A RECURRENCE FORMULA FOR $\zeta(2n)$

HUAN-TING KUO

In the present paper I shall give a new recurrence formula for $\zeta(2n)$. It differs from other similar formulas in one very important respect. In order to calculate $\zeta(4n)$, for example, other formulas use the values of $\zeta(2)$, $\zeta(4)$, $\zeta(6)$, $\cdots$ up to $\zeta(4n-2)$, while mine only requires the values of $\zeta(2)$, $\zeta(4)$, $\zeta(6)$, $\cdots$, $\zeta(2n)$. So, for large values of n its advantage is manifest.

This new recurrence formula is

$$\begin{aligned} C_{2n} = {} & \frac{\pi(2\pi)^{2n-1}}{4\{(n-1)!\}^2(2n-1)} \\ (1) \quad & + \frac{1}{(n-1)!}\sum_{k=0}^{[n/2]}(-1)^k\frac{C_{2k}(2\pi)^{2n-2k}}{(n-2k)!(2n-2k)} \\ & + \frac{1}{\pi}\sum_{k=0}^{[n/2]}\sum_{j=0}^{[n/2]}(-1)^{k+j}\frac{C_{2k}C_{2j}(2\pi)^{2n-2k-2j+1}}{(n-2k)!(n-2j)!(2n-2k-2j+1)}, \end{aligned}$$

where n is a positive integer, and where for simplicity we have put $C_0=-1/2$ and $C_{2k}=\zeta(2k)=\sum_{n=1}^{\infty} 1/n^{2k}$, $k=1, 2, 3, \cdots$.

To prove it, let us note that

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty}\frac{\sin nx}{n}, \qquad 0 < x < 2\pi.$$

Repeated integration between limits 0 and x gives

$$(2) \quad \frac{\pi x^{2m-2}}{2(2m-2)!} + \sum_{k=0}^{m-1}(-1)^k\frac{C_{2k}x^{2m-2k-1}}{(2m-2k-1)!} = (-1)^{m-1}\sum_{n=1}^{\infty}\frac{\sin nx}{n^{2m-1}},$$

$$(3) \quad \frac{\pi x^{2m-1}}{2(2m-1)!} + \sum_{k=0}^{m}(-1)^k\frac{C_{2k}x^{2m-2k}}{(2m-2k)!} = (-1)^{m}\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2m}},$$

where the C's have the same meaning as above. An application of Parseval's theorem to (2) would yield (1) with $n=2m-1$ and a similar application to (3) would yield (1) with $n=2m$. Thus (1) is true for all positive integral values of n.

Now, let us apply the formula to calculate, for example, $\zeta(2)$ and $\zeta(4)$.

We have

Received by the editors April 16, 1948.

$$\zeta(2) = C_2 = \frac{\pi(2\pi)}{4} + \left(-\frac{1}{2}\right)\cdot\frac{(2\pi)^2}{2} + \frac{(-1/2)(-1/2)(2\pi)^3}{3\pi} = \frac{\pi^2}{6},$$

and

$$\begin{aligned}\zeta(4) = C_4 = &\frac{\pi(2\pi)^3}{4\cdot 3} + \left(-\frac{1}{2}\right)\frac{(2\pi)^4}{2\cdot 4} - \frac{\pi^2(2\pi)^2}{6\cdot 2}\\ &+ \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{(2\pi)^5}{\pi\cdot 2\cdot 2\cdot 5}\\ &+ (-1)\left(-\frac{1}{2}\right)\left(\frac{\pi^2}{6}\right)\frac{2\cdot(2\pi)^3}{2\cdot 3\pi} + \frac{\pi^4(2\pi)}{36\pi} = \frac{\pi^4}{90},\end{aligned}$$

which are in accordance with the usual results.

NATIONAL WUHAN UNIVERSITY