A NOTE ON THE ERGODIC THEOREMS

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Introduction, definitions and remarks. The purpose of this note is to give an example of a measurable transformation of a measure space onto itself for which the individual ergodic theorem holds while the mean ergodic theorem does not hold.

Let S be a measure space of finite measure, m the measure defined on the measurable subsets of S, and T a 1-1 point transformation of S onto itself which is measurable (both T and T^{-1} transform measurable sets into measurable sets). Let the points of S be denoted by y and let $f(y)$ be any real valued function defined on S. We denote by $F_h(y)$ the average $(1/h)\sum_{i=0}^{h-1} f(T^i y)$.

We shall say that the individual ergodic theorem holds for $f(y)$ if the sequence of averages $\{F_h(y)\}$ converges to a finite limit almost everywhere. If the individual ergodic theorem holds for every integrable function $f\in L_1(m)$ we shall say that the individual ergodic theorem holds (with respect to m).[1]

We shall say that the mean ergodic theorem holds in $L_p(m)$ $(p\geqq 1)$ for a function $f\in L_p(m)$ if $F_h(y)\in L_p(m)$ for $h=1, 2, \cdots$ and the sequence $\{F_h(y)\}$ converges in the norm of $L_p(m)$. If the mean ergodic theorem holds in $L_p(m)$ for every function $f(y)\in L_p(m)$ then we shall say that the mean ergodic theorem holds in $L_p(m)$.

The following relations between the two ergodic theorems are known: If T is measure preserving, both the individual [**1**] and the mean [**4**][2] ergodic theorems hold. Without assuming that T is measure preserving, the mean ergodic theorem in $L_p(m)$ for any $p\geqq 1$ implies the individual ergodic theorem for all functions in $L_p(m)$ ([**2**, p. 1061], see also [**3**, p. 539] for the case $p=1$).

The question arises whether, conversely, the individual ergodic theorem implies the mean ergodic theorem in $b_p(m)$ for some $p\geqq 1$. This question has significance only when $L_p(m)$ is transformed into itself by the transformation induced on it by T. For in this case and only in this case is it true that for any $f\in L_p(m)$ the averages $\{F_h\}$ also belong to $L_p(m)$ for $h=1, 2, 3, \cdots$.[3] We answer this question in

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[1] The words in the parenthesis will be omitted if there is no reason for ambiguity.

[2] Only the case $p=2$ is proved in [**4**]; see [**2**, p. 1053] for all $p\geqq 1$. The numbers in brackets refer to the bibliography at the end of the paper.

[3] It is easy to give examples for which the individual ergodic theorem does hold while $L_p(m)$ is not transformed into itself. Such an example for instance is given if T is periodic while m is non-atomic and T^{-1} is singular.

the negative by constructing for each given $p \geqq 1$ an example of a transformation of a measure space (S, m) onto itself for which (1) the individual ergodic theorem holds, (2) the mean ergodic theorem does not hold in $L_p(m)$, and (3) every function in $L_p(m)$ is transformed into a function belonging to $L_p(m)$.

REMARK. Even though the individual ergodic theorem does not imply the mean ergodic theorem with respect to the original measure m it is known [**2**, p. 1059] that in case the individual ergodic holds, it is possible to introduce a new measure μ defined on the measurable sets of S such that μ has no more null sets than m and μ is also invariant under T. It follows then from the statements made above that the mean ergodic theorem holds in $L_p(\mu)$ for every $p \geqq 1$.

