A BOUND FOR THE MEAN VALUE OF A FUNCTION

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Let $f(t)$ be a bounded measurable function defined when $0 \leqq t \leqq \pi$. The Fourier sine series associated with $f(t)$ is

$$\sum_{n=1}^{\infty} b_n \sin nt, \qquad b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt \, dt.$$

We shall be interested in this paper in establishing a bound for the mean value[1]

$$a = \frac{1}{\pi} \int_0^{\pi} f(t) dt$$

when $f(t)$ is such that one of the coefficients b_n vanishes.

We can suppose without essential loss of generality that $|f(t)| \leqq 1$. Since $b_{2n} = 0$ whenever $f(t)$ is constant, it is clear that the only conclusion on a that can be drawn from the inequality $|f(t)| \leqq 1$ and the equality $b_{2n} = 0$ is that $|a| \leqq 1$, and this conclusion is valid whether b_{2n} vanishes or not. Hence we shall restrict attention to b_{2n+1}. For the same reason we shall not discuss the vanishing of the coefficient

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt \, dt$$

of the Fourier cosine series of $f(t)$.

Suppose that $b_{2n+1} = 0$. Define a positive number y by the equation $y = \sin [\sec^{-1}(2n+2)]$ where the $\sec^{-1}$ lies between 0 and $\pi/2$. Let E be the sum of the intervals

$$\frac{2p\pi + \sin^{-1} y}{2n+1} \leqq t \leqq \frac{(2p+1)\pi - \sin^{-1} y}{2n+1} \qquad (p = 0, 1, \cdots, n),$$

where the $\sin^{-1}$ lies between 0 and $\pi/2$. Then it is clear that

$$(1) \qquad \begin{aligned} &\sin (2n+1)t \geqq y && \text{if } t \text{ is in } E, \\ &\sin (2n+1)t < y && \text{if } t \text{ is not in } E. \end{aligned}$$

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[1] The importance of the concept of mean value in the study of Fourier series can be seen by consulting Bohr [1, pp. 7–29]. Numbers in brackets refer to the bibliography at the end of the paper.

Now let $f_0(t)=-1$ if t is in E and $f_0(t)=+1$ if t is not in E. It follows from the definitions of y and E that

$$\int_0^\pi f_0(t)\sin(2n+1)t\,dt = 0,$$

and that the mean value of $f_0(t)$ is

$$\begin{aligned} a_0 &= 1-(2/\pi)\text{ meas } E \\ &= [4(n+1)/(2n+1)\pi]\sec^{-1}(2n+2) - 1/(2n+1). \end{aligned}$$

We shall now prove that if $f(t)$ is an arbitrary real-valued measurable function such that $|f(t)|\leqq 1$ and $b_{2n+1}=0$, then $|a|\leqq a_0$. Let $g(t)=f(t)-f_0(t)$. Then $0\leqq g(t)$ on E and $0\geqq g(t)$ on the complement cE of E. By virtue of relations (1) we conclude that

$$\int_E g(t)\sin(2n+1)t\,dt \geqq y\int_E g(t)dt,$$

$$\int_{cE} g(t)\sin(2n+1)t\,dt \geqq y\int_{cE} g(t)dt.$$

Adding and remembering that $b_{2n+1}=0$ for both f and f_0 we see that

$$0 \geqq y\int_0^\pi g(t)dt,$$

with equality if and only if $g(t)=0$ almost everywhere. Since $y>0$, we have that

$$\int_0^\pi f(t)dt \leqq \int_0^\pi f_0(t)dt, \tag{2}$$

with equality if and only if $f(t)=f_0(t)$ almost everywhere.

Now let $h(t)=f(t)+f_0(t)$. Then $0\geqq h(t)$ on E, $0\leqq h(t)$ on cE, and so

$$\int_E h(t)\sin(2n+1)t\,dt \leqq y\int_E h(t)dt,$$

$$\int_{cE} h(t)\sin(2n+1)t\,dt \leqq y\int_{cE} h(t)dt,$$

$$0 \leqq y\int_0^\pi h(t)dt,$$

$$-\int_0^\pi f(t)dt \leqq \int_0^\pi f_0(t)dt, \tag{3}$$

with equality if and only if $f(t) = -f_0(t)$ almost everywhere. Combining the inequalities (2) and (3) we conclude that *when* $f(t)$ *is a real-valued measurable function such that* $|f(t)| \leqq 1$ *and* $b_{2n+1}=0$, *then*

$$(4) \qquad |a| = \left| \frac{1}{\pi} \int_0^\pi f(t)dt \right| \leqq \frac{4(n+1)}{(2n+1)\pi} \sec^{-1}(2n+2) - \frac{1}{2n+1},$$

with equality if and only if $f(t) = \pm f_0(t)$ *almost everywhere.*

In particular, if $b_1=0$, then $|a| \leqq 1/3 = .3333$, while if $b_3=0$, then $|a| \leqq .7855$. The right-hand side of the inequality (4) approaches unity as n approaches infinity.

This conclusion may be extended to complex functions $f(t)$ as follows. Let $f(t) = f_1(t) + if_2(t)$, where $f_1(t)$ and $f_2(t)$ are real. There exist real numbers x and y such that

$$x^2 + y^2 = 1, \qquad x\int_0^\pi f_2(t)dt + y\int_0^\pi f_1(t)dt = 0.$$

Hence it is true that the mean value of $f(t)$ has the same absolute value as the mean value of the real function $xf_1(t) - yf_2(t)$. This real function has a Fourier coefficient b_{2n+1} equal to zero since this is true for both $f_1(t)$ and $f_2(t)$ and is bounded by one since $f(t)$ is and $x^2+y^2 = 1$. Since the inequality (4) is valid for $xf_1 - yf_2$, it is therefore true for $f(t)$. Moreover since equality for $xf_1 - yf_2$ implies that $xf_1 - yf_2 = \pm f_0(t)$, equality for $f(t)$ implies that $f(t) = cf_0(t)$ where c is a constant of absolute value unity.

Bibliography

1. H. Bohr, *Almost periodic functions*, New York, Chelsea, 1947.

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