THE IRREGULARITY OF AN ALGEBRAIC SURFACE AND A THEOREM ON REGULAR SURFACES

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1. **Introduction.** In a recent paper [3][2] O. Zariski and the writer have shown that the arithmetic genus of a field Σ of algebraic functions of two variables can be invariantly defined with the aid of the Hilbert characteristic function associated with a pair of models of Σ. The first objective of this note is to show that the irregularity of Σ can be defined in a similar manner. The second objective is to obtain more general results on the existence of integral bases for regular surfaces than were obtained in [2].

2. **The irregularity.** We consider a field Σ of algebraic functions of two independent variables over a ground field k. The field k is assumed to be of characteristic zero and to be maximally algebraic in Σ. We use the notations and definitions of [3]. In particular, if U and V are normal models of Σ we let $\{A_m\}$ ($\{B_n\}$) denote the system of curves cut out on U (V) by the hypersurfaces of order m (n) of its ambient space. The dimension increased by one of the complete system $|A_m+B_n|$ (which system is regarded as lying on the join W of U and V) is denoted by $r(m, n)$. The transformation $T: U \rightarrow V$ is said to be proper [3, definition 2] if $T(A)$ is normal for a generic $A \in \{A_1\}$.

LEMMA 1. *If the transformation $T: U \rightarrow V$ is proper then there exists an integer n_0 such that for $n \geqq n_0$ and $m \geqq i$ the complete system $|A_m+B_n|$ cuts a complete series on the generic curve A_i of the system $|A_i|$, where i is an arbitrary positive integer.*

PROOF. Let A be a nonsingular irreducible hyperplane section of U such that $T(A)$ is normal. (Such hyperplane sections exist in view of the Bertini-Zariski theorems [7] and [9] and the fact that T is proper.) If $\bar{r}(m, n)$ is the r-function associated with the pair $(A, T(A))$ in the sense of [3] (article 2), then by formula 4.1 of [3], there exists an integer n_0 such that when $n \geqq n_0$ the function $r(m, n)$ satisfies the addition formula

$$(2.1) \qquad r(m, n) = r(m-1, n) + \bar{r}(m, n).$$

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[2] Numbers in brackets refer to the bibliography.

If π is the genus of A, if μ and ν are respectively the orders of A and $T(A)$, and if $n\nu > 2\pi - 2$, then the function $\bar{r}(m, n)$ is given by the formula $\bar{r}(m, n) = m\mu + n\nu - \pi + 1$, so that by repeated application of (2.1),

$$(2.2) \qquad r(m, n) = r(m - i, n) + i(m\mu + n\nu - \pi + 1) - 2^{-1}i(i - 1)\mu.$$

If μ_i and π_i are the degree and genus of the system $|A_i|$ the well known formulas of Noether yield

$$(2.3) \qquad \begin{aligned} \mu_i &= i^2\mu, \\ \pi_i &= i\pi + 2^{-1}i(i - 1)\mu - i + 1, \end{aligned}$$

so that (2.2) becomes

$$(2.4) \qquad r(m, n) = r(m - i, n) + i(m\mu + n\nu) - \pi_i + 1.$$

Let A_i be a generic[3] member of $|A_i|$ and observe that $|A_m + B_n|$ cuts a series of order $(m\mu + n\nu)i$ on A_i. If necessary we increase n_0 so that $n_0\nu > 2\pi - 2$. It then follows from the second of formulas (2.3) that if $m \geqq i$ and $n \geqq n_0$ the inequality, $(m\mu + n\nu)i > 2\pi_i - 2$, holds so that the series cut on A_i by $|A_m + B_n|$ is non-special. If d is its deficiency, then the dimension of this series is $(m\mu + n\nu)i - \pi_i - d$, and since the residual system of $|A_m + B_n|$ with respect to A_i is the complete system $|A_{m-i} + B_n|$,

$$(2.5) \qquad r(m, n) = r(m - i, n) + (m\mu + n\nu)i - \pi_i - d + 1.$$

Equations (2.4) and (2.5) together imply that $d = 0$, q.e.d.

It is well known that if C_1 and C_2 are generic members of a linear system $|C|$ of curves on an algebraic surface then the characteristic series cut by $|C|$ on C_i $(i = 1, 2)$ will have the same deficiency. This deficiency is therefore a character of the system $|C|$. We quote the following lemma proved in [4] (page 70).

