a L-S-deformation on* Π *through the closed p-curves* $AF^t\mu$ *on* Π.

4. **The cases $f\approx 0$ and f not ≈ 0.** Let f be a closed p-curve on Π. Let x_0 and $-x_0$ be the points in M which cover $f(1)$. Let a be either one of the points x_0, $-x_0$. Given f and a it follows from the local top. character of the projection A of M onto Π that there exists a unique continuous mapping of ϕ_a^f of the θ-axis into M such that (with $z=e^{i\theta}$)

$$(4.1)\qquad \phi_a^f(0) = a \qquad A\phi_a^f(\theta) = f(z) \qquad (-\infty < \theta < +\infty).$$

It follows from (4.1) that

$$(4.2)\qquad A\phi_a^f(\theta + 2\pi) = A\phi_a^f(\theta).$$

Recall that for points x and y in M, $A(x) = A(y)$ if and only if $x = \pm y$. From (4.2) then

$$(4.3)\qquad \phi_a^f(\theta + 2\pi) = \pm\, \phi_a^f(\theta).$$

The sign in (4.3) is independent of θ and is + if and only if $f \approx 0$ in accordance with the following lemma.

LEMMA 4.1. *If f is a closed p-curve on* Π *and a a point in M such that $A(a) = f(1)$, and if ϕ_a^f is the unique continuous mapping of the θ-axis into M such that* (4.1) *holds given a, then a necessary and sufficient condition that*

$$(4.4)\qquad \phi_a^f(\theta + 2\pi) = \phi_a^f(\theta) \qquad (-\infty < \theta < +\infty)$$

is that $f \approx 0$ on Π.

If (4.4) holds the equation

$$(4.5)\qquad F_a^f(z) = \phi_a^f(\theta) \qquad (z = e^{i\theta})$$

defines a single-valued mapping F_a^f of C into M. Thus F_a^f is a closed p-curve on M, and as such is deformable on M through a continuous family F_a^{ft}, $0 \leqq t \leqq 1$, of closed p-curves on M into a p-curve, whose carrier is a point of M. The A-projections AF_a^{ft} into Π of these p-curves deform f on Π into a p-curve on Π whose carrier is a point. Hence $f \approx 0$ on Π if (4.4) holds.

Conversely suppose that $f \approx 0$ on Π, or more specifically that f on Π is deformed into a p-curve whose carrier is a point, through a family f^t $(0 \leqq t \leqq 1)$ of closed p-curves on Π. With ϕ_a^f given in terms of f by (4.1) a continuous mapping ϕ_a^{ft} of the θ-axis into M can be determined by continuation with respect to increasing t, with

$$\phi_a^{f0}(\theta) \equiv \phi_a^f(\theta)$$

initially, and

$$(4.6)\qquad A\phi_a^{ft}(\theta) \equiv f^t(z) \qquad [\text{for } z = e^{i\theta}]$$

where $\phi_a^{ft}(\theta)$ is continuous in (θ, t) for $0 \leqq t \leqq 1$ and arbitrary θ.

As a consequence of (4.6)

$$\phi_a^{ft}(2\pi) = \pm \phi_a^{ft}(0) \qquad (0 \leqq t \leqq 1) \tag{4.7}$$

where the sign in (4.7) is independent of t on $[0, 1]$. For t sufficiently near 1 on $[0, 1]$ the $+$ sign must hold in (4.7) since the carrier of f^1 in (4.6) is a point and A is locally top. Hence the $+$ sign must hold in (4.7) when $t=0$ as well. Thus (4.4) holds when $f \approx 0$ on Π.

Antecedents and μ-antecedents on M of p-curves on Π. If f is a closed p-curve on Π, any closed p-curve F on M such that $AF=f$ will be called an antecedent on M to f on Π. If f not ≈ 0 no closed p-curve F on M can be antecedent to f, because (4.4) cannot hold in this case. However we shall verify the following. When f not ≈ 0 there always exists (Lemma 4.2) an R-invariant closed p-curve F on M such that $AF\mu=f$. Such a closed p-curve F will be called a *μ-antecedent* of f.

LEMMA 4.2. *Let f be a closed p-curve on* Π. *If $f \approx 0$ on Π there exist just two closed p-curves F^f antecedent on M to f on* Π. *If f not ≈ 0 there exist just two closed p-curves F^f, μ-antecedent on M to f on* Π.

The two antecedents (μ-antecedents) F and F^* of f satisfy the relation $F(z) = -F^*(z)$.

Case I. $f \approx 0$. We start with ϕ_a^f as defined in (4.1), with $a=x_0$ or $-x_0$ where $A(x_0)=f(1)$. Let F_a^f then be defined as in (4.5). In case I, (4.4) holds, so that F_a^f is closed on M. The A-projection of F_a^f is f in accordance with (4.1). There are accordingly at least two closed p-curves F_a^f ($a=x_0$ or $-x_0$) antecedent to f on Π.

Any other closed p-curve F on M such that $AF=f$ must satisfy the condition

$$F(1) = a \qquad (a = x_0 \text{ or } - x_0).$$

By virtue of the continuity of the mapping F^f and the top. character of A, F is thereby uniquely determined by f and a, and so must equal F_a^f. There are accordingly just two closed p-curves F antecedent on M to f on Π when $f \approx 0$.

Case II. f not ≈ 0. We start again with ϕ_a^f as defined in (4.1). In Case II we define F_a^f as a p-curve on M such that for $w=e^{i\alpha}$

$$F_a^f(w) = \phi_a^f(2\alpha) \qquad [-\infty < \alpha < \infty]. \tag{4.8}$$

That F_a^f is single-valued for each w in C, and R-invariant, follows from the relation

$$\phi_a^f(\alpha + 2\pi) = - \phi_a^f(\alpha) \tag{4.9}$$

which holds by virtue of Lemma 4.1 and (4.3). In fact

$$F_a^f(-w) = \phi_a^f(2[\alpha + \pi]) = -\phi_a^f(2\alpha) = -F_a^f(w)$$

for every w in C. Relation (4.8) implies that

$$F_a^f\mu(z) = \phi_a^f(\theta) \qquad \lfloor z = e^{i\theta}, 0 \leqq \theta < 2\pi \rfloor \tag{4.10}$$

and it follows from (4.1) that $AF_a^f\mu = f$. While $F_a^f\mu$ is discontinuous at $z = 1$ on account of (4.9), $AF_a^f\mu$ is continuous at $z = 1$. In Case II there accordingly exists two closed p-curves F_a^f $(a = \pm x_0)$, μ-antecedent on M to f on Π.