The example for $p = 1$. Let S be the totality of all points on the circumferences $c_1, c_2, \cdots$ of a sequence of circles. Let the length of c_n be $1/2^n$. Let the measure m and the family of measurable sets in S be the obvious ones determined by the Lebesgue measure on each of the circumferences. On each circumference c_n we fix the polar coordinates $\rho = (1/2^n \cdot 2\pi) \cdot e^{i\theta}$. Let $x = \theta/2\pi$. Let us divide c_n into $2n+2$ arcs, the end points of the arcs being $x = 0$, $x = 1/2$, and $x = \pm 1/2^{k+1}$, $k = 1, 2, \cdots, n$. The arcs are

$$A_{nk}: \frac{1}{2^k} \geqq x \geqq \frac{1}{2^{k+1}},$$

$$A_{n,n+1}: \frac{1}{2^{k+1}} \geqq x \geqq 0,$$

while if $n+2 \leqq k \leqq 2n+2$, A_{nk} is the reflection in $\theta = 0$ of $A_{n,2n+3-k}$.

We define T as follows: For the points of A_{nk} let T be the unique transformation given by $x' = ax + b$, $a > 0$, which transforms A_{nk} onto $A_{n,k+1}$ for $k = 1, \cdots, 2n-1$, and A_{nk} onto $A_{n,1}$ for $k = 2n+2$. T is clearly a 1:1 point transformation of S onto itself. Moreover it is easily seen that T is measurable and pointwise periodic with the period of $2n+2$ for the points of c_n. T also satisfies the following two conditions:

(1) $m(T^{-1}A) \leqq 2m(A)$ for every measurable set A in S.

(2) The set of ratios

$$R_A^n = \frac{1}{2n+2} \cdot \sum_{i=0}^{2n-1} \frac{m(T^{-i}A)}{m(A)}$$

where $n = 1, 2, \cdots$ and A varies over all measurable sets $A \subseteq S$ is not a bounded set of numbers. (1) follows from the fact that T^{-1} is determined by a linear transformation with a stretching factor of at most 2. (In fact, it is either 1/2, 1, or 2.)

To prove (2) consider the sequence of sets $A_{n,n+1}$. Since $\bigcup_{i=0}^{2n-1} m(T^{-i}A_{n,n+1}) = c_n$ we have that

$$\frac{1}{2n+2} \cdot \sum_{i=0}^{2n-1} m(T^{-i}A_{n,n+1}) = \frac{1}{2n+2} \cdot \frac{1}{2^n} \cdot$$

On the other hand $m(A_{n,n+1}) = (1/2^{n+1}) \cdot (1/2^n)$ and hence

$$R^n_{A_{n,n+1}} = \frac{2^{n+1}}{2n+2},$$

which is an unbounded sequence.

We can now show that the above example satisfies the required conditions specified in the introduction.

(a) The individual ergodic theorem holds. In fact let $f(y)$ be any real-valued function defined on S, then since T is pointwise periodic the sequence $\{F_h(y)\}$ converges to a finite limit for every y, that is, the individual ergodic theorem holds for every real-valued function defined on S. A fortiori it holds (with respect to m).

(b) $L_1(m)$ is transformed into itself by T: In fact, it can be easily seen that $\bar{m}(A) = m(T^{-1}A)$ is a completely additive non-negative set function (that is, a measure) defined on the measurable sets of S. It can also be shown by considering approximating sums to the integrals that if $f \in L_1(m)$ then

$$\int_S |f(Ty)| \, dm = \int_S |f(y)| \, d\bar{m}.$$

By (1), $\bar{m}(A) \leqq 2m(A)$ for every measurable set A and hence

$$\int_S |f(y)| \, d\bar{m} \leqq 2 \int_S |f(y)| \, dm < \infty$$

from which follows that $f(Ty) \in L_1(m)$, that is, $L_1(m)$ is transformed into itself by T.

(c) The mean ergodic theorem does not hold in $L_1(m)$. To prove this statement we use the following result due to Miller and Dunford [**3**, p. 539]: Suppose that the mean ergodic theorem did hold in $L_1(m)$; then there would exist a positive constant c independent of A and h such that

$$\text{(i)} \qquad \frac{1}{h} \cdot \sum_{i=0}^{h-1} m(T^{-i}A) < c \cdot m(A)$$

for all measurable sets A and $h=1, 2, \cdots$. But (i) is in contradiction with (2) above. Hence the mean ergodic theorem does not hold in $L_1(m)$.