LEMMA 2. *If the system $|C + D|$ cuts out a complete series on a generic curve $C \in |C|$, then the deficiency of the characteristic series of $|C|$ is not greater than that of $|D|$.*

If we let $\alpha(m)$ and $\beta(n)$ denote the deficiencies of the characteristic series of the complete systems $|A_m|$ and $|B_n|$ respectively then Lemmas 1 and 2 together yield the following theorem.

THEOREM 1. *If the transformation $T: U \rightarrow V$ is proper then there exists*

[3] The term "generic" is here used only to signify that A_i is irreducible and has genus π_i, where π_i is the genus of $|A_i|$. Unless otherwise specified the term will be used in this sense throughout the text.

an integer n_0 such that $\alpha(m) \leqq \beta(n)$ when $n \geqq n_0$ and m is arbitrary.

PROOF. By Lemma 1 there is an integer n_0 such that $|A_m+B_n|$ cuts a complete series on the generic curve of $|A_m|$ when $n \geqq n_0$ and m is arbitrary. By Lemma 2 it follows that $\alpha(m) \leqq \beta(n)$, q.e.d.

COROLLARY 1. *If both $T: U \to V$ and $T^{-1}: V \to U$ are proper then there exist integers m_0 and n_0 such that $\alpha(m) = \beta(n)$ if $m \geqq m_0$ and $n \geqq n_0$.*

It follows that the deficiency $\alpha(m)$ of the characteristic series of the complete system determined by the m-fold of the system of hyperplane sections of any normal model U of Σ is independent of m if m is sufficiently large. Indeed, if V is any model of Σ in regular correspondence with U (so that both $T: U \to V$ and $T^{-1}: V \to U$ are proper [3]) and if m_0 and n_0 are the integers determined by the pair (U, V) in the sense of Corollary 1, then $\alpha(m) = \beta(n_0)$ for all $m \geqq m_0$. Moreover, $\alpha(m_0)$ is the maximum value of $\alpha(m)$ since $\alpha(m) \leqq \beta(n_0) = \alpha(m_0)$ for all m. This maximum value assumed by the function $\alpha(m)$ is therefore a character of the model U. We denote it by $\delta(U)$. The non-negative numerical character $\delta(U)$ is a *relative invariant* of U, for if U and V are in regular correspondence then $\delta(U) = \alpha(m_0) = \beta(n_0) = \delta(V)$.

COROLLARY 2. *If U and V are normal models of Σ such that*[4] *$U < V$ then $\delta(U) \leqq \delta(V)$.*

PROOF. If U and V are normal and if $U < V$ then $T: U \to V$ is proper. Hence by Theorem 1 there exists an integer n_0 such that $\delta(U) \leqq \beta(n_0) \leqq \delta(V)$, q.e.d.

COROLLARY 3. *If U and V are nonsingular models of Σ then $\delta(U) = \delta(V)$.*

PROOF. It is shown in [8] that there exist models U_1 and V_1 such that the correspondence $T_1: U_1 \to V_1$ is regular and such that U_1 (V_1) is obtained from U (V) by a sequence of quadratic transformations with simple centers. Since such quadratic transformations and their inverses are proper [3, Lemma 3] and since T_1 and T_1^{-1} are proper Corollary 1 implies that $\delta(U) = \delta(U_1) = \delta(V_1) = \delta(V)$, q.e.d.

Since the character $\delta(U)$ has the same value for all nonsingular models of Σ we can regard it as a character of Σ. It is this character

[4] The notation $U < V$ is used to indicate that the birational transformation T^{-1}: $V \to U$ has no fundamental points on V; or equivalently, the local ring $Q(P')$ contains the local ring $Q(P)$, when $P(\subset U)$ and $P'(\subset V)$ are a pair of corresponding points in the birational correspondence T.

which we define to be the irregularity q of the field Σ, and Σ is said to be regular or irregular according as $q=0$, or $q>0$. The use of the term "irregularity" for this character of Σ is justified by the well known work of Castelnuovo (see [**4**, chap. IV]) who has obtained the above results by different methods.

