SPECIAL PROPERTIES OF MEASURE PRESERVING TRANSFORMATIONS

STEPHEN P. DILIBERTO

1. **Summary.** In studying problems[1] concerned with the qualitative description of bounded trajectories, in a region free of singular points, associated with a flow

$$dx_i/dt = P_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n) \tag{1.1}$$

the P_i being holomorphic in the x_i, we considered the possibility of finding "point conditions" on the P_i which would insure a smooth behavior on the part of a trajectory—or more specifically, on any motion in its limit set. For example, the restriction that the transformation defined by equations (1.1) preserve n measure is expressible by the point condition

$$\sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i} = 0.$$

Being unable to ensure the behavior desired by this condition, we sought stronger conditions. In particular we asked, "What are the conditions on the P_i such that the transformations defined by (1.1) preserves p measure where p is restricted to values $1 \leqq p \leqq n-1$?" The answer to this question and also to the question, "Is this a restrictive condition?" is contained in the following theorem.

THEOREM I. *The condition that the flow defined by*

$$dx_i/dt = P_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

the P_i being holomorphic, preserve p measure, p any (fixed) integer between 1 *and $n-1$, is that*

$$\partial P_i/\partial x_j = -\partial P_j/\partial x_i$$

for all i and j. These conditions imply that the motion is rigid.

It is an open and apparently difficult question as to whether every point transformation (we are considering only homeomorphisms)—of a sufficiently differentiable class—of E^3 onto itself is obtainable from a flow, that is, from the solutions of a system of first order equations of the type (1.1). (In E^2 there are point transformations which are *not* embeddable in flows.) Thus it is natural to ask for how

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[1] *On systems of ordinary differential equations*, which will appear elsewhere.

large a class of transformations the above result holds. We have a partial sharpening of Theorem I to the following theorem.

THEOREM II. *The only point transformations, of class C^3, of Euclidean three space into itself which preserve area are rigid motions (and reflections).*

2. **A determinant expansion.** We state next, without proof, a well known property about the product of an $n\times p$ by $p\times n$ matrix:

LEMMA. *Let*

$$A = (a_{ij}) = \begin{Bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \cdot & \cdot\ \cdot & \cdot \\ a_{p1} & \cdots & a_{pn} \end{Bmatrix}, \qquad B = (b_{ij}) = \begin{Bmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2p} \\ \cdot & \cdot\ \cdot & \cdot \\ b_{n1} & \cdots & b_{np} \end{Bmatrix},$$

$$C = A\cdot B = (c_{ij}) = \begin{Bmatrix} c_{11} & \cdots & c_{1p} \\ \cdot & \cdot\ \cdot & \cdot \\ c_{p1} & \cdots & c_{pp} \end{Bmatrix}, \qquad c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Then

$$|C| = \sum_{(l_i)} \begin{vmatrix} a_{1l_1} & \cdots & a_{1l_p} \\ \cdot & \cdot\ \cdot & \cdot \\ a_{pl_1} & \cdots & a_{pl_p} \end{vmatrix} \begin{vmatrix} b_{l_11} & \cdots & b_{l_1p} \\ \cdot & \cdot\ \cdot & \cdot \\ b_{l_p1} & \cdots & b_{l_pp} \end{vmatrix}.$$

This lemma says that the determinant of the product of $p\times n$ matrix by an $n\times p$ matrix is expressible as the sum of products of p square matrices, the first factor being a p square determinant from A selected by choosing columns $l_1, \cdots, l_p$ and the second factor being a p square determinant from B selected by choosing rows $l_1, \cdots, l_p$ from B, the summation being over all ways, $C_{n,p}$, of choosing p numbers from a set of n (and we take $l_1<l_2< \cdots <l_p$).