Any other closed p-curve F on M such that $AF\mu = f$ must satisfy the condition $F(1) = a$, $[a = \pm x_0]$ and by a process of continuation be uniquely determined by the relation $AF\mu = f$ as the R-invariant closed p-curve F_a^f. The p-curves $F_a^f[a = \pm x_0]$ are accordingly the only closed p-curves μ-antecedent on M to f on Π when f not ≈ 0.

The preceding lemma can be extended to deformations D on M as follows.

LEMMA 4.3. *Corresponding to any continuous deformation D on Π of a closed p-curve f such that $f \approx 0$ [f not ≈ 0] on Π, and to either of the two antecedents [μ-antecedents] F^f of f, there exists a unique deformation Δ of F^f on M such that $A\Delta = D$ when $f \approx 0$, while Δ is R-invariant and $A\Delta\mu = D$ when f not ≈ 0.*

We term the deformation Δ of the lemma *antecedent* on M to D on Π when $A\Delta = D$, and μ-antecedent when $A\Delta\mu = D$.

The proof of Lemma 4.3 is so similar to that of Lemma 4.2 that it need only be indicated.

Let f^t, $0 \leqq t \leqq 1$, be the deform of f under D at the time t. Let $X_0(t)$ and $-X_0(t)$ be the two points in M which cover $f^t(1)$ on Π, X_0 being chosen so that it is continuous for t in $[0, 1]$. Let Θ represent the θ-axis. Let a be either of the functions $\pm X_0$. There exists a unique continuous mapping Φ_a of $\Theta \times J$ into M such that (with $z = e^{i\theta}$)

$$\Phi_a(0, t) = a(t), \qquad A\Phi_a(\theta, t) = D(z, t) \tag{4.11}$$

for each point (θ, t) on $\Theta \times J$. Equation (4.11) replaces (4.1) while the analogue of (4.3) is

$$\Phi_a(\theta + 2\pi, t) = \pm \Phi_a(\theta, t) \tag{4.12}$$

where the sign is $+$ if and only if $f \approx 0$ on Π.

Case I. $f \approx 0$. In this case the required deformation Δ is defined by the equation

$$\Delta_a(z, t) = \Phi_a(\theta, t) \qquad [z = e^{i\theta}].$$

Then $A\Delta_a = D$ in accordance with (4.11).

Case II. f not ≈ 0. One here sets

$$\Delta_a(w, t) = \Phi_a(2\alpha, t) \qquad [w = e^{i\alpha}] \tag{4.13}$$

and observes that Δ_a is single-valued and R-invariant by virtue of (4.12), the sign $-$ prevailing in (4.12). Finally (4.13) implies that

$$\Delta_a[\mu(z), t] = \Phi_a(\theta, t) \qquad (z = e^{i\theta}, 0 \leqq \theta < 2\pi)$$

for each t on $[0, 1]$, so that $A\Delta_a\mu = D$.

In either case the uniqueness of a deformation Δ satisfying the lemma when an antecedent (μ-antecedent) F^t of f is given, follows as in the proof of Lemma 4.3.

If F is an R-invariant closed p-curve on M the class of all R-invariant closed p-curves which admit R-invariant L-S-deformations into F on M will be called an R-invariant *L-S-homotopy class* on M.

The following theorem reduces the problem of determining the L-S-homotopy classes on Π to a problem on M. It is a consequence of the preceding lemmas including Lemma 3.1.

THEOREM 4.1. *A necessary and sufficient condition that L-S-closed p-curves f_1 and $f_2 \approx 0$ [not ≈ 0] on Π be in the same L-S-homotopy class on Π is that an antecedent [μ-antecedent] F^{f_1} and F^{f_2} of f_1 and f_2 respectively be in the same L-S-homotopy class [R-invariant L-S-homotopy class] on M.*

In case $f \approx 0$ on Π models for the L-S-homotopy classes of f can accordingly be inferred from those on the 2-sphere M. Such models on M are given in Theorem 4.2 of Morse [1].

The model p-curve k on Π, *and* Γ *on* M. We shall introduce a simple closed p-curve k on Π, with carrier on Π covered by a great semicircle on M. More definitely we suppose that k has *a* μ-antecedent Γ on M given by the mapping

$$x_1 + ix_2 = z, \qquad x_3 = 0 \qquad [z \in C], \tag{4.14}$$

of the circle C into M. For each integer $n > 0$ let closed p-curves $k^{(n)}$ on Π and $\Gamma^{(n)}$ on M be defined by the equations

$$k^{(n)}(z) = k(z^n), \qquad \Gamma^{(n)}(z) = \Gamma(z^n) \qquad [z \in C].$$

The p-curves $k^{(1)}$ and $k^{(3)}$ have Γ and $\Gamma^{(3)}$ as μ-antecedents on M, while $k^{(2)}$ and $k^{(4)}$ have Γ and $\Gamma^{(2)}$ as antecedents on M. Theorem 4.2 of Morse [1] gives the following.

THEOREM 4.2. *Any L-S-closed p-curve $f \approx 0$ on Π is in the L-S-homotopy class of $k^{(2)}$ or $k^{(4)}$ on Π, while $[k^{(2)}] \neq [k^{(4)}]$ on Π.*

That $[k^{(2)}] \neq [k^{(4)}]$ on Π follows from the fact that $[\Gamma] \neq [\Gamma^{(2)}]$ on M. For the equality $[k^{(2)}] = [k^{(4)}]$ on Π would imply that $[\Gamma] = [\Gamma^{(2)}]$ on M by virtue of Theorem 4.1.

Given $f \approx 0$ on Π the problem of determining to which of the two homotopy classes, $[k^{(2)}]$ or $[k^{(4)}]$, f belongs is equivalent to the problem of determining to which of the two homotopy classes, $[\Gamma]$ or $[\Gamma^{(2)}]$ on M, an antecedent F of f belongs on M. This problem is resolved by the determination of the M-order $p(F)$, of F, as shown in §4 of Morse [1]. In fact

$$[F] = [\Gamma] \quad \text{or} \quad [\Gamma^{(2)}]$$

according as $p(F) = 1$ or 2 mod 2. As shown in Morse [1], $p(F)$ is a topological invariant of M and F, and in particular is invariant under any "R-invariant" homeomorphism T of M, that is, one for which

$$T(-x) = -T(x) \qquad (x \in M),$$

and is accordingly a top. invariant of Π. Finally if F and F' are L-S-closed p-curves on M

$$p(F) = p(F')$$

if and only if $[F] = [F']$ on M, and accordingly if and only if $[AF] = [AF']$ on Π.

