

COMBINATORIAL HOMOTOPY. I

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1. Introduction. This is the first of a series of papers, whose aim is to clarify the theory of “nuclei” and “ n -groups” and its relation to Reidemeister’s¹ *Überlagerungen*. Here we give a new definition of “ n -groups,” or n -types as we now propose to call them. This is stated in terms of $(n-1)$ -homotopy types, which were introduced by R. H. Fox.² In a later paper we shall show that this is equivalent to the definition in terms of elementary transformations, which was given in [1]. The series of n -types ($n=1, 2, \dots$) is a hierarchy of homotopy, and a fortiori of topological invariants. That is to say, if two complexes,³ K, L , are of the same n -type, then they are of the same m -type for any $m < n$, where $n \leq \infty$ and the ∞ -type means the homotopy type. If $\dim K, \dim L \leq n$ then K, L are of the same homotopy type if they are of the same $(n+1)$ -type. Two complexes are of the same 2-type if, and only if, their fundamental groups are isomorphic. Moreover any (discrete) group is isomorphic to the fundamental group of a suitably constructed complex. Therefore the classification of complexes according to their 2-types is equivalent to the classification of groups by the relation of isomorphism. Thus the n -type ($n > 2$) is a natural generalization of a geometrical equivalent of an abstract group.⁴

Following up this idea we look for a purely algebraic equivalent of an n -type when $n > 2$. An important requirement for such an algebraic system is “realizability,” in two senses. In the first instance this means that there is a complex which is in the appropriate relation to a given one of these algebraic systems, just as there is a complex whose fundamental group is isomorphic to a given group. The second kind, whose importance is underlined by theorems in [5; 6] and in this paper, is the “realizability” of homomorphisms, chain mappings, etc., by maps of the corresponding complexes. Thus realizability means that the algebraic representation is not subject to conditions which can only be expressed geometrically.

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¹ See [1], [3] and [8, p. 177]. Numbers in brackets refer to the references cited at the end of the paper.

² See [9, p. 343] and [10, p. 49].

³ I.e., CW-complexes, as defined in §5 below.

⁴ I.e., the class of groups which are isomorphic to a given group.

What has been achieved so far is a purely algebraic description of the homotopy type of any 3-dimensional complex (see [3]) and of any finite, simply-connected, 4-dimensional complex. An account of the former will be given in Paper II of this series and of the latter in [5]. This and Theorem 6 below lead to an algebraic description of the 3-type of any complex and of the 4-type of any simply-connected, finite complex. The theorem on the realizability of chain-mappings, which applies to the 3-dimensional complexes, will be generalized to what we call J_m -complexes, which are defined in §3 below.

The algebraic apparatus used in [5] and in Paper II is inadequate for the classification of homotopy classes of maps of such complexes. Each homotopy class of maps induces a unique equivalence class of algebraic maps. Moreover each algebraic map, of the appropriate algebraic type, has a geometrical realization. But in general the homotopy class of the latter is not unique. For example in the case of a simply-connected, 3-dimensional complex our algebraic system simply consists of the ordinary chain groups and all maps of a 3-sphere in a 2-sphere lead to the same equivalence class of chain mappings. Thus our results are complementary to much of the recent work on the algebraic classification of mappings (e.g. [11]).

In this presentation we abandon simplicial complexes in favor of cell complexes. This first part consists of geometrical preliminaries, including some elementary propositions concerning what we call *closure finite complexes with weak topology*, abbreviated to *CW-complexes*, which are defined in §5 below. There are two main reasons why we do not confine ourselves to finite complexes. The first is that we want to include such simple spaces as open manifolds or, more generally, infinite but locally finite complexes. The second is that we have a great deal to do with covering complexes. We do not restrict ourselves to locally finite complexes because this restriction would be troublesome when considering "mapping cylinders" of infinite complexes. Also we want to allow for such operations as shrinking an infinite subcomplex (e.g., a tree containing all the 0-cells) into a point.