We make use of this lemma to generalize the well known product formula for Jacobians. Let X_i and x_i be functions of class C^2 where

$$X_i = X_i(x_1, \cdots, x_n), \qquad i = 1, \cdots, p,$$
$$x_i = x_i(u_1, \cdots, u_p), \qquad i = 1, \cdots, n;$$

then, the Jacobian of the X_i with respect to the u_i is expressible as

$$\frac{\partial(X_1, \cdots, X_p)}{\partial(u_1, \cdots, u_p)} = \left|\frac{\partial X_i}{\partial u_j}\right| = \begin{vmatrix} \partial X_1/\partial u_1 & \cdots & \partial X_1/\partial u_n \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \partial X_n/\partial u_1 & \cdots & \partial X_n/\partial u_n \end{vmatrix}$$

but
$$\partial X_i/\partial u_j = \sum_{k=1}^{n} (\partial X_i/\partial x_k)\cdot(\partial x_k/\partial u_j)$$

and so $\partial(X_1, \cdots, X_p)/\partial(u_1, \cdots, u_p)$ has the same form as $|C|$ in the last lemma. Thus

$$\frac{\partial(X_1, \cdots, X_p)}{\partial(u_1, \cdots, u_p)} = \sum_{(l_i)} \frac{\partial(X_1, \cdots, X_p)}{\partial(u_{l_1}, \cdots, u_{l_p})}\cdot\frac{\partial(x_{l_1}, \cdots, x_{l_p})}{\partial(u_1, \cdots, u_p)}.$$

3. **Proof of Theorem I.** Beginning with equation (1.1), let

$$x_i = f_i(x_{10}, \cdots, x_{n0}, t) \tag{3.1}$$

be the solution of that equation which at time t_0 passes through the point $(x_{10}, \cdots, x_{n0})$. We take a regular surface to be given by

$$x_i = g_i(u_1, \cdots, u_p), \qquad i = 1, \cdots, n, \tag{3.2}$$

where the rank of the functional matrix

$$\begin{pmatrix} \partial g_1/\partial u_1 \cdots \partial g_1/\partial u_p \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \partial g_n/\partial u_1 \cdots \partial g_n/\partial u_p \end{pmatrix}$$

is exactly p and where the g_i are analytic in the u_i and defined for values of u_i in an open convex p-cell, P, in the E^p space of the u_i. Then the p area, A_p, of this surface is given by

$$A_p = \int \cdots \int \left\{ \sum_{(l_i)} \frac{\partial(g_{l_1}, \cdots, g_{l_p})^2}{\partial(u_1, \cdots, u_p)} \right\}^{1/2} du_1, \cdots, du_p. \tag{3.3}$$

If we use capital letters for the transform of the surface (3.2), the transformation being given by (3.1), its equation is, at "time t,"

$$X_i = f_i(g_1, \cdots, g_n, t), \qquad i = 1, \cdots, n. \tag{3.4}$$

Its area $A_p(t)$ is given by the integral

$$A_p(t) = \int \cdots \int \left\{ \sum_{(l_i)} \frac{\partial(X_{l_1}, \cdots, X_{l_p})^2}{\partial(u_1, \cdots, u_p)} \right\}^{1/2} du_1 \cdots du_p. \tag{3.5}$$

But by hypothesis $A_p(t) \equiv A_p$, and since this must be true for each piece of the surfaces we must have that the integrands in (3.5) and (3.3) are equal. Using the lemma of the last section this equality becomes

$$\sum_{(l_i)} \frac{\partial(g_{l_1}, \cdots, g_{l_p})^2}{\partial(u_1, \cdots, u_p)} = \sum_{(m_i)} \left\{ \sum_{(n_i)} \frac{\partial(X_{m_1}, \cdots, X_{m_p})}{\partial(x_{n_1}, \cdots, x_{n_p})} \cdot \frac{\partial(g_{n_1}, \cdots, g_{n_p})}{\partial(u_1, \cdots, u_p)} \right\}^2. \tag{3.6}$$

We now proceed to draw consequences of (3.6) by making special choices of the g_i of (3.2). For a particular choice we do two things: (a) verify that this reduction is true for $t=t_0$; (b) impose the condition that this be an identity in t_0, that is, see that these equations, after differentiation with respect to t_0, are verified at $t_0=t$.