We turn accordingly to the case f not ≈ 0 on Π.

5. **L-S-homotopy classes when f not ≈ 0 on Π.** Let F be an R-invariant L-S-closed p-curve on M. In accordance with Theorem 4.1 we seek a model for the R-invariant L-S-homotopy classes of F on M. To that end we refer to the semi-circle C_1 defined by $z = e^{\theta i}$ for $0 \leqq \theta < \pi$, and to the residual semi-circle C_2 defined by $z = e^{\theta i}$ when $\pi \leqq \theta < 2\pi$. Let F_1 and F_2 be submappings of F defined by the equations (a superimposed bar indicates closure)

$$F_1(z) = F(z) \qquad [z \in \overline{C}_1],$$

$$F_2(z) = F(z) \qquad [z \in \overline{C}_2]$$

and term F_1 the *kernel* of F_1 and F_2 the *kernel residue.* We shall be concerned with various continuous mappings of $\overline{C}_1$ into M and will term such mappings p-arcs on M.

Various elementary p-arcs and p-curves on M will be defined and analyzed for later use. In defining kernels F_1 the path which $F(z)$ traces as z traces $\overline{C}_1$ will be given. These paths will be ordered finite sequences of simple, sensed arcs successively joined to form a continuous curve. The paths used will be rectifiable. From a path α a kernel $F_1 = \{\alpha\}$ will be formed by making z in $\overline{C}_1$ correspond to that point $F_1(z)$ in α which divides α in the same ratio with respect to arc length as that in which z divides $\overline{C}_1$ with respect to arc length. The kernel residue F_2 will be defined by setting

$$F_2(-z) = -F_1(z) \qquad [z \in \overline{C}_1].$$

In order that F so defined be L-S it is sufficient that F_1 be L-S and that the images on M under F of sufficiently small neighborhoods of $z=1$ in C be simple.

A symbolism is needed for a path α which is a *product*

$$\alpha = a_1 a_2 \cdots a_n$$

of simple, regular, sensed, closed curves $a_1, \cdots, a_n$, with a_k positively tangent to a_{k+1} $(k=1, \cdots, n-1)$ at a prescribed point P_k. If a is a simple, sensed arc or closed curve, and P and Q are two points in a, $a(P, Q)$ shall denote the subarc (if any exists) of a leading from P to Q on a. With this understood α shall denote the path defined by the sequence of simple arcs (with $a_1(P_1, P_1)$ the arc a_1 cut at P_1),

$$\begin{aligned} &a_1(P_1, P_1)a_2(P_1, P_2) \cdots a_{n-1}(P_{n-2}, P_{n-1}), \\ &a_n(P_{n-1}, P_{n-1})a_{n-1}(P_{n-1}, P_{n-2}) \cdots a_2(P_2, P_1). \end{aligned} \tag{5.1}$$

We admit the possibility that a_1 is not a closed curve, but rather the closure of a simple arc, while $a_2 \cdots a_n$ remain simple closed curves. In such a case P_1 is to be an inner point of a_1. If P and P' are the initial and final points of a_1, the preceding sequence (5.1) is to be altered by replacing $a_1(P_1, P_1)$ by $a_1(P, P_1)$ and $a_1(P_1, P')$ is to be added to the sequence.

The elementary p-arcs on M to be used in defining model kernels F_1 on M can now be defined. Let γ be the simple arc

$$x_1 = \cos\theta, \qquad x_2 = \sin\theta, \qquad x_3 = 0 \qquad (0 \leqq \theta < \pi) \tag{5.2}$$

taken in the sense of increasing θ. Let λ be a small sensed circle of diameter <1, with $x_3 \geqq 0$ thereon, positively tangent to γ at the mid point $(0, 1, 0)$ of γ. Let λ^{-1} be the reflection of λ in the plane $[x_3=0]$. For n a positive integer λ^n shall formally symbolize $\lambda \cdots \lambda$ with n factors λ, while λ^{-n} shall formally symbolize $\lambda^{-1} \cdots \lambda^{-1}$ with n factors λ^{-1}. Let q be any nonvanishing integer. We introduce a product path $\gamma\lambda^q$ in which $(0, 1, 0)$ is the point of contact of successive factors. Then $\{\gamma\lambda^q\}$ is a well-defined p-arc on M which, taken as a kernel F_1, leads to a L-S-closed p-curve F on M. We shall prove the following lemma.

LEMMA 5.1. *The p-arc $\{\gamma\lambda^q\}$ admits a L-S-deformation on M in which sufficiently short initial and final subarcs of γ remain simple with invariant carriers, and in which $\{\gamma\lambda^q\}$ is deformed into $\{\gamma\}$ when q is even, and into $\{\gamma\lambda\}$ when q is odd.*

For simplicity we begin with $\gamma\lambda^2$. The circle λ can be deformed on M through circles tangent to γ at (0, 1, 0) into λ^{-1}, so that $\{\gamma\lambda^2\}$ is L-S-deformable on M into $\{\gamma\lambda\lambda^{-1}\}$. The point of contact of λ^{-1} with λ can be continuously regressed on λ to the point of maximum x_3 on λ, varying λ^{-1} through circles λ_t^{-1}, $0 \leqq t \leqq 1$, of fixed radius. Then λ_1^{-1} is the terminal circle in this deformation of λ^{-1}. Observe that $\lambda\lambda_1^{-1}$ is a figure eight with $x_3 > 0$ thereon, except at the point of contact of λ with γ at (0, 1, 0). It is clear that $\{\gamma\lambda\lambda_1^{-1}\}$ is L-S-deformable into $\{\gamma\}$, and that the whole deformation of $\{\gamma\lambda^2\}$ into $\{\gamma\}$ can be so made that sufficiently short initial and final arcs of γ remain simple with invariant carriers.