We recall from [6] that a space, P , *dominates* a space, X , if, and only if, there are maps, $\lambda: X \rightarrow P$, $\lambda': P \rightarrow X$, such that $\lambda'\lambda \simeq 1$. We show that the theorems in [6] can be extended to non-compact spaces, which are dominated by CW-complexes. To do this we give a new and possibly⁵ more restrictive definition of n -homotopy. According to Fox, maps $f, g: X \rightarrow Y$, of any space X in a space Y , are n -homotopic, written $f \simeq_n g$, if, and only if, $f\phi \simeq g\phi$, for every map,

⁵ See §7 below.

$\phi: K^n \rightarrow X$, of every finite simplicial complex, K^n , of at most n dimensions. We alter this by defining $f \simeq_n g$ if, and only if, $f\phi \simeq g\phi$ for every map, $\phi: K^n \rightarrow X$, of every CW-complex, K^n , of at most n dimensions. As in [6] we describe a map, $f: X \rightarrow Y$, as an n -homotopy equivalence if, and only if, it has an n -homotopy inverse, meaning a map, $g: Y \rightarrow X$, such that $gf \simeq_n 1, fg \simeq_n 1$. A homotopy equivalence is similarly defined in terms of ordinary homotopy. We shall use the symbol

$$X \equiv_n Y$$

to mean that X and Y are of the same n -homotopy type (i.e. there is an n -homotopy equivalence $f: X \rightarrow Y$) and $X \equiv Y$ will mean that X and Y are of the same homotopy type.

Let α be the class of all connected spaces, each of which is dominated by some CW-complex.⁶ Let $\alpha_0 \subset \alpha$ be the subclass consisting of spaces which are locally simply-connected in the weak sense. That is to say, if $X \in \alpha_0$ there is a basic set of neighborhoods in X , such that any closed curve in one of these neighborhoods is contractible to a point in X . The universal covering space, \tilde{X} , of such a space has the usual properties. If X is dominated by a CW-complex of finite dimensionality we shall use ΔX to denote the minimum dimensionality of all CW-complexes which dominate X . If none of the CW-complexes, which dominate X , has finite dimensionality we write $\Delta X = \infty$. We restate the theorems in [6], which now refer to spaces $X, Y \subset \alpha$ and to the new definition of n -homotopy. Let $f_n: \pi_n(X) \rightarrow \pi_n(Y)$ be the homomorphism induced by a map $f: X \rightarrow Y$ ($n = 1, 2, \dots$) and let $N = \max(\Delta X, \Delta Y) \leq \infty$.

THEOREM 1. *The map $f: X \rightarrow Y$ is a homotopy equivalence if, and only if, $f_n: \pi_n(X) \rightarrow \pi_n(Y)$ is an isomorphism onto for every n such that $1 \leq n < N + 1$.*

THEOREM 2. *The map $f: X \rightarrow Y$ is an $(N - 1)$ -homotopy equivalence if, and only if, f_n is an isomorphism onto for every n such that $1 \leq n < N$.*

Let $X, Y \subset \alpha_0$ and let \tilde{X}, \tilde{Y} be the universal covering spaces of X, Y , with base points $x_0 \in X$ and $y_0 = fx_0 \in Y$. As explained in [6], the map $f: X \rightarrow Y$ induces homomorphisms $H_n(\tilde{X}) \rightarrow H_n(\tilde{Y})$, where H_n indicates the n th homology group, defined in terms of singular cycles. The argument given in [6] shows that $H_n(\tilde{X}) = 0, H_n(\tilde{Y}) = 0$ if $n > N$.

⁶ A space in α must satisfy the usual union and intersection axioms for closed and open sets but need not satisfy any separation axioms. For example, a single point need not be a closed subset. It follows from (M) in §5 below that any connected CW-complex is arcwise connected. Hence it is easily proved that every space in α is arcwise connected.