(A) *First choice of the* g_i. Let

$$\begin{aligned} x_i &= u_i, & i &= 1, \cdots, p, \\ x_i &= 0, & i &> p. \end{aligned} \tag{3.7}$$

We now verify (a) for (3.6) for this choice of the g_i. First,

$$\frac{\partial(x_{l_1}, \cdots, x_{l_p})}{\partial(u_1, \cdots, u_p)} = \begin{cases} 1, & l_i = i, \; i = 1, 2, \cdots, p, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, (3.6) reduces to

$$1 = \sum_{(l_i)} \frac{\partial(X_{l_1}, \cdots, X_{l_p})}{\partial(x_1, \cdots, x_p)}. \tag{3.8}$$

Now at $t_0=t$

$$\frac{\partial(X_{l_1}, \cdots, X_{l_p})}{\partial(x_1, \cdots, x_p)} = \begin{cases} 1 & \text{if } l_i = i, \; i = 1, \cdots, p, \\ 0 & \text{otherwise.} \end{cases}$$

This last fact may be seen as follows: the expression $\partial X_k/\partial x_j$ is the derivative of the kth component X_k, of a solution of (1.1) with respect to the jth component of its initial point. According to (3.1)

$$X_i = f_i(x_{10}, \cdots, x_{n0}, t)$$

and when $t=t_0$, $X_i(t_0)=f_i(x_1, \cdots, x_n, t_0)=x_i$. Thus $\partial X_k/\partial x_j\big|_{t_0=t}=\delta_{kj}$, explaining the value of the last Jacobian. From this it follows that at $t_0=t$ (3.8), and hence (3.6), is verified.

We now verify (3.8) (that is, (3.6) reduced by this choice of the g_i) for condition (b). To do this we must differentiate (3.8) with respect to t_0 and then put $t_0=t$. The differentiation will produce a sum of products, and when t_0 is put equal to t all products containing a factor of the form

$$\frac{\partial(X_{l_1}, \cdots, X_{l_p})}{\partial(x_1, \cdots, x_p)}, \qquad \text{some } l_j \neq j,$$

will be zero. Hence, *eliminating all such terms before differentiation*, (3.8) reduces on the right to one type of term, namely, those for which $l_i = i$ $(i=1, \cdots, p)$

$$0 = \sum_{i=1}^{p} \left. \frac{\partial(X_1, \cdots, dX_i/dt_0, \cdots, X_p)}{\partial(x_1, \cdots, x_p)} \right|_{t_0=t}.$$

And when $t_0 = t$, $dX_i/dt_0|_{t=t_0} = dX_i/dt = P_i(x_1, \cdots, x_n) = P_i$ so that condition (b) yields

$$0 = \sum_{i=1}^{p} \frac{\partial P_i}{\partial x_i}. \tag{3.9}$$

If we now choose a set of g_i as

$$\begin{aligned} x_1 &= 0, & & \\ x_i &= u_i, & i &= 2, \cdots, p+1, \\ x_i &= 0, & i &> p+1, \end{aligned}$$

we shall, by the same route as above, arrive at the relationship

$$\sum_{i=2}^{p+1} \frac{\partial P_i}{\partial x_i} = 0.$$

This equation combined with (3.9) yields $\partial P_1/\partial x_1 = \partial P_{p+1}/\partial x_{p+1}$ and in general we shall have (after suitable repetitions of this argument) that $\partial P_i/\partial x_i = \partial P_k/\partial x_k$. These combine with (3.9) to furnish the condition

$$\partial P_i/\partial x_i = 0, \qquad i = 1, \cdots, n. \tag{3.10}$$

(B) *Second choice of* g_i. Put

$$\begin{aligned} x_i &= u_i, & i &= 1, \cdots, p-1, \\ x_p &= x_{p+1} = u_p, & & \\ x_i &= 0, & i &> p+1. \end{aligned} \tag{3.11}$$

We note first of all that

$$\frac{\partial(x_{l_1}, \cdots, x_{l_p})}{\partial(u_1, \cdots, u_p)} = \begin{cases} 1, & l_i = i, \ i = 1, \cdots, p-1, \ l_p = p \text{ or } p+1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, if we use (3.11) and these last results, (3.6) reduces to

$$(3.12)\quad 2 = \sum_{(l_i)} \left\{ \frac{\partial(X_{l_1}, \cdots, X_{l_p})}{\partial(x_1, \cdots, x_p)} + \frac{\partial(X_{l_1}, \cdots, X_{l_p})}{\partial(x_1, \cdots, x_{p-1}, x_{p+1})} \right\}^2$$

and this is easily seen to verify condition (a). Namely when $t_0 = t$ the only nonzero terms are those for which

$$(l_1, \cdots, l_p) = \begin{cases} 1, 2, \cdots, p, \\ 1, 2, \cdots, p-1, p+1. \end{cases}$$