In the same way, it is clear that for $q > 2$ $[\gamma\lambda^q\}$ is L-S-deformable successively into

$$\{\gamma\lambda^{q-2}\lambda\lambda_1^{-1}\} \qquad \{\gamma\lambda^{q-2}\},$$

so that an induction with respect to q shows that the lemma is true if $q > 2$. A reflection in the plane at $x_3 = 0$ makes it appear that for $q < 0$, $\{\gamma\lambda^q\}$ is L-S-deformable in the required manner into $\{\gamma\}$ when q is even, and into $\{\gamma\lambda^{-1}\}$, when q is odd. But the above deformation of λ into λ^{-1} shows that $\{\gamma\lambda^{-1}\}$ is L-S-deformable into $\{\gamma\lambda\}$, and the proof of the lemma is complete.

The succeeding proofs will be simplified if one can suppose that the mappings F of C into M are regular, that is, that the representation of the point $F(z)$ in terms of the parameter θ defining $z = e^{\theta i}$ has a form

$$F(z) = [a_1(\theta), a_2(\theta), a_3(\theta)] \tag{5.3}$$

in which a_i $(i = 1, 2, 3)$ has a continuous derivature $\dot{a}_i$ and

$$\dot{a}_1^2(\theta) + \dot{a}_2^2(\theta) + \dot{a}_3^2(\theta) \neq 0. \tag{5.4}$$

This and more is needed, and is supplied by the following lemma.

LEMMA 5.2. *Let e be a positive constant. Any R-invariant L-S-closed p-curve F on M admits an R-invariant L-S-deformation on M into a p-curve F' on M with no point $F(z)$ thereby displaced a distance more* *than e, and with F′ regular.*

The method of proof of this lemma is entirely similar to the methods used in proving Theorems 28.2 and 28.3 of Morse [2], except for the conditions of R-invariance of the p-curves used. Disregarding this condition for the moment recall that the component deformations used in Morse [2] are local in character, involving among other procedures the use of conformal transformations. All this is essentially the same on the sphere M. Short straight arcs used in the plane are here replaced by short geodesics on M. If the successive local deformations D are applied to sufficiently restricted arcs h on M, it will be possible to accompany each D by a simultaneous deformation of the reflection h' of h in the origin through a reflection of the deforms h^t of h under D. In this way the resultant deformations will be made R-invariant as required.

The order $Q(F_1, E)$. We shall refer to the given system of coordinates (x_1, x_2, x_3) on M as the system E. The points

$$Z_1 = (0, 0, 1), \qquad Z_{-1} = (0, 0, -1)$$

will be called the *poles* of E. A p-curve or arc on M whose carrier does not intersect the poles of E will be termed *E-pole free.* A p-curve on Π will be termed E-pole free if no point of its carrier is covered by a pole of E on M. Let F be an R-invariant closed p-curve of M which is E-pole free. With $F(z)$ of the form

$$F(z) = [x_1(z), x_2(z), x_3(z)] \qquad (z \in C)$$

we set

$$Q(F_1, E) = \underset{C_1}{\text{variation}} \left[\frac{\text{arc } x_1(z) + ix_2(z)}{\pi} \right] \tag{5.5}$$

as z traverses C_1 from $z=1$ to $z=-1$. Observe that $x_1(z)$ and $x_2(z)$ do not vanish simultaneously since F is E-pole free. Thus $Q(F_1, E)$ is well defined. Moreover $Q(F_1, E)$ is an odd integer since

$$x_1(-1) = -x_1(1), \qquad x_2(-1) = -x_2(1).$$

We define $Q(F, E)$ similarly with C_1 replaced by C in (5.5), and obobserve that $Q(F, E) = 2Q(F_1, E)$.

The following lemma is of the nature of a procedural simplification.

LEMMA 5.3. *Let F be an R-invariant closed p-curve on M. The order $Q(F_1, E)$ is an invariant of any R-invariant deformation of F on M in which the deforms F^t of F remain E-pole free. The R-invariant L-S-homotopy class of F contains p-curves F^* such that*

$$(5.6) \qquad Q(F^*, E) = 2.$$

The first affirmation of the lemma is immediately clear. In establishing the concluding statement of the lemma no generality will be lost if F is assumed regular.

We shall deform an arc of F_1 over one of the poles of E. More definitely we start with an open simple arc g of F_1 and deform the middle third g_1 of g, leaving the carrier of the residue of g invariant in order that the deforms g^t of g may cause no failure of F_1^t to remain L-S, apart from a failure of g^t itself to remain L-S. We deform g_1 through tongue shaped curves g_1^t with two end points fixed on g, and with semi-circular tips τ^t. We suppose τ^t moves across $(0, 0, -1)$ so that at the moment of crossing $(0, 0, -1)$ is at the mid point of τ^t. By virtue of such a crossing $Q(F_1^t, E)$ will change by 2 or -2 according as the sense of τ^t just after the moment t_0 of crossing is or is not the sense in which arc (x_1+ix_2) increases on τ^t. By an appropriate deformation in which the tongue remains L-S, either case can be made to happen. It should be observed that the tongue can be made self-intersecting provided it remains L-S. Since any finite number of such tongues can be used, it is clear that (5.6) can be made to hold provided the deformation of F_1 through the above p-arcs F_1^t be converted into an R-invariant L-S deformation of F by deforming the kernel residue F_2 of F through p-arcs F_2^t for which

$$(5.7) \qquad F_2^t(-z) = -F_1^t(z) \qquad [z \in C_1].$$

Canonical p-curves on M. These curves are special p-curves introduced to simplify the proof of Theorem 5.1. Such p-curves are to be regular R-invariant closed p-curves with the following properties:

(I) $F(\pm 1) = (\pm 1, 0, 0)$.

(II) *The positive tangent to the path of F at the points corresponding to $z = \pm 1$ on C shall have the direction cosines* $(0, \pm 1, 0)$ *respectively.*

(III) *The p-curve F shall be E-pole free.*

(IV) *The order* $Q(F, E) = 2$.

It follows from Lemmas 5.2 and 5.3 that there exists a canonical p-curve F in the R-invariant L-S-homotopy class of any given R-invariant L-S-p-curve. Cf. proof of Lemma 7.1. We term a kernel F_1 of a canonical p-curve F, a *canonical kernel* F_1. Canonical kernels lie on the open sub-manifold of M

$$M_1 = M - Z_1 - Z_{-1} \qquad [Z_{\pm 1} = (0, 0, \pm 1)].$$

We shall make several uses of the following mapping.

