

ON THE EXISTENCE OF THE INVERSE OPERATION IN ALTERNATION GROUPOIDS¹

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Introduction. An Abelian quasigroup S may be defined as a set of elements a, b, c, \dots for which the following postulates hold:

Postulate I. There is an equivalence relation in S , denoted by “=” —that is, equality of elements is reflexive, symmetric, and transitive. There is a binary operation in S , denoted by $\#$ or, when convenient, by the notation of multiplication. It is understood that this implies S is closed with respect to $\#$, and that $\#$ is uniquely defined in S —that is, if $a = b$ and $c = d$, then $ac = bd$.

Postulate II. $\#$ is an alternation—that is, if a, b, c , and d are elements in S , $(ab)(cd) = (ac)(bd)$.

Postulate III. Each element in S is proper (see §2 for definitions).

D. C. Murdoch [1, p. 516]² pointed out that an Abelian quasigroup is a natural generalization of an Abelian group. The same author [2, Theorem 11] proved that in an Abelian quasigroup it is always possible to define a new operation under which the elements of the quasigroup form an Abelian group. R. H. Bruck [3, Theorem 12] in a sense completed the theory by showing how all Abelian quasigroups may be derived from Abelian groups.

In this paper a set of elements S for which postulates I and II hold is called an alternation groupoid. A series of extensions of an alternation groupoid is described which for certain groupoids leads to an Abelian quasigroup (Corollary 5.5). This imbedding process has as a special case the well known procedure for imbedding commutative semigroups in groups or, more generally, the procedure for imbedding a space S , in which there is a commutative and associative operation, in a space S' in such a way that each “regular” element in S has an inverse in S' (see, for example, [4, p. 24]). It should be noted that A. Malcev [5, §2] has given an example of a noncommutative semigroup which cannot be imbedded in a group.

Some interesting examples of Abelian quasigroups and alternation groupoids are given in §1.

1. Examples of alternation groupoids. The examples of Abelian

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² Numbers in brackets refer to references cited at the end of the paper.

quasigroups that have appeared in print fall mainly in two classes—those based on a finite set of elements and those based on a properly chosen set of numbers by defining $x \# y$ as $ax + by + c$, or $cx^a y^b$, where a , b , and c are constants (in particular, $\#$ can be ordinary addition, subtraction, multiplication, division, or the operation of taking an arithmetic or geometric mean). These examples give little indication of the variety of form found in alternation groupoids.

Moreover, the geometric interest of alternation groupoids can be given greater emphasis. For instance, each Abelian quasigroup furnishes an analogue of the theorem that the midpoints of a quadrilateral determine a parallelogram. To see this, let S be the set of points in the plane and let $\#$ be the operation of taking the midpoint. Consider the associated group operation defined by Murdoch—for fixed elements r and s in S , $x \circ y$ is defined as uv where $x = ur$ and $y = sv$. Under \circ , the elements of S form a commutative group which has sr as its unit element. Thus $ur \circ sv = uv \circ sr$ and sr , sv , uv , and ur are the vertices of a parallelogram. In the case of example *E* below, this process gives an interesting construction for an ordinary parallelogram. From example *F*, we obtain an odd construction for projective addition by following the steps in the definition of operation \circ .

A. Let S be the set of positive integers. Let α, β, γ , and δ be integers such that $\alpha, \gamma, \alpha + \beta$, and $\gamma + \delta$ are positive and $\alpha\delta + \beta = \gamma\beta + \delta$. Define $x \# y$ as $\max [\alpha x + \beta, \gamma y + \delta]$.

B. Let a and b be points in the projective plane. Let S be the set of points of the plane not on the line ab . Define $x \# y$ as the point of intersection of the lines xa and yb .

C. Let a be a point in the complex plane and S the set of points in the plane excluding a and ∞ . Let $x \# y = \overline{x - a}(y - a) / \overline{y - a} + a$, the reflection of x with respect to the line ay .

D. Let a and S be chosen as in the preceding example but let $x \# y = (y - a)\overline{y - a} / \overline{x - a} + a$, the inverse of x with respect to the circle of center a which passes through y .