Proceeding as before, we verify (b) by differentiating (3.12)—and simplifying beforehand by dropping out all terms for which

$$(l_1, \cdots, l_p) \neq \begin{cases} 1, 2, \cdots, p, \\ 1, 2, \cdots, p-1, p+1, \end{cases}$$

since these will drop out when t_0 is put equal to t. This gives us

$$\begin{aligned} 0 = & \left\{ \frac{\partial(X_1, \cdots, X_p)}{\partial(x_1, \cdots, x_p)} + \frac{\partial(X_1, \cdots, X_p)}{\partial(x_1, \cdots, x_{p-1}, x_{p+1})} \right\} \\ & \cdot \sum_{i=1}^{p} \left\{ \frac{\partial(X_1, \cdots, dX_i/dt_0, \cdots, X_p)}{\partial(x_1, \cdots, x_p)} \right. \\ & \qquad \left. + \frac{\partial(X_1, \cdots, dX_i/dt_0, \cdots, X_p)}{\partial(x_1, \cdots, x_{p-1}, x_{p+1})} \right\} \\ & + \left\{ \frac{\partial(X_1, \cdots, X_{p-1}, X_{p+1})}{\partial(x_1, \cdots, x_p)} + \frac{\partial(X_1, \cdots, X_{p-1}, X_{p+1})}{\partial(x_1, \cdots, x_{p-1}, x_{p+1})} \right\} \\ & \cdot {\sum_{i=1}^{p+1}}' \left\{ \frac{\partial(X_1, \cdots, dX_i/dt_0, \cdots, X_{p-1}, X_{p+1})}{\partial(x_1, \cdots, x_p)} \right. \\ & \qquad \left. + \frac{\partial(X_1, \cdots, dX_i/dt_0, \cdots, X_{p-1}, X_p)}{\partial(x_1, \cdots, x_{p-1}, x_p)} \right\} \end{aligned}$$

where $\sum'$ means the value $i = p$ is omitted. Putting $t = t_0$ reduces this to

$$\begin{aligned} 0 = & \sum_{i=1}^{p} \left\{ \frac{\partial(X_1, \cdots, P_i, \cdots, X_p)}{\partial(x_1, \cdots, x_p)} + \frac{\partial(X_1, \cdots, P_i, \cdots, X_p)}{\partial(x_1, \cdots, x_{p-1}, x_{p+1})} \right\} \\ (3.13) \quad & + {\sum_{i=1}^{p+1}}' \left\{ \frac{\partial(X_1, \cdots, P_i, \cdots, X_{p-1}, X_{p+1})}{\partial(x_1, \cdots, x_p)} \right. \\ & \qquad \left. + \frac{\partial(X_1, \cdots, P_i, \cdots, X_{p-1}, X_p)}{\partial(x_1, \cdots, x_{p-1}, x_{p+1})} \right\}. \end{aligned}$$

The first term of the first brace is $\partial P_i/\partial x_i$ and the second term is zero except when $i=p$ when it is $\partial P_p/\partial x_{p+1}$; the first term of the second brace is zero except for $i=p+1$ when it is $\partial P_{p+1}/\partial x_p$ and the second term is $\partial P_i/\partial x_i$. Thus if we remove the $\partial P_i/\partial x_i$ which are zero, (3.13) gives us

$$\frac{\partial P_p}{\partial x_{p+1}} + \frac{\partial P_{p+1}}{\partial x_p} = 0.$$

A similar procedure based on corresponding choices of the g_i produces

$$\frac{\partial P_k}{\partial x_l} = -\frac{\partial P_l}{\partial x_k} \qquad \text{for all } k,\ l \tag{3.14}$$

(actually for $k \neq l$ but if we allow $k=l$ then (3.14) will include (3.10)).