COROLLARY 4. *If U is any normal model of Σ then $\delta(U)\leqq q$.*

PROOF. There exists a nonsingular model V such that $U<V$ (see [**6**]). Hence $\delta(U)\leqq\delta(V)=q$, q.e.d.

3. **Regular models.** Let U be a normal model of Σ, and let $\{A_m\}$ be the system cut out on U by the hypersurfaces of order m of its ambient space.

LEMMA 3. *There exists an integer n_0 such that if $n\geqq n_0$ the complete system $|rA_n|$ cuts out a complete series on the generic curve of $|A_n|$ for any integer $r\geqq 2$.*

PROOF. We regard U as being in regular birational correspondence with itself under the identity correspondence and we identify the systems $\{A_m\}$ and $\{B_m\}$ of the preceding article. By Lemma 1 there exists an integer n_0 such that $|A_{m+n}|$ cuts a complete series on a generic $A_m\in|A_m|$ when $n\geqq n_0$. It follows that $|A_{n+ns}|$ cuts a complete series on a generic A_n if $n\geqq n_0$ and s is an arbitrary positive integer. Hence $|A_{rn}|$ $(=|rA_n|)$ cuts a complete series on A_n if $n\geqq n_0$ and $r\geqq 2$, q.e.d.

THEOREM 2. *If U is a normal model of Σ such that the relative invariant $\delta(U)$ is zero, then there exists an integer h_0 such that if U_h is any derived arithmetically normal model of U belonging to a character of homogeneity $h\geqq h_0$, then the generic hyperplane section of U_h is arithmetically normal. In fact, any irreducible nonsingular hyperplane section of U_h is an arithmetically normal curve.*

PROOF. Let h_0 be the integer n_0 determined in Lemma 3 and let h be a character of homogeneity[5] of U such that $h\geqq h_0$. If $\{A_n\}$ denotes as usual the system cut out on U by the hypersurfaces of order n of its ambient space, and if U_h is the derived arithmetically normal model of U belonging to the character h, then the hypersurfaces of order r in the ambient space of U_h will cut out the complete system $|A_{hr}|$ on U_h, $r=1, 2, \cdots$ [**1**]. Since $h\geqq h_0$, it follows that $|A_{hr}|$ will cut out a complete series on the generic curve of $|A_h|$ if $r\geqq 2$. If necessary,

[5] For the definition and properties of characters of homogeneity see [**5**, articles 20 and 21].

we increase h_0 so that for $h \geqq h_0$ the deficiency of the characteristic series of $|A_h|$ will equal $\delta(U)$. Then since $\delta(U)$ is zero, the system $|A_h|$ will cut a complete series on a generic curve of $|A_h|$. It therefore follows that $|A_{hr}|$ cuts a complete series on a generic curve $A_h \in |A_h|$ for $r=1, 2, \cdots$, so that A_h is arithmetically normal [1].

To show that any nonsingular irreducible hyperplane section of U_h is arithmetically normal we observe that the term "generic" is used to signify an irreducible member A_h of $|A_h|$ of genus π_h as was pointed out in footnote 3. In view of the fact that $|A_h|$ is cut out on U_h by the hyperplanes of its ambient space, it is a straightforward matter to show that any two irreducible nonsingular hyperplane sections of U_h have the same genus.[6] It follows that any such member of $|A_h|$ is generic in the sense in which we have used the term, q.e.d.

Models of Σ which satisfy the hypothesis of Theorem 2 (that is, normal models U such that $\delta(U)=0$) will be called *regular* models of Σ. If Σ is a regular field, then since $0 \leqq \delta(U) \leqq q$, it follows that every normal model of Σ is a regular model. Whether or not irregular fields possess regular models is an open question.