E. Let S be the set of points in the Euclidean plane and a, b , and c fixed elements of S . Let $x \# y$ be defined as the point z which determines a triangle xyz strictly similar to the triangle abc in such a way that vertices x, y, z correspond respectively to vertices a, b, c .

F. Let S be the set of points on the projective line, s a fixed element of S , and r a fixed real number. Let $x \# y$ be defined as the point z for which the double ratio $(sxyz) = r$. Defined in numerical terms $x \# y = [xy - s\{x(1 - r) + yr\}] / [xr + y(1 - r) - s]$ and when $r = 1/2$ we see that the arithmetic and harmonic means are special cases of this alternation.

2. Definitions and notation. In the pages following S denotes an alternation groupoid. It is well known that if S has a unit element the alternation is commutative and associative and that, conversely, if an operation is commutative and associative it is an alternation.

An element a in S is called left regular if, for all elements x and y in S , $ax = ay$ implies $x = y$. In much of the material which follows we consider only "left" conditions and their consequences. We shall omit statements with corresponding "right" or "left and right" conditions. Here, for example, right regularity of an element has an obvious definition and to say an element is regular means that it is both left and right regular.

An element a in S is called left proper if, for all elements b in S , $ax = b$ has a unique solution, x , in S . It follows that each element left proper in S is left regular in S . An element θ in S is called left null if $\theta x = \theta$ for all elements x in S . If S has more than one element, a left null element is not left regular.

An element, $\phi(x)$, is a power of x if it belongs to the set of elements $\Phi(x)$ defined recursively by the rules:

- (i). x belongs to $\Phi(x)$,
- (ii). If y and z belong to $\Phi(x)$, yz belongs to $\Phi(x)$.

Murdoch has shown that in an alternation groupoid a power of a power of x is a power of x and that the following laws hold:

$$\phi(xy) = \phi(x)\phi(y), \quad \phi_1\phi_2(x) = \phi_2\phi_1(x).$$

An element of S is said to be left regular* if all of its powers are left regular. We denote by L , L^* , and L^{**} respectively the sets of elements in S which are left regular, left regular*, and left proper in S . The sets of elements with the corresponding right properties are denoted by R , R^* , and R^{**} respectively. Finally, we let $P = L^{**} \cap R^{**}$ and $Q = L^* \cap R^*$.

3. Relations between subsets of S .

THEOREM 3.1. L^* is an alternation groupoid.

PROOF. It is sufficient to show that L^* is closed under $\#$. Let $\phi(ab)$ be a power of ab where $a, b \in L^*$. Assume that for some $x, y \in S$, $\phi(ab)x = \phi(ab)y$. Then

$$\begin{aligned} [\phi(aa)\phi(aa)][\phi(bb)x] &= [\phi(aa)\phi(bb)][\phi(aa)x] \\ &= \phi\{(aa)(bb)\}[\phi(aa)x] = \phi\{(ab)(ab)\}[\phi(aa)x] \\ &= [\phi(ab)\phi(aa)][\phi(ab)x] = [\phi(ab)\phi(aa)][\phi(ab)y]. \end{aligned}$$

Similarly,

$$[\phi(aa)\phi(aa)][\phi(bb)y] = [\phi(ab)\phi(aa)][\phi(ab)y].$$

We may cancel to obtain $\phi(bb)x = \phi(bb)y$ and $x = y$. Hence L^* is closed under $\#$.

COROLLARY 3.2. *L is an alternation groupoid if and only if $L = L^*$.*

THEOREM 3.3. *L^{**} is an alternation groupoid.*

PROOF. We must show that L^{**} is closed under $\#$. For h and $k \in L^{**}$ and $a \in S$, we find a solution x of the equation $(hk)x = a$ by choosing y, z, w and x so that $hy = a, hz = h, kw = y$, and $x = zw$.

COROLLARY 3.4. $L^{**} \subseteq L^* \subseteq L$.

PROOF. The second inclusion relation follows from the definitions of L and L^* . Since the theorem implies that all powers of a left proper element are left proper and since a left proper element is left regular, the first inclusion relation is valid.