A mapping W of M_1 into a complex w-plane. Under this mapping x in M_1 has an image $w = W(x)$ in the w-plane where

$$W(x) = \exp [x_3 + 2i \text{ arc } (x_1 + ix_2)] \qquad (x \in M_1). \tag{5.8}$$

This mapping can be equivalently given in the form

$$|w| = \exp [x_3], \qquad \text{arc } w = 2 \text{ arc } (x_1 + ix_2). \tag{5.9}$$

The mapping W is single-valued and continuous, and carries M_1 into a ring in the w-plane on which

$$e^{-1} < |u| < e. \tag{5.10}$$

Each point w in this ring has just two distinct points on M_1 of the form

$$(a_1, a_2, a_3) \qquad (-a_1, -a_2, a_3) \quad [(a_1, a_2) \neq (0, 0)]\}$$

as antecedents. The mapping W is locally top. The inverse W^{-1} is single-valued on a two-sheeted Riemann surface covering the ring (5.10) twice without branch points.

Canonical p-curves in the w-plane. If F_1 is a canonical kernel on M, there exists a unique regular, closed p-curve Ω mapping the circle C into the ring (5.10) on the w-plane, and such that

$$\Omega(z) = WF_1\mu(z) \qquad [z \in C]. \tag{5.11}$$

Such a p-curve has the following properties, paralleling the properties I–IV of canonical p-curves on M.

(I′) $\Omega(1) = 1$.

(II′) *The positive tangent to the path of Ω at the point corresponding to $z = 1$ is parallel to the positive v-axis ($u + iv = w$).*

(III′) *The carrier of Ω is on the ring* (5.10).

(IV′) *The ordinary plane order of Ω with respect to $w = 0$ is* 1.

Conversely any closed regular p-curve Ω in the w-plane which is canonical in the above sense determines a unique canonical kernel F_1 on M such that (5.11) holds. We then term F the μ-antecedent on M of Ω in the w-plane. Any L-S-deformation of a canonical kernel on M_1 or p-curve on the ring (5.10) through such curves will be called *canonical.*

Let Ω then be a canonical p-curve Ω on the ring (5.10) and F its canonical μ-antecedent on M. Any canonical L-S-deformation Ω^t, $0 \leqq t \leqq 1$, of Ω on the ring (5.10) implies a canonical L-S-deformation F_1^t, $0 \leqq t \leqq 1$, of F_1 on M such that

$$\Omega^t = WF_1^t\mu.$$

Such deformations of Ω on the ring (5.10) are 0-deformations in the sense of Theorem 33.1 of Morse [2]. Since the ordinary plane order q of Ω is 1 the proof[4] of Lemma 33.1 and Theorem 33.1 shows that Ω admits a canonical L-S-deformation on the ring (5.10) into a p-curve Ω^1 whose μ-antecedent on M has the kernel $\{\gamma\}$ or $\{\gamma\lambda^{p-1}\}$ according as the angular order (cf. Morse [2]) p of Ω is 1 or not 1. Lemma 5.1 thus permits the following conclusion.

LEMMA 5.4. *A canonical kernel F_1 on M admits a canonical L-S-deformation on M into $\{\gamma\}$ or $\{\gamma\lambda\}$.*

Observe that $\{\gamma\}$ is the kernel Γ_1 of the R-invariant p-curve Γ on M defined at the end of §4. Recall that $A\Gamma\mu = k^{(1)}$. Observe further that the circle λ can be deformed through circles which remain tangent to γ at (0, 1, 0) into a great circle C' on which $x_3 = 0$, so that $\{\gamma\lambda\}$ is L-S-deformable among canonical kernels into $\{\gamma C'\}$. This is the kernel of $\Gamma^{(3)}$. Recall that $A\Gamma^{(3)}\mu = k^{(3)}$. We are thus led to the basic theorem.

THEOREM 5.1. *Any L-S-closed p-curve f not ≈ 0 on Π is in the L-S-homotopy class of $k^{(1)}$ or $k^{(3)}$.*

We have merely to review the various steps which lead to this result. In the first place the given f has an R-invariant closed p-curve F as a μ-antecedent on M. Cf. Lemma 4.2. Such an F admits an R-invariant L-S-deformation into a canonical p-curve F^*. Cf. Lemmas 5.2 and 5.3, and the proof of Lemma 7.1.

The kernel F_1^* admits a L-S-deformation through canonical kernels F_1^{*t} into $\{\gamma\}$ or $\{\gamma\lambda\}$ in accordance with Lemma 5.4, and hence into the canonical kernel of Γ or $\Gamma^{(3)}$. On extending these canonical kernels on M by reflection as in (5.7) we infer that F^* admits a L-S-deformation on M through R-invariant p-curves F^{*t} into Γ or $\Gamma^{(3)}$. The p-curves $AF_1^{*t}\mu$ on Π are closed and L-S, and deform $AF_1^*\mu$ into $k^{(1)}$ or $k^{(3)}$. In résumé, $f = AF_1\mu$ is first L-S-deformed on Π into $AF_1^*\mu$ and then into $k^{(1)}$ or $k^{(3)}$.

This completes the proof of the theorem.

It remains to show that $k^{(1)}$, $k^{(2)}$, $k^{(3)}$, $k^{(4)}$ are in distinct L-S-homotopy classes on Π. Part of this result is already clear. For the property of a p-curve f being null homotop. on Π is invariant of arbitrary continuous deformations of f on Π and in particular invariant of L-S-deformations. Thus the null homotop. p-curves $k^{(2)}$ and $k^{(4)}$

[4] In the proof of Lemma 33.1 suppose that a line element E of g at Q is tangent to a circle C with center at $w = 0$. One can hold E fast in the deformation. A preliminary L-S-deformation should be used to make g convex towards the origin near Q. One then proceeds as before identifying Q with the point $s = a$ of the proof.

are not in the L-S-homotopy classes of $k^{(1)}$ and $k^{(3)}$. Moreover $[k^{(2)}] \neq [k^{(4)}]$ as affirmed in Theorem 4.2. We must finally show that

$$[k^{(3)}] \neq [k^{(1)}]$$

and going somewhat deeper characterize the classes $[k^{(3)}]$ and $[k^{(1)}]$ topologically.

In §6 a numeral invariant $d(f, E)$ of a L-S-homotopy class $[f]$ is defined in case f not ≈ 0 and f is E-pole free. In §7, $d(f, E)$ is replaced by a topological invariant $d_S(f)$ defined for an arbitrary top. image S of the projective plane, thereby freeing $d(f, E)$ from its dependence on the special coordinate system E and the special representation Π of a projective plane.