THEOREM 3. *The map $f: X \rightarrow Y$ is a homotopy equivalence if, and only if, each of the induced homomorphisms $f_1: \pi_1(X) \rightarrow \pi_1(Y)$ and $H_n(\tilde{X}) \rightarrow H_n(\tilde{Y})$ is an isomorphism onto.*

Let $N < \infty$.

THEOREM 4. *The map $f: X \rightarrow Y$ is an $(N-1)$ -homotopy equivalence if*
 (a) *each of the induced homomorphisms $f_1: \pi_1(X) \rightarrow \pi_1(Y)$, $H_n(\tilde{X}) \rightarrow H_n(\tilde{Y})$ is an isomorphism onto, for $n=0, \dots, N-2$, and also*
 (b) *the induced homomorphism $H_{N-1}(\tilde{X}) \rightarrow H_{N-1}(\tilde{Y})$ is onto and*
 (c) *$f_{N-1}: \pi_{N-1}(X) \rightarrow \pi_{N-1}(Y)$ is an isomorphism into.*

Conversely if $f: X \rightarrow Y$ is an $(N-1)$ -homotopy equivalence,⁷ so is the lifted map, $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$, and $H_n(\tilde{X}) \rightarrow H_n(\tilde{Y})$ is an isomorphism onto for $n=0, \dots, N-1$.

2. n -types. Let K, L be CW-complexes. A map $f: K \rightarrow L$ is said to be *cellular* if, and only if, $fK^n \subset L^n$ for each $n \geq 0$, where K^n, L^n are the n sections of K, L . A homotopy $f_i: K \rightarrow L$ is said to be *cellular* if, and only if, the maps f_0, f_1 are cellular and $f_i K^n \subset L^{n+1}$ for each $n \geq 0$. According to (L), in §5 below, any map $K \rightarrow L$ is homotopic to a cellular map and if $f_0 \simeq f_1: K \rightarrow L$, where f_0, f_1 are cellular, then there is a cellular homotopy $f_i: K \rightarrow L$. Let $f_0, f_1: L \rightarrow X$ be maps of L in any space X and let $f_0 \simeq_n f_1$. Then $f_0 \phi \simeq f_1 \phi$ for any map, $\phi: K^n \rightarrow L$, of any CW-complex, K^n , of at most n dimensions. In particular $f_0|L^n \simeq f_1|L^n$. But ϕ is homotopic to a cellular map, $\phi': K^n \rightarrow L$, and if $f_0|L^n \simeq f_1|L^n$ we have $f_0 \phi \simeq f_0 \phi' \simeq f_1 \phi' \simeq f_1 \phi$. Therefore $f_0 \simeq_n f_1$ if, and only if, $f_0|L^n \simeq f_1|L^n$. If $\dim L \leq n$ then $f_0 \simeq_n f_1$ obviously means $f_0 \simeq f_1$. It will be convenient to allow n to take the value ∞ on the understanding that $L^\infty = L$ and that $f_0 \simeq_\infty f_1$ and $K \equiv_\infty L$ mean $f_0 \simeq f_1$ and $K \equiv L$.

We shall say that CW-complexes, K, L , are of the same n -type ($1 \leq n \leq \infty$) if, and only if, $K^n \equiv_{n-1} L^n$. If $\dim K, \dim L \leq n$, then $X^{n+1} = X^n = X$ ($X = K$ or L) and $K^{n+1} \equiv_n L^{n+1}$ means $K \equiv L$.

THEOREM 5. *If K and L are of the same n -type ($2 \leq n \leq \infty$) then they are of the same m -type for any $m < n$.*

Let $f: K^n \rightarrow L^n$ be an $(n-1)$ -homotopy equivalence and let $g: L^n \rightarrow K^n$ be an $(n-1)$ -homotopy inverse of f . We may assume that f and g are cellular maps and also that there are cellular homotopies, $\xi_i: K^{n-1} \rightarrow K^n, \eta_i: L^{n-1} \rightarrow L^n$, such that $\xi_0 = gf|K^{n-1}, \xi_1 = 1, \eta_0 = fg|L^{n-1}$,

⁷ By taking X to be a 2-sphere and Y to be a complex projective plane one sees that $f: X \rightarrow Y$ need not be an $(N-1)$ -homotopy equivalence ($N=4$) even though the induced homomorphisms f_1 and $H_n(\tilde{X}) \rightarrow H_n(\tilde{Y})$ are isomorphisms onto for $n=1, \dots, N-1$.