THEOREM 3.5. *If $ab \in L$, then $b \in L$. If in addition $R \neq 0, a \in L$.*

PROOF. If we have $bx = by$, $(ab)[(ab)x] = [a(ab)](bx) = [a(ab)](by) = (ab)[(ab)y]$ and $x = y$. If $r \in R$ and $ax = ay$, $(ab)(xr) = (ab)(yr)$, and $x = y$.

COROLLARY 3.6. *If $ab \in L^*, b \in L^*$. If in addition $R \neq 0, a \in L^*$.*

THEOREM 3.7. *If $L^* \neq 0$ and $R^* \neq 0$, then $L^* = R^* = Q$.*

PROOF. Consider elements l and r in L^* and R^* respectively and p , a power of r . We first prove that $lp \in L$. Suppose $(lp)x = (lp)y$. Then

$$\begin{aligned} [(ll)x][(lp)(pp)] &= [(lp)x][(lp)(pp)] \\ &= [(lp)y][(lp)(pp)] = [(ll)y][(lp)(pp)] \end{aligned}$$

and $x = y$. By Theorem 3.5, $lp \in L$ implies $p \in L$. Hence $r \in L^*$ and $R^* \subseteq L^*$. Similarly, we prove $L^* \subseteq R^*$.

LEMMA 3.8. *If $hk \in L^{**}$, $h(hk) \in L$.*

PROOF. Let $hk = l$. If $(hl)x = (hl)y$ choose z and w so $lz = x, lw = y$. Then

$$\begin{aligned} [(ll)(ll)][l(lz)] &= [(ll)(ll)][(hl)(kz)] \\ &= [(ll)(hl)][(lk)x] = [(ll)(lk)][(hl)y]. \end{aligned}$$

Similarly,

$$[(l)(l)][l(lw)] = [(l)(lk)][(hl)y].$$

We cancel to obtain $z = w$. Hence $x = y$.

THEOREM 3.9. *If $hk \in L^{**}$, $k \in L^{**}$.*

PROOF. Let $l = hk$. To solve the equation $kx = a$ choose y and x so $ly = (hl)a$ and $lx = y$. Then $(hl)(kx) = (hk)(lx) = ly = (hl)a$. By the previous lemma, $kx = a$.

COROLLARY 3.10. *All elements left proper in S are left proper in L^{**} .*

COROLLARY 3.11. *If $L^{**} \neq 0$ and $R^{**} \neq 0$, then $L^{**} = R^{**} = P$.*

PROOF. To prove, say, $R^{**} \subseteq L^{**}$, consider $r \in R^{**}$. Since there exists an element x such that $xr = l$, it follows from the theorem that $r \in L^{**}$.

LEMMA 3.12. *If $L^{**} \neq 0$, $a \in L^*$, $b \in L$, and $c \in R$, then $ab \in L$ and $bc \in R$.*

PROOF. If $l \in L^{**}$, l and $al \in L^*$. If $(ab)x = (ab)y$, let $lz = x$ and $lw = y$. Then $(al)(bz) = (al)(bw)$, $z = w$, and $x = y$. Thus $ab \in L$. It follows as a particular case of this result that $lb \in L$. Hence if $x(bc) = y(bc)$, $(lb)(zc) = (lb)(wc)$, $z = w$, and $x = y$.

LEMMA 3.13. *If $L^{**} \neq 0$ and $R \neq 0$, $L \subseteq R$.*

PROOF. Let $b \in L$ and $c \in R$. By Lemma 3.12, $bc \in R$. By the "right" dual of Theorem 3.5, $bc \in R$ implies $b \in R$.

LEMMA 3.14. *If $R \neq 0$, $L^{**} \subseteq R^*$.*

PROOF. Let $r \in R$ and let $\phi(l)$ be a power of $l \in L^{**}$. By Lemma 3.12, $\phi(l)r \in R$. Hence $\phi(l) \in R$ and $l \in R^*$.

THEOREM 3.15. *If $L^{**} \neq 0$ and $R \neq 0$, $L^{**} \subseteq Q \subseteq L \subseteq R$.*

PROOF. The inclusion relations are consequences of Corollary 3.4, Lemmas 3.13 and 3.14, and Theorem 3.7.