6. **The difference order $d(f, E)$ when f not ≈ 0.** Let F be an R-invariant closed p-curve on M. In the case in which F is E-pole free an angular order $P(F_1, E)$ of the kernel F_1 will be defined. For this purpose it is necessary that M receive an orientation from E.

The E-orientation of M. Corresponding to the coordinate system E of M, M will be oriented as follows. Let $C(x)$ be an arbitrarily small circle on M with center at x in M. As previously, let $Z_{\pm 1} = (0, 0, \pm 1)$. The positive sense of $C(Z_{-1})$ shall be such that a continuous branch of the multiple-valued function

$$\text{arc } (y_1 + iy_2) \qquad [y = (y_1, y_2, y_3)]$$

increases as y traces $C(Z_{-1})$ in its positive sense. The sense of $C(x)$ at other points x in M will be obtained by a continuous variation of $C(x)$ from $C(Z_{-1})$. In particular it should be noted that as $C(Z_1)$ is traced in its positive sense by a point y any continuous branch of arc $(y_1 + iy_2)$ decreases.

Reference directions for the measurement of angles at a point x of $M_1 = M - Z_1 - Z_{-1}$ must be defined. For each x in M_1 let $C^1(x)$ be the circle through x parallel to the plane on which $x_3 = 0$. Let the positive sense of $C^1(x)$ be that of increasing arc $(y_1 + iy_2)$ for y in $C^1(x)$. The reference direction at x shall be the positive tangent to $C^1(x)$ at x. The sign of an angle at x measured from the reference direction will be determined by the orientation of M at x as defined by $C(x)$.

The angular order $P(F_1, E)$. Let F be R-invariant L-S, closed and E-pole free. Set $f = AF_1\mu$. Let ϵ_1 be a positive constant so small that the submappings of F on which $z = e^{i\theta}$ with $\alpha \leqq \theta \leqq \alpha + \epsilon_1$ are top. mappings for each constant α. Given $z = e^{i\theta}$ let z_ϵ denote the point $e^{(\theta+\epsilon)i}$. We suppose that $0 < \epsilon < \epsilon_1$. Let $H_F(z, \epsilon)$ denote the angle at the point $x = F(z)$ in M, measured from the reference direction at x to the positive tangent at x to the great circle on M leading from $F(z)$ to

$F(z_\epsilon)$. For fixed ϵ let $H_F(z, \epsilon)$ be chosen so as to vary continuously with z in $\bar{C}_1$. Since $F(-z) = -F(z)$ it is clear that

$$H_F(-1, \epsilon) = -H_F(1, \epsilon) \pmod{2\pi} \tag{6.1}$$

so that

$$H_F(-1, \epsilon) + H_F(1, \epsilon) = 2r\pi \tag{6.2}$$

where r is an integer. We set

$$P(F_1, E) = \frac{H_F(-1, \epsilon) + H_F(1, \epsilon)}{\pi} \tag{6.3}$$

and note the following.

The value of $P(F_1, E)$ is independent mod 4 of the choice of ϵ in $(0, \epsilon_1)$, of the choice of F between the two μ-antecedents of f, and of the choice of H_F among the possible continuous branches of this angle function. For $H_F(z, \epsilon)$ can be chosen as to vary continuously with (z, ϵ) for ϵ in $(0, \epsilon_1)$ and z in $\bar{C}_1$, so that the left member of (6.2) is independent of ϵ in $(0, \epsilon_1)$. If F and F^* are the two μ-antecedents of f, $F(z) = -F^*(z)$, so that one can take

$$H_{F^*}(z, \epsilon) = -H_F(z, \epsilon).$$

Hence

$$P(F_1^*, E) = -P(F_1, E) = P(F_1, E) \pmod{4}.$$

Finally a change of the continuous branch of H_F will change the left member of (6.2) by an integral multiple of 4π and so leave $P(F_1, E)$ unchanged mod 4.

The difference order $d(f, E)$. Let f not ≈ 0 be a L-S-closed p-curve on Π which is E-pole free. Let F be a μ-antecedent on M of f on Π. We set

$$d(f, E) \equiv Q(F_1, E) - P(F_1, E) \pmod{4}$$

and observe that $d(f, E)$ is independent of the choice of F as μ-antecedent of f, and of any R-invarient L-S-deformation of F on M_1.

One sees that

$$\begin{aligned} d(k^{(1)}, E) &\equiv Q(\Gamma_1, E) - P(\Gamma_1, E) = 1 - 0 \pmod{4}, \\ d(k^{(3)}, E) &\equiv Q(\Gamma_1^{(3)}, E) - P(\Gamma_1^{(3)}, E) = 3 - 0 \pmod{4}. \end{aligned} \tag{6.4}$$

An immediate conclusion is that $k^{(1)}$ admits no L-S-deformation on Π into $k^{(3)}$ through p-curves which are E-pole free. To remove the latter condition, deformations must be made through the poles of E and the effect on $d(f, E)$ determined. For this purpose p-curves on Π whose μ-antecedents on M are broken geodesics are useful.

Admissible broken geodesics on M. In Morse and Heins [1] use has been made of L-S-curves composed of sequences of a finite number of straight arcs. The analogous p-curves on M are sequences of a finite number of geodesic arcs each less than π in length, with nonzero angles at the vertices (the junction points of successive arcs). A p-curve on M of this character will be called an admissible broken geodesic. Admissible broken geodesics are L-S. A deformation on M of an R-invariant closed p-curve F through admissible broken geodesics F^t will be termed *admissible* if the number of vertices is independent of t, if the vertices vary continuously with t and remain distinct on any p-curve F^t, if the point $F^t(1)$ is a vertex of F^t, and if each p-curve F^t is R-invariant. The methods of Morse and Heins [1] suffice to prove the following lemma.

LEMMA 6.1. *Let e be a positive constant. Any R-invariant L-S-p-curve F on M admits an R-invariant L-S-deformation into an admissible broken geodesic displacing each point F(z) on M at most e in this process.*

Any two R-invariant broken geodesic closed p-curves F and F′ which are in the same R-invariant L-S-homotopy class, can be admissibly deformed on M into each other through R-invariant broken geodesics, provided a suitable number of vertices are initially added to F and F′.