4. **Integral bases.** Let $\mathfrak{o}=k[x_0, x_1, \cdots, x_n]$ be the integrally closed ring of homogeneous coordinates along an arithmetically normal model W of Σ. A triple (y_0, y_1, y_2) of elements of $\mathfrak{o}$ will be called an *admissible* set of independent variables for $\mathfrak{o}$ if (a) y_i is homogeneous of degree one, and (b) $\mathfrak{o}$ is integrally dependent on $k[y_0, y_1, y_2]$. The elements y_0, y_1, y_2 will then be algebraically independent over k, and the quotients y_1/y_0, y_2/y_0 will form a transcendence base for Σ/k. If $\nu=[\Sigma: k(y_1/y_0, y_2/y_0)]$, and if there exist ν elements $\omega_1, \omega_2, \cdots, \omega_\nu$ in $\mathfrak{o}$ which are linearly independent over $k[y_0, y_1, y_2]$ and form a modular base for $\mathfrak{o}$ over $k[y_0, y_1, y_2]$, then the set (y_0, y_1, y_2) will be called a *regular set* of independent variables for $\mathfrak{o}$. The elements $\omega_1, \omega_2, \cdots, \omega_\nu$ are said to be an independent integral base for $\mathfrak{o}$ over $k[y_0, y_1, y_2]$. The ring $\mathfrak{o}$ will be called a *regular ring* if every admissible set of independent variables is regular. It is evident that if the set (y_0, y_1, y_2) is regular then the set (x_0, x_1, x_2), $x_i=\sum a_{ij}y_j$, $a_{ij}\in k$, $|a_{ij}|\neq 0$, is also regular.

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[6] Let $\mathfrak{o}$ be the ring of homogeneous coordinates on U_h and let $\chi(n)$ be the number of independent homogeneous elements of degree n in $\mathfrak{o}$. If A_h is an irreducible hyperplane section of U_h of order ν and genus p, then the fact that the prime ideal of A_h in $\mathfrak{o}$ is a principal ideal, together with the fact that A_h is nonsingular, implies that $\chi(n)=2^{-1}n(n-1)\nu+(\nu-p+1)n+c$, where c is a constant. (See [2, article 6] for details.) Since the function $\chi(n)$ is independent of the curve A_h used to compute it, the fact that all irreducible nonsingular members of $|A_h|$ have the same genus is established.

Let U be a regular model of Σ and let U_h be a derived arithmetically normal model of U such that every irreducible nonsingular hyperplane section of U_h is arithmetically normal. Let $\mathfrak{o}$ be the ring of homogeneous coordinates on U_h and let (y_0, y_1, y_2) be an admissible set of independent variables for $\mathfrak{o}$. Since $\mathfrak{o}$ depends integrally on $k[y_0, y_1, y_2]$, it follows that the ideal $\Sigma\mathfrak{o}y_i$ is irrelevent so that the net of curves, $c_0y_0+c_1y_1+c_2y_2=0$, $c_i\in k$, has no base points on U_h. Since U_h has only a finite number of singularities it follows [9] that the generic curve of this net is nonsingular. Since the quotients y_1/y_0 and y_2/y_0 form a transcendence base for Σ/k, the general curve of the net is irreducible [7]. Hence, after applying a linear transformation to the quantities y_i if necessary, we can assume that (a) the ideals $\mathfrak{o}y_i$ are prime, (b) the curves $y_i=0$, $i=0, 1, 2$, intersect pair by pair at simple points of U_h, and (c) the curves $y_i=0$ are arithmetically normal. It is shown in [2, article 8] that these conditions are sufficient to insure the existence of an independent integral base for $\mathfrak{o}$ over $k[y_0, y_1, y_2]$.[7] Hence every admissible set of independent variables in $\mathfrak{o}$ is regular, and $\mathfrak{o}$ is a regular ring. We can therefore assert the following theorem.

THEOREM 3. *If U is a regular model of Σ then the ring of homogeneous coordinates along a derived arithmetically normal model of U belonging to a sufficiently high character of homogeniety is a regular ring.*

5. **Regular fields.** Let Σ be a regular field and let ξ_1, ξ_2 be an arbitrary transcendence base for Σ/k. The question has been raised by O. Zariski as to whether or not the integral closure $\mathfrak{T}$ in Σ of the ring $k[\xi_1, \xi_2]$ always has an independent integral base over $k[\xi_1, \xi_2]$. Although we cannot answer this question, the following theorem throws some light on the problem.