THEOREM 3.16. *If $P \neq 0$, $P \subseteq Q = L = R$.*

PROOF. If $P \neq 0$, neither L nor R^{**} is null. By the dual of Theorem 3.15, $R^{**} \subseteq Q \subseteq R \subseteq L$. From these relations, from Theorem 3.15, and from Corollary 3.11, the conclusion of this theorem follows.

As an immediate consequence of our definitions, we have the following relations.

THEOREM 3.17. *If S_1 and S_2 are alternation groupoids and $S_1 \subseteq S_2$, then $L_1 \supseteq S_1 \cap L_2$ and $L_1^* \supseteq S_1 \cap L_2^*$.*

4. The imbedding theorem. This section is devoted to a proof of Theorem 4.1. If S has no left regular* element we may satisfy the theorem by choosing S_∞ as S . Hence throughout the section we assume $L^* \neq 0$. Notations such as L_∞^* , the set of left regular* elements in S_∞ , have an obviously meaning. We show in Theorem 4.11 that the following theorem cannot be made stronger by replacing left regularity* with mere left regularity.

THEOREM 4.1. *An alternation groupoid S can be imbedded in an alternation groupoid S_∞ with the following properties:*

- (A). L^* is isomorphic to a subset of L_∞^* .
- (B). Left regular* elements in S_∞ are left proper in S_∞ .
- (C). S_∞ is a minimal extension of S into an alternation groupoid with properties A and B.

DEFINITION. S_1 , the first left extension of S , is a groupoid whose elements are ordered pairs (a, b) , where $a \in L^*$ and $b \in S$. In S_1 we define a binary operation $\#$ and the equality of elements as follows:

- (i). $(a, b) \# (c, d) = (ac, bd)$.
- (ii). $(a, b) = (c, d)$ if and only if, in S , $(aa)d = (ac)b$.

When the notation (a, b) is used it is assumed a is left regular*. By $\phi(a, b)$ we shall mean $\phi((a, b))$. This power obviously has the value $(\phi(a), \phi(b))$.

LEMMA 4.2. $\#$ is a binary operation in S_1 .

PROOF. S_1 is clearly closed with respect to $\#$. To show the operation is uniquely defined, we assume $(a, b) = (c, d)$ and $(e, f) = (g, h)$. It follows that

$$[(ae)(ae)](dh) = [(aa)d][(ee)h] = [(ac)b][(eg)f] = [(ae)(cg)](bf),$$

$$(ae, bf) = (cg, dh), \text{ and } (a, b)(e, f) = (c, d)(g, h).$$

LEMMA 4.3. *If $(a, b) = (c, d)$, then $(xa)d = (xc)b$ for all x in S . Conversely, if $x \in L^*$ and $(xa)d = (xc)b$, then $(a, b) = (c, d)$.*

PROOF. Since $(aa)d = (ac)b$,

$$[(aa)a][(xc)b] = [(ax)a][(ac)b] = [(ax)a][(aa)d] = [(aa)a][(xa)d].$$

Since $(aa)a$ is left regular, the first statement is proved. The second statement has a similar proof.

LEMMA 4.4. *The relation " $=$ " is an equivalence relation in S_1 .*

PROOF. That the equality is reflexive is trivial; that it is symmetric

follows from the previous lemma. To prove transitivity, assume $(a, b) = (c, d)$ and $(c, d) = (e, f)$. It follows that

$$\begin{aligned} [(cc)(cc)][(aa)f] &= [(cc)(aa)][(cc)f] \\ &= [(ca)(ca)][(ce)d] = [(ca)(ce)][(ca)d] \\ &= [(cc)(ae)][(cc)b] = [(cc)(cc)][(ae)b]. \end{aligned}$$

Cancellation of $(cc)(cc)$ gives the condition that $(a, b) = (e, f)$.

We have shown that S_1 satisfies postulate I. We can show without difficulty that postulate II is also satisfied, and we state the following theorem by way of summary.

THEOREM 4.5. S_1 is an alternation groupoid.

THEOREM 4.6. $(a, x) \in L_1^*$ [resp. R_1^*] if and only if $x \in L^*$ [resp. R^*].