$\eta_1 = 1$, where 1 stands for both identical maps $K^{n-1} \rightarrow K^n$, $L^{n-1} \rightarrow L^n$. Let $f^m: K^m \rightarrow L^m$, $g^m: L^m \rightarrow K^m$ be the maps induced by f, g (i.e. $f^m p = f p$, $g^m q = g q$, where $p \in K^m$, $q \in L^m$). Since $\xi_t K^{m-1} \subset K^m$ it follows that $g^m f^m | K^{m-1} \simeq 1$ (in K^m). Therefore $g^m f^m \simeq_{m-1} 1$. Similarly $f^m g^m \simeq_{m-1} 1$, whence $K^m \equiv_{m-1} L^m$ and the theorem is proved.

It follows from Theorem 5 that the n -type is an invariant of the homotopy type (i.e. is the same for two complexes of the same homotopy type). It is a fortiori a topological invariant. Since a CW-complex is locally contractible, according to (M) in §5 below, each component is arcwise connected. Therefore two CW-complexes have the same 1-type if, and only if, they have the same cardinal number of components. It also follows from the results in §5 below, and the argument used in the finite case,⁸ that connected CW-complexes, K, L , have the same 2-type if, and only if, $\pi_1(K) \approx \pi_1(L)$. We now assume, until §4, that any given complex is connected and that $n \geq 2$ in statements concerning n -homotopy and n -types.

These ideas provide a generalization of a result due to Hopf,⁹ on the relation between the fundamental group and the second homology group of a polyhedron. For let X be any space and let $G_n(X)$ be the residue group

$$G_n(X) = H_n(X) - S_n(X),$$

where $S_n(X) \subset H_n(X)$ is the sub-group whose elements are represented by spherical cycles. Let $f, g: X \rightarrow Y$ be maps of X in a space Y . It may be proved that,¹⁰ if $f \simeq_{n-1} g$, then f and g induce the same homomorphism $G_n(X) \rightarrow G_n(Y)$. It follows that, if $X \equiv_{n-1} Y$, then $G_n(X) \approx G_n(Y)$. In case X, Y are CW-complexes this also follows from:

THEOREM 6. *If $K_1^n \equiv_{n-1} K_2^n$ then*

$$K_1^n \cup \Sigma_1^n \equiv K_2^n \cup \Sigma_2^n,$$

where Σ_i^n ($i=1, 2$) is a set of (disjoint) n -cells, whose closures are n -spheres, attached to K_i^n at a single point, Σ_i^n being a finite set of cells if K_1^n, K_2^n are finite complexes.

This is Theorem 13 in¹¹ [1], restated in terms of CW-complexes

⁸ See [16, pp. 217, 213], and [18, 3.1, 3.2 on p. 29].

⁹ See [17] and [18]. See also [1, Theorems 12 and 13, pp. 266, 269].

¹⁰ This is easily proved on the assumption that any homology class in $H_n(X)$ has a representative (singular) cycle, which is also a continuous cycle (i.e., the image in a map, $K^n \rightarrow X$, of a cycle carried by a finite, n -dimensional, simplicial complex K^n).

¹¹ [1], p. 269 for finite and p. 324 for infinite complexes.

and the new definition of the n -type. We shall give another proof in a later paper.

Another corollary of Theorem 6, and of Theorem 19 in [1] is:

COROLLARY 1. *If $K_1^n \equiv_{n-1} K_2^n$ then¹²*

$$\pi_n(K_1) + M_1 \approx \pi_n(K_2) + M_2,$$

where $+$ indicates direct summation and M