From condition (3.14) we can derive very easily that the motion must be rigid, for differentiating (3.14) with respect to x_k shows that

$$\frac{\partial^2 P_l}{\partial x_k^2} = -\frac{\partial}{\partial x_k}\cdot\frac{\partial P_k}{\partial x_l} = -\frac{\partial}{\partial x_l}\cdot\frac{\partial P_k}{\partial x_k} \equiv 0.$$

Thus the P_i will be linear functions of the x_i and by (3.14) skew symmetric. Equation (1.1) written in vector form is then

$$\frac{dx}{dt} = Ax + c$$

where x, dx/dt, c are column vectors and A a skew-symmetric matrix ($A=(a_{ij})$ where $a_{ij}=-a_{ji}$). If one defines a new vector z by the relationship $Bz=x$, B being nonsingular, then z satisfies the vector differential equation

$$\frac{dz}{dt} = B^{-1}AB\cdot z + B^{-1}c.$$

And since A is skew-symmetric we can find an orthogonal B such that

$$B^{-1}AB = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & A_k \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & A_k & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

depending on whether n is even or odd and where

$$A_i = \begin{pmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{pmatrix}$$

where α_i is a constant (and real). The variables thus separate and the equations of the components of z can be integrated in pairs directly (trivially if $A_i=(0)$. For example, if $\alpha\neq 0$

$$\frac{dx_1}{dt} = \alpha x_2 + c_1, \qquad \frac{dx_2}{dt} = -\alpha x_1 + c_2$$

when integrated will give a rotation of angular velocity α about the point $(-c_2/\alpha, c_1/\alpha)$.

4. **Proof of Theorem II.** Let the correspondence $y_i=f_i(x_1, x_2, x_3)$ $(i=1, 2, 3)$, $f_i\in C^3$, be an area-preserving map of E^3 onto itself. Since the surface S: $x_i=g_i(u_1, u_2)$ $(i=1, 2, 3)$ $(g_i\in C^3$, $(u_1, u_2)\in R$, R a convex, open two-cell) and its image under this map have equal areas we must have

$$\iint_R \left\{ \sum_i \frac{\partial(g_i g_{i+1})^2}{\partial(u_1, u_2)} \right\}^{1/2} du_1 du_2$$

$$= \iint_R \left\{ \sum \frac{\partial(f_i(g_r) f_{i+1}(g_s))^2}{\partial(u_1, u_2)} \right\}^{1/2} du_1 du_2.$$

Moreover the integrands must be equal, hence, by the lemma of §2,

$$\sum_{i=1}^{3} \frac{\partial(g_i, g_{i+1})^2}{\partial(u_1, u_2)} = \sum_{j=1}^{3} \left(\sum_{k=1}^{3} \frac{\partial(f_j, f_{j+1})}{\partial(x_k, x_{k+1})} \frac{\partial(g_k, g_{k+1})}{\partial(u_1, u_2)} \right)^2.$$

This expression must be an identity in the g_i. Choosing $g_1=u_1$, $g_2=u_2$ and $g_3=$const. (and making two similar choices) yields

$$1 = \sum_{i=1}^{3} \frac{\partial(f_i, f_{i+1})^2}{\partial(x_j, x_{j+1})}, \qquad j = 1, 2, 3;$$

choosing $g_1=u_1$, $g_2=g_3=u_2$ (and making two similar choices) yields

$$2 = \sum_{i=1}^{3} \left(\frac{\partial(f_i, f_{i+1})}{\partial(x_j, x_{j+1})} + \frac{\partial(f_i, f_{i+1})}{\partial(x_{j+2}, x_{j+3})} \right)^2, \qquad j = 1, 2, 3.$$

Combination of these last two restrictions yields

$$0 = \sum_{i=1}^{3} \frac{\partial(f_i, f_{i+1})}{\partial(x_j, x_{j+1})} \frac{\partial(f_i, f_{i+1})}{\partial(x_{j+2}, x_{j+3})}, \qquad j = 1, 2, 3.$$

The first and last of these relations tell us immediately that the 3 vectors $A_i = (\partial(f_1, f_2)/\partial(x_i, x_{i+1}), \partial(f_2, f_3)/\partial(x_i, x_{i+1}), \partial(f_3, f_1)/\partial(x_i, x_{i+1}))$ are normal and orthogonal. A_i represents the normal to a surface element which is the image of a plane element perpendicular to the $(i+2)$th coordinate axis. Thus we know that the families of planes parallel to the coordinate planes must be carried into a triply orthogonal family of surfaces. Thus excluding the rigid motions it follows from the proof of Liouville's theorem on three-dimensional conformal maps[2] that the map must be an inversion. But such maps do not preserve areas of spheres concentric to center of inversion and so are excluded—leaving the rigid motions (and reflections, of course).

UNIVERSITY OF CALIFORNIA AT BERKELEY

[2] Blaschke, *Differentialgeometrie*, 3d ed., Berlin, Springer, p. 100.