PROOF. Proofs of both statements are similar—that the conditions are sufficient is easily seen. To prove necessity assume, say, $x \in L^*$. There exist elements y and z and a power of x such that $y \neq z$ but $\phi(x)y = \phi(x)z$. Then $\phi(a, x)(a, y) = (\phi(a)a, \phi(x)y) = (\phi(a)a, \phi(x)z) = \phi(a, x)(a, z)$. If $\phi(a, x) \in L_1$, $(a, y) = (a, z)$ and $y = z$. Hence $\phi(a, x) \notin L_1$ and $(a, x) \in L_1^*$.

COROLLARY 4.7. $R_1^* = 0$ if and only if $R^* = 0$.

LEMMA 4.8. $(a, ax) = (b, by)$ if and only if $x = y$.

THEOREM 4.9. If $e \in L^*$, the correspondence $x \sim (e, ex)$ defines an isomorphism of S to a subset of S_1 under which L^* is isomorphic to a subset of L_1^* .

PROOF. That the correspondence determines an isomorphism follows from Lemma 4.8. From Theorems 3.1 and 4.6 and from Corollary 3.6, $(e, ex) \in L_1^*$ if and only if $x \in L^*$.

Whenever it proves convenient, we identify S with the subset of S_1 to which it is isomorphic.

We have shown that S_1 satisfies property (A) of Theorem 4.1. That we cannot expect it to satisfy properties (B) and (C) follows from consideration of the example where S is the set of real numbers on a finite interval and where $x \# y$ is defined as $(x+y)/2$. S_1 does, however, satisfy a weakened form of Theorem 4.1 given below.

THEOREM 4.10. S_1 can be uniquely characterized as a minimal extension of S into an alternation groupoid which satisfies property (A) of Theorem 4.1 and the following property: If $a \in L^*$ and $b \in S$, then $ax = b$ has a solution x in S_1 .

PROOF. Since $(e, ea)(a, b) = (e, eb)$, (a, b) is the required solution. If S is imbedded in a space T with this property and property (A), let T_1 be the subset of T made up of elements which are solutions of the equations $ax = b$ where $a \in L^*$ and $b \in S$. Each element of T_1 can be represented as an ordered pair $[a, b]$. It is a routine matter to show that the correspondence $(a, b) \sim [a, b]$ maps S_1 isomorphically on T_1 .

DEFINITION. S_n , the n th left extension of S , is defined as the first left extension of S_{n-1} , $n = 2, 3, \dots$. S_∞ , the left extension of S , is defined as the union of the spaces S_n .

Thus by repeating the imbedding process just discussed, we obtain $S \subseteq S_1 \subseteq S_2 \subseteq \dots \subseteq S_\infty$. It is evident that S_∞ is an alternation groupoid and that $L^* \subseteq L_\infty^*$. If $\alpha \in L_\infty^*$ and $\beta \in S_\infty$, there exists (see Theorem 4.9) an integer n such that $\alpha \in L_n^*$ and $\beta \in S_n$. By Theorem 4.10, $\alpha x = \beta$ has a solution x in $S_{n+1} \subseteq S_\infty$ and thus $\alpha \in L_\infty^{**}$. Finally it is not difficult to show S_∞ is the minimal extension described in Theorem 4.1, and the proof of that theorem is now complete.

If we measure the worth of an extension T of S by the number of elements of S which are left proper in T , the following theorem shows that the left extension of S , described in this section, is the best possible. The theorem is a direct consequence of Theorem 3.17 and Corollary 3.4.

THEOREM 4.11. *There exists no extension of an alternation groupoid S into an alternation groupoid T such that an element not left regular* in S is left proper, or even left regular*, in T .*

THEOREM 4.12. *The properties of being right regular*, right proper and left proper are preserved under left extension.*

PROOF. That right regularity* is preserved follows from Theorems 3.7 and 4.6 (it can be shown that if $L^{**} \neq 0$, even right regularity is preserved). To show right properness is preserved it will be sufficient to find an element of S_1 which when multiplied on the right by (e, er) gives (a, f) , where $r \in R^{**}$, e and $a \in L^*$, and $f \in S$. Choose g and d so $f = g(rr)$ and $a = dr$. Applying Theorem 3.7 and the dual of Corollary 3.6, we have $a \in R^*$, $d \in R^*$, and $d \in L^*$. Hence (d, g) is an element in S_1 . It proves to be the element we sought. That left properness is preserved is proved similarly.