THEOREM 4. *If (ξ_1, ξ_2) is a transcendence base for the regular field Σ/k, then there exist integers h such that the integral closure in Σ of the ring $R_h=k[\xi_1^h, \xi_2^h]$ has an independent integral base over R_h.*

PROOF. For any h the integral closure in Σ of R_h coincides with the integral closure $\mathfrak{T}$ of R_1. The ring $\mathfrak{T}$ is a finite integral domain, so that there exist elements $\xi_3, \xi_4, \cdots, \xi_n$ in $\mathfrak{T}$ such that $\mathfrak{T}=k[\xi_3, \xi_4, \cdots, \xi_n]$. Let η_0 be a transcendental over Σ and let $\eta_i=\eta_0\xi_i$, $i=1, 2$. If

$$(5.1)\quad \xi_i^\nu + a_{i1}(\xi_1, \xi_2)\xi_i^{\nu-1} + \cdots + a_{i\nu}(\xi_1, \xi_2) = 0, \qquad i = 3, 4, \cdots, n,$$

[7] Condition (b) above is somewhat weaker than the corresponding condition (b) given in [2, article 8]. However, the stronger form was used only as a matter of convenience to simplify the details of the proof, as an examination of the proof will show.

is the equation of integral dependence for ξ_i over R_1 and if m is an integer which is not less than the greatest of the numbers $j^{-1}\cdot\deg a_{ij}$, $i=3, 4, \cdots, n; j=1, 2, \cdots, \nu$, then on multiplying (5.1) by $\eta_0^{m\nu}$ we find that $\eta_0^m\cdot\xi_i$ depends integrally on $k[\eta_0, \eta_1, \eta_2]$. By the transitivity of integral dependence it follows that $\eta_0^m\cdot\xi_i$ depends integrally on $k[\eta_0^m, \eta_1^m, \eta_2^m]$. We let $y_i=\eta_i^m$, $i=0, 1, 2$; $y_j=\eta_0^m\cdot\xi_j$, $j=3, 4, \cdots, n$. The quantities $y_0, y_1, \cdots, y_n$ can be regarded as the coordinates of the general point of a model W of Σ. Moreover, the ring $k[y]$ of homogeneous coordinates on W depends integrally on $k[y_0, y_1, y_2]$.

Let ρ be a character of homogeneity of W and let U be the derived normal model of W belonging to the character ρ. Since Σ is regular, the model U is regular and hence possesses a derived normal model U_σ which has a regular ring $\mathfrak{o}=k[x_0, x_1, \cdots, x_s]$ of homogeneous coordinates. We put $g=\rho\cdot\sigma$ and observe that g is a character of homogeneity of W and that U_σ is a derived normal model W_g of W belonging to the character g.

The quantities y_0^g, y_1^g, y_2^g are homogeneous of degree g when the degree is measured with respect to W, but they are of degree one when measured with respect to W_g. These elements are in $\mathfrak{o}$, and it is not difficult to see that every element of $\mathfrak{o}$ depends integrally on $k[y_0^g, y_1^g, y_2^g]$. In fact, every element of $\mathfrak{o}$ depends integrally on $k[y]$, so that $\mathfrak{o}$ is integral over $k[y_0, y_1, y_2]$. Since this latter ring is integral over $k[y_0^g, y_1^g, y_2^g]$, it follows that (y_0^g, y_1^g, y_2^g) is an admissible set of independent variables for $\mathfrak{o}$. Since $\mathfrak{o}$ is a regular ring, the set (y_0^g, y_1^g, y_2^g) is a regular set of independent variables.

After applying a nonsingular linear transformation with coefficients in k we can assume that $x_i=y_i^g$, $i=0, 1, 2$. Then $x_i/x_0=\xi_i^{gm}$, $i=1, 2$. If $h=gm$, then the ring $\mathfrak{o}_0=k[x_1/x_0, x_2/x_0, \cdots, x_s/x_0]$ depends integrally on R_h, and since $\mathfrak{o}_0$ is integrally closed, $\mathfrak{o}_0=\mathfrak{T}$. By [2, Theorem 2.1] the fact that $\mathfrak{o}$ has an independent modular base over $k[x_0, x_1, x_2]$ consisting of $\mu=[\Sigma: k(x_1/x_0, x_2/x_0)]$ elements implies that $\mathfrak{o}_0$ has an independent integral base over $k[x_1/x_0, x_2/x_0]$, that is, $\mathfrak{T}$ has an independent integral base over R_h, q.e.d.

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