It is possible to prove the last statement also directly by exhibiting functions in $L_1(m)$ for which the mean ergodic theorem does not hold in $L_1(m)$. In fact let $\bar{f}(y)$ be defined as follows: $\bar{f}(y)=2^n$ on $A_{n,n+1}$, $n=1, 2, \cdots, \bar{f}(y)=0$ everywhere else on S, then $\bar{f}(y)\in L_1(m)$ but $\{\bar{F}_h(y)\}$ is not convergent in $L_p(m)$, for if it were then the limit function $\bar{f}^*(y)$ of $\{\bar{F}_h(y)\}$ would have to belong to $L_1(m)$. But $\bar{f}^*(y)$ is seen to be equal to $(1/2n+2)\cdot 2^n$ for $y\in c_n$. We have

$$\int_S |\bar{f}^*(y)|\,dm = \sum_{n=1}^{\infty}(1/2n+2)$$

which is a divergent series, that is, $\bar{f}^*\notin L_1(m)$ and hence $\{F_h(y)\}$ is not convergent in $L_1(m)$.

The example for $p\geqq 1$. Let p be a fixed integer $\geqq 1$. Let S be the same sequence of circumferences $c_1, c_2, \cdots$ as before. We divide each c_n into $2(n\cdot 2^p-n+1)$ arcs, the end points being $x=0$, $x=1/2$ and $x=r/2^{kp+1}$, $k=1, 2, \cdots, n$, $r=1, 2, \cdots, 2^p-1$. Again we define T by the transformation given by $x'=ax+b$, $a>0$, which transforms each arc into the next adjacent one.

T is again seen to be a 1-1 measurable pointwise periodic transformation of S onto itself with the period $2(n\cdot 2^p-n+1)$ for the points of c_n. As before it follows that (a) the individual ergodic theorem holds. (b) $L_p(m)$ is transformed into itself since there is a bound (the bound being 2^p) on the stretching factor of T^{-1}. (c) The mean ergodic theorem does not hold in $L_p(m)$. To prove this last statement we use the following generalization of Miller and Dunford's result stated above: Let t be any real number $\geqq 1$, then if the mean ergodic theorem holds in $L_t(m)$ there exists a constant C independent of A and h such that

$$\text{(ii)}\qquad \left[\frac{1}{h}\cdot\sum_{i=0}^{h-1} m(T^{-i}A)\right]^t < C\cdot m(A).$$

The proof is almost the same as for the special case $t=1$. If, however, we consider the sequence of sets A_{nq}, where $q=n(2^p-1)+1$ and where the enumeration of the arcs on each c_n is analogous to that used in the case $p=1$, we can easily see that the sequence of ratios

$$R_n = \left[\frac{1}{2q}\sum_{i=0}^{2q-1} m(T^{-i}A_{nq})\right]^p \Big/ m(A_{nq})$$

is an unbounded sequence. This is in contradiction with (ii) for $t=p$. Hence the mean ergodic theorem does not hold in $L_p(m)$. Again the statement made in (c) may be proved directly by exhibiting functions in $L_p(m)$ for which the mean ergodic theorem does not hold in $L_p(m)$.

Let p_1 be any fixed number not less than 1. Let p be the first integer not less than p_1. Then the example constructed above for p is also a valid example for p_1, for the individual ergodic theorem clearly holds and $L_{p_1}(m)$ is transformed into itself for the same reasons as before, while it follows from the fact that $m(A) \leqq 1$ for every measurable set A and the fact that $p \geqq p_1$ that the same sequence of sets which violates (ii) for the case $t=p$ also violates (ii) for $t=p_1$. Hence the mean ergodic theorem does not hold in $L_{p_1}(m)$.

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