Before leaving this section we note the inclusion relations which have been established by Theorems 3.17, 4.1, 4.12, and Corollary 3.4.

THEOREM 4.13. $L^{**} \subseteq L_\infty^{**} = L_\infty^*$, $R^{**} \subseteq R_\infty^{**}$, $L^* = S \cap L_\infty^*$, and $R^* = S \cap R_\infty^*$.

5. **General extensions of S .** It is necessary to introduce new notation. Let S_0 be S . Let $S_{\infty 0}$ be S_∞ and let S_{n0} be S_n , the n th left extension of S . Let $S_{0\infty}$ and S_{0n} be respectively the right extension and the n th right extension of S . Finally let S_{mn} be $(S_{m0})_{0n}$ where m and n are any non-negative integers or where either m and n or both are replaced by the symbol ∞ . It is evident that $S_{n+10} = (S_{n0})_{10}$ and $S_{0n+1} = (S_{0n})_{01}$. It is consistent with what we have done in §4 to define $S_{10} = S [S_{01} = S]$ in case $L^* = 0 [R^* = 0]$.

LEMMA 5.1. S_{11} is isomorphic to $(S_{01})_{10}$.

PROOF. If $L^* = 0, L_{01}^* = 0$ by the dual of Corollary 4.7, and $(S_{01})_{10} = S_{01} = (S_{10})_{01} = S_{11}$. Similarly if $R^* = 0, S_{11} = (S_{01})_{10}$. We now assume $L^* = R^* \neq 0$ (see Theorem 3.7) and let $x_n \in S_{11}$. We define a correspondence $x_n \sim y_n$ where $y_n \in (S_{01})_{10}$ by means of six equations. By Theorem 4.10 and its dual, there exist elements r_n in R_{10}^*, p_n in S_{10}, a_n and b_n in L^* , and c_n and g_n in S , such that $x_n r_n = p_n$ in $S_{11}, a_n r_n = c_n$ in S_{10} , and $b_n p_n = g_n$ in S_{10} . Since a_n and r_n are regular* in S_{10}, c_n is regular* in S_{10} and hence in S (see Theorems 3.1, 3.17, and 4.12). Let s_n and q_n be elements in S_{01} such that $s_n a_n = b_n$ and $q_n c_n = g_n$ in S_{01} . By the dual of Corollary 3.6, s_n is right regular* and hence regular* in S_{01} . There exists an element y_n in $(S_{01})_{10}$ such that $s_n y_n = q_n$ in $(S_{01})_{10}$. That a different choice of elements r_n, p_n, a_n, b_n, c_n and g_n will lead to the same element y_n will follow from our proof that $x_m = x_n$ if and only if $y_m = y_n$. We apply repeatedly Lemma 4.3, Theorem 4.10, and their duals. Assume, say, $x_1 = x_2$. By the definition of equality (in the 1st right extension of S_{10}), $p_2(r_1 r_1) = p_1(r_2 r_1)$ in S_{10} . But

$$[b_1(a_2 a_1)] [p_1(r_2 r_1)] = g_1(c_2 c_1)$$

and

$$[b_2(a_1 a_1)] [p_2(r_1 r_1)] = g_2(c_1 c_1).$$

Hence, in S ,

$$[\{b_1(a_1 a_1)\} \{b_1(a_2 a_1)\}] \{g_2(c_1 c_1)\} = [\{b_1(a_1 a_1)\} \{b_2(a_1 a_1)\}] \{g_1(c_2 c_1)\}$$

or, restated,

$$\{(b_1 b_1) g_2\} [\{(a_1 a_2) c_1\} \{(a_1 a_1) c_1\}] = \{(b_1 b_2) g_1\} [\{(a_1 a_1) c_2\} \{(a_1 a_1) c_1\}].$$

This equality implies that the solutions α and β of the equations

$$\begin{aligned} \alpha [(a_1 a_1) c_2] &= (b_1 b_1) g_2, \\ \beta [(a_1 a_2) c_1] &= (b_1 b_2) g_1 \end{aligned}$$

are equal in S_{01} . These solutions are $(s_1s_1)q_2$ and $(s_1s_2)q_1$. Their equality implies that $y_1 = y_2$ in $(S_{01})_{10}$. Since these steps are reversible, the correspondence of x_n to y_n is one to one. It remains to prove that the correspondence is an isomorphism. If $x_1 \sim y_1$ and $x_2 \sim y_2$, let $x_1x_2 = x_3$, $r_1r_2 = r_3$, $p_1p_2 = p_3$, and so on. It follows that $x_1x_2 = x_3 \sim y_3 = y_1y_2$.