With this lemma as an aid, the following theorem can be proved:

THEOREM 6.1. *If f not ≈ 0 and f' not ≈ 0 are two closed L-S-p-curves on Π in the same homotopy class on Π and if f and f' are E-pole free, then*

$$d(f, E) = d(f', E) \pmod 4. \tag{6.5}$$

The theorem follows at once from the definition of $d(f, E)$ if f can be L-S-deformed into f' on Π through p-curves which are E-pole free. In any other case we can suppose, without loss of generality, that the μ-antecedents F and F' on M, of f and f' respectively on Π, are admissible broken geodesics which are E-pole free. In accordance with Lemma 6.1, F can be admissibly deformed into F' through a family F^t of broken geodesics. If use is made of the freedom of small displacements of the vertices of F^t, we can be assured that F^t is E-pole free except for a finite set of values $t_1, \cdots, t_n$ of t, that no vertex of F^{t_i} $(i=1, \cdots, n)$ is at a pole of E, and that F^{t_i} has just one point in common with the poles of E.

The conventions as to the measurement of angles are such that as a geodesic arc of F_1^t moves across the pole (0, 0, 1)

$$\Delta P(F_1^t, E) = \Delta Q(F_1^t, E) = \pm 2$$

so that $d(f^t, E)$ is unchanged by such a passage. When a geodesic arc of F_1^t moves across the pole $(0, 0, -1)$

$$\Delta P(F_1^t, E) = -\Delta Q(F_1^t, E) = \pm 2.$$

The difference $d(f^t, E)$ mod 4 is accordingly invariant as t increases from 0 to 1. This completes the proof of the theorem.

Reference to (6.4) gives the following corollary of the theorem.

COROLLARY 6.1. *The models $k^{(1)}$ and $k^{(3)}$ on Π are not in the same L-S-homotopy class on Π.*

By virtue of Theorems 5.1 and 6.1 any L-S-p-curve f not ≈ 0 on Π which is E-pole free is in the L-S-homotopy class of $k^{(1)}$ or $k^{(3)}$ according as $d(f, E) = 1$ or 3 mod 4. This determination of the L-S-homotopy class of f depends upon our special model Π of the projective plane and upon the coordinate system E. We shall remove this dependency.

7. **Invariant orders and models.** We begin with the following lemma:

LEMMA 7.1. *Any simple closed p-curve f not ≈ 0 on Π can be deformed on Π into any other such p-curve on Π through simple, closed p-curves.*

It will be sufficient to show that f can be deformed into $k^{(1)}$ in the manner required. If F is a μ-antecedent of f it will be sufficient to show that F can be deformed on M into Γ through R-invariant simple, closed p-curves on M. The required deformation will be defined as a sequence of five deformations.

(1) We first deform F in the required manner into an R-invariant, simple, regular closed p-curve $F^{(1)}$. With obvious precautions to maintain a simple curve the proof of Lemma 5.2 will suffice.

(2) We next rotate M in such a manner that $F^{(1)}$ is deformed into a p-curve $F^{(2)}$ for which $F^{(2)}(1) = 1$.

(3) A suitable rotation of M about the x_1 axis will then carry $F^{(2)}$ into a p-curve $F^{(3)}$ which is tangent to Γ at the point $(1, 0, 0)$.

(4) If $F_1^{(3)}$ is not E-pole free its kernel $F_1^{(3)}$, with its simple projection on Π, intersects $(0, 0, 1)$ in a point $F_1^{(3)}(z)$ for just one value of z on C_1. A suitable L-S-deformation of $F_1^{(3)}$ near this point of intersection and a corresponding R-invariant L-S-deformation of $F^{(3)}$ will yield a simple closed p-curve $F^{(4)}$ which is E-pole free. Moreover

$$Q(F^{(4)}, E) = \pm 2 \tag{7.0}$$

as one sees on projecting M_1 stereographically from $(0, 0, 1)$ onto the plane tangent to M at (0, 0, -1). Finally we can suppose that the $+$ sign holds in (7.0). For if one keeps $F_1^{(4)}$ simple and regular and deforms a tongue once over (0, 0, 1), the order (7.0) of $F^{(4)}$ will be changed from -2 to 2, if initially -2.

(5) The resultant p-curve $F^{(4)}$ is canonical in the sense of §5. Use can be made of the mapping W of M_1 into the ring (5.10) of the w-plane. On this ring there exists a canonical p-curve Ω such that

$$\Omega = WF_1^{(4)}\mu.$$

In particular Ω has the plane order 1 relative to $w=0$ in the w-plane. It follows that there exists a deformation of Ω on the ring (5.10) through simple, canonical p-curves Ω^t, $0 \leqq t \leqq 1$, on the ring into the p-curve $w=z=e^{\theta i}$. On M, F can accordingly be L-S-deformed through simple canonical p-curves into Γ.

Hence f can be deformed on Π in the manner required into $k^{(1)}$.

Various methods (including conformal mapping) are available to prove the following.

(i) Let g^t, $0 \leqq t \leqq 1$, be a 1-parameter family of simple closed p-curves in the (u, v)-plane of which g^1 is the circle $C: z=e^{i\theta}$. Let G^t be the closure of the interior of g^t. There exists a continuous 1-parameter family of top. mappings T^t of G^t into the closed disc bounded by C, such that T^t maps $g^t(z)$ into z and G^1 is the identity.

Recall that an *isotopic* deformation of a manifold S is defined by a continuous 1-parameter family of top. mappings of S onto S. With this understood we state the following lemma. In proving this lemma it will be convenient to denote the carrier of a p-curve F by $|F|$.

LEMMA 7.2. *Any homeomorphism H of Π can be isotopically deformed into the identity on Π.*

Let K be an R-invariant homeomorphism of M such that $AK=H$. Set $F=K^{-1}\Gamma$. By virtue of Lemma 7.1 there exists a continuous 1-parameter family of R-invariant, simple, closed p-curves F^t on M which deform F into Γ. Let Σ^t, $0 \leqq t \leqq 1$, be a continuous 1-parameter family of closed domains on M bounded by the respective Jordan curves $|F^t|$. Observe that Σ^1 is a hemisphere of M bounded by $|\Gamma|$. It follows from (i) that there exists a continuous 1-parameter family of top. mappings T^t of Σ^t onto the hemisphere Σ^1, which, in particular, map the Jordan curve $|F^t|$ onto the circle $|\Gamma|$ in such a manner that $\Gamma(z)$ is the image of $F^t(z)$ and T^1 is the identity. The mappings T^t can be extended over M by reflection, that is, so that

$$T^t(-x) = -T^t(x) \qquad (X, t) \in (M \times J).$$

So extended T^t, $0 \leqq t \leqq 1$, defines an isotopic deformation of T^0 into the identity T^1.