For convenience, we identify S_{11} with $(S_{01})_{10}$ and use equality below instead of an isomorphism notation.

LEMMA 5.2. $(S_{0n})_{m0} = (S_{m0})_{0n}$ for $m, n = 0, 1, 2, \dots$.

PROOF. As in Lemma 5.1 we may assume $L^* = R^* \neq 0$. Clearly the equality holds if either m or $n = 0$. We assume $m \neq 0$ and $n \neq 0$ and proceed by induction from the result of Lemma 5.1. If $S_{h1} = (S_{01})_{h0}$, then $S_{h+1\ 1} = (S_{h+1\ 0})_{01} = [(S_{h0})_{10}]_{01} = [(S_{h0})_{01}]_{10} = [(S_{01})_{h0}]_{10} = (S_{01})_{h+1\ 0}$, and $S_{m1} = (S_{01})_{m0}$ for $m = 1, 2, \dots$. Similarly, if $S_{mh} = (S_{0h})_{m0}$, $S_{m\ h+1} = (S_{0\ h+1})_{m0}$, and the lemma is proved.

THEOREM 5.3. *The right extension of the left extension of S is isomorphic to the left extension of the right extension of S .*

PROOF. It is easy to validate the two equalities below which have not yet been established. $S_{\infty\infty} = (S_{\infty 0})_{0\infty} = \bigcup_n (\bigcup_m S_{m0})_{0n} = \bigcup_n \bigcup_m (S_{m0})_{0n} = \bigcup_m \bigcup_n (S_{0n})_{m0} = \bigcup_m (\bigcup_n S_{0n})_{m0} = (S_{0\infty})_{\infty 0}$.

The following theorem follows principally from Theorems 4.1, 4.13, and their duals.

THEOREM 5.4. *In Theorem 4.1 we may replace S_∞ by $S_{\infty\infty}$ and left regularity* properties by the corresponding regularity* properties.*

COROLLARY 5.5. *An alternation groupoid S may be imbedded in an Abelian quasigroup if and only if all elements of S are regular.*

We may classify alternation groupoids according to whether or not they have left regular* or right regular* elements. Members in different classes, as we have seen, have distinctly different types of extensions.

I. $L^* = R^* = 0$.

Here $S_{\infty\infty} = S$ has no left or right proper element. An example: example A of §2 with $\alpha = 5, \beta = -2, \gamma = 3$, and $\delta = -1$. In this example, 1 is regular and $2 \in R$.

II. $L^* \neq 0, R^* = 0$.

Here $S_{\infty\infty} = S_\infty$ and in S_∞ we have $L_{\infty}^{**} = L_{\infty}^* \subseteq L_\infty$ and $R_\infty = 0$. Example: let S be the set of points (x, y) in the square $-1 \leq x \leq 1, -1 \leq y \leq 1$ and define $(x_1, y_1) \# (x_2, y_2)$ as $(x_1/2 + x_2/2, y_2)$.

III. The situation dual to that of II.

IV. $L^* = R^* \neq 0$.

Here $S_{\infty\infty}$ is such that $P_{\infty\infty} = Q_{\infty\infty} = L_{\infty\infty} = R_{\infty\infty}$. (If $S_{\infty} \neq S_{\infty\infty}$, $L_{\infty}^{**} = Q_{\infty} \subseteq L_{\infty} \subseteq R_{\infty}$ and $R_{\infty}^{**} = 0$.) Example: example A of §2 with $\alpha = 3$, $\beta = -2$, $\gamma = 2$, and $\delta = -1$. In this example, $1 \in Q$ and $2 \in R$.

6. Conditions which insure that $L = L^*$. Since a necessary and sufficient condition that S have an extension with left proper elements is that $L^* \neq 0$, it is of interest to find what conditions insure that left regular elements are left regular* or, in other words, that the left regular elements are closed under $\#$. This closure property holds in many special alternation groupoids—for example, an idempotent groupoid or a groupoid in which $\#$ is associative. The following theorem shows in general how badly an alternation groupoid must behave if $L \neq L^*$.

THEOREM 6.1. *If S is an alternation groupoid in which $L \neq 0$ and $L \neq L^*$, then:*

- (1). *S is infinite.*
- (2). *There is no proper element in S .*
- (3). *If S has a right proper element, S has left regular elements which are not right regular.*
- (4). *If S has a left proper element, then either S has no right regular element or S has right regular elements which are not left regular.*
- (5). *S has left regular elements whose squares are not left regular.*
- (6). *There are elements in S neither left regular nor left null.*

PROOF. One proves property (1) just as one proves every finite integral domain is a field. Assume S contains a finite number of elements $x_1, x_2, x_3, \dots, x_n$ where x_1 , say, $\in L$. Since no two elements in the set $x_1x_1, x_1x_2, \dots, x_1x_n$ are equal, this set consists of the elements of S in some order. Hence $x_1 \in L^{**}$, and by Corollary 3.4, $x_1 \in L^*$.

Property (2) is a direct consequence of Theorem 3.16. To prove property (5) we show that if all squares of left regular elements are left regular, L is closed under $\#$. Let a and $b \in L$ and let $(ab)x = (ab)y$. Then $[(aa)(aa)][(bb)x] = [(ab)(aa)][(ab)x] = [(aa)(aa)][(bb)y]$. Since bb, aa , and $(aa)(aa) \in L$, $x = y$.

We prove property (3) by showing that if it does not hold property (5) does not hold. Let $a \in L$ and assume $L \subseteq R$. Then, by the dual of Lemma 3.12, $aa \in L$. Property (4) is proved similarly. Property (6) is a consequence of Theorem 6.6 to a proof of which the remainder of this section is devoted.

Let N, N_1 and N_2 denote respectively the sets of null, left null, and right null elements in S . The following lemma holds for any groupoid satisfying postulate I of the introduction.

LEMMA 6.2. *If $N_1 \neq 0$ and $N_2 \neq 0$, there exists a null element θ and $N = N_1 = N_2 = (\theta)$.*

PROOF. If $\theta_1 \in N_1$ and $\theta_2 \in N_2$, $\theta_1 = \theta_1 \theta_2 = \theta_2$. Thus each left null element equals every right null element and, in particular, all null elements are equal.

LEMMA 6.3. *If an alternation groupoid S has more than one element and if $ab \in L$, then neither a nor $b \in N_1$.*

PROOF. If $a \in N_1$, $ab = a \in N_1$. If $b \in N_1$, $(ab)[(ab)a] = [a(ab)](ba) = [a(ab)](bb) = (ab)[(ab)b]$ and $a = b \in N_1$.

LEMMA 6.4. *If $S = L \cup N_1$, if $N_1 \neq 0$, and if $N_1 \neq S$, there exists a null element θ in S .*

PROOF. Let $x \in S$, $a \in L$, θ_1 and $\theta_2 \in N_1$. By Lemma 6.3, $x\theta_1$ and $x\theta_2 \in N_1$. Hence $a\theta_1 = (a\theta_1)(\theta_2 a) = (a\theta_2)(\theta_1 a) = a\theta_2$, and $\theta_1 = \theta_2$. Thus all left null elements are equal, say, to θ . In particular $x\theta$, being left null, equals θ .

As a corollary to Lemmas 6.2 and 6.4 we have:

LEMMA 6.5. *If $S = L \cup N_1$, if $L \neq 0$, and if $L \neq S$, there exists a null element θ such that $L \cup (\theta) = S$.*

THEOREM 6.6. *If $S = L \cup N_1$, $L = L^*$.*

PROOF. If $L = 0$ or $L = S$, the theorem holds. If neither equality holds, $N_1 = (\theta)$ by Lemma 6.5. Then a and $b \in L$ implies $ab \in L$, for if $ab = \theta$, $ab = a\theta$, and $b = \theta$.

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