It remains to deform K isotopically into T^0. By definition $KF=\Gamma$ so that $K(x)=T^0(x)$ when x is in the Jordan curve $|F|$. By a theorem of Tietsze there is an isotopic deformation of the mapping K restricted to Σ^0, into T^0, likewise restricted to Σ^0, leaving $|F|$ pointwise fixed. This deformation can be extended to all of K by reflection in the origin, so as to yield an R-invariant isotopic deformation of K into T^0. Hence K is isotopically deformable into the identity through R-invariant top. mappings of M onto M.

The lemma follows.

Proof of Theorem 1.1 *of the introduction.* The p-curve k whose multiple tracings $k^{(1)}$, $k^{(2)}$, $k^{(3)}$, $k^{(4)}$ appear in Theorems 4.2 and 5.1 is a simple, closed p-curve on Π with k not ≈ 0 on Π. It follows from Lemma 7.1 that in these theorems k can be replaced by any other simple, closed p-curve h such that h not ≈ 0. This completes the proof of the fundamental Theorem 1.1.

Further invariance of $d(f, E)$. We now admit any coordinate system E' obtained from E by a rotation of E about the origin, or by a reflection of E' in the origin. We have seen that $d(f, E)$ is independent of the L-S-deformation class of f provided only that $d(f, E)$ is well defined, that is, provided that f is E-pole free. The following theorem shows the essential top. invariance of $d(f, E)$.

THEOREM 7.1. *Let f be a L-S-closed p-curve on Π, H a homeomorphism of Π and $f'=Hf$ the transform of f under H. If E and E' are admissible coordinate systems such that f and f' are respectively E and E'-pole free, then*

$$d(f, E) = d(f', E'). \tag{7.1}$$

We shall first show that

$$d(f, E) = d(Hf, E) \tag{7.2}$$

provided f and Hf are E-pole free. Relation (7.2) follows from Lemma 7.2 according to which H can be isotopically deformed into the identity, thus implying a L-S-deformation of Hf into f. From the invariance of $d(f, E)$ under such deformations of f, (7.2) must hold.

We shall next show that

$$d(f, E) = d(f, E') \tag{7.3}$$

provided f is E and E'-pole free. To that end let T be the orthogonal transformation by virtue of which $E'=TE$. It is trivial that

$$d(f, E) = d(Tf, TE). \tag{7.4}$$

But f and f' are E'-pole free so that

$$d(Tf, TE) = d(f, TE) = d(f, E')$$

according to (7.2). Hence (7.3) holds.

To establish (7.1) let E'' be chosen (as is possible) so that f and f' are E''-pole free. By hypothesis f is E-pole free, and f' is E'-pole free. Hence

$$d(f, E) = d(f, E'') = d(f', E'') = d(f', E')$$

in accordance with (7.2) and (7.3). This completes the proof of the theorem.

Definition of an invariant S-order of f when f not ≈ 0. Let S be an arbitrary top. model of the projective plane, and f an L-S-closed p-curve on S with f not ≈ 0 on S. Then $d(Zf, E)$ is independent mod 4 of the choice of Z among top. mappings of S onto Π and of the choice of E among admissible rectangular coordinate systems for M provided Zf is E-pole free.

For each L-S-closed p-curve of not ≈ 0 *on* S *we set*

$$d(Zf, E) = d_S(f) \pmod 4$$

provided Zf *is* E*-pole free, and term* $d_S(f)$ *the S-difference order of* f.

The fundamental nature of the top. invariance of $d_S(f)$ is specified in the following theorem.

THEOREM 7.2. *The S-difference order* $d_S(f)$ *of a L-S-closed p-curve* f *not* ≈ 0 *on* S *is independent of the choice of* f *in its L-S-homotopy class. If* S *is mapped top. onto* S' *under a mapping* K *and if* $f' = Kf$, *then*

$$d_S(f) = d_{S'}(f') \pmod 4. \tag{7.5}$$

The difference order $d_S(f)$ *has but two possible values* 1 *and* 3, mod 4. *A necessary and sufficient condition that* $d_S(f) = 1$ mod 4 *is that the L-S-homotopy class of* f *contain a simple, closed p-curve* f_1 *not* ≈ 0 *on* S.

Let Z and Z' be arbitrary top. mappings of S and S' respectively onto Π. Then by definition

$$d_S(f) = d(Zf, E), \qquad d_{S'}(f') = d(Z'f', E')$$

provided E and E' are admissible coordinate systems for M such that Zf and $Z'f'$ are respectively E and E'-pole free. Observe that

$$Z'f' = (Z'KZ^{-1})(Zf)$$

and that the transformation

$$Z'KZ^{-1} = H,$$

is a top. mapping of Π onto Π. Hence

$$d(Zf, E) = d(Z'f', E')$$

in accordance with (7.1). Thus (7.5) holds.

The first statement in the theorem is a consequence of Theorem 6.1.

To establish the last statement in the theorem suppose first that $d_S(f)=1$. Recall that $[f]=[k^{(1)}]$ or $[k^{(3)}]$ by Theorem 5.1. But we have seen in (6.4) that

$$d(k^{(1)}, E) = 1, \qquad d(k^{(3)}, E) = 3 \qquad \text{(mod 4)}$$

so that $[f]=[k^{(1)}]$. Thus $k^{(1)}$ is a simple closed p-curve in $[f]$ as affirmed.

Conversely, suppose that $[f]$ contains a simple, closed p-curve f_1. Recall that $[f_1]=[k^{(1)}]$ as a consequence of Lemma 7.1. Hence

$$d_S(f) = d(k^{(1)}, E) = 1 \qquad \text{(mod 4)}.$$

This completes the proof of the theorem.

The part of Theorem 1.1 which concerns the case f not ≈ 0 can be completed as follows.

THEOREM 7.3. *If f not ≈ 0 is a L-S-closed p-curve on the top. image S of a projective plane $[f]=[k^{(1)}]$ or $[k^{(3)}]$ according as $d_S(f)=1$ or* 3 mod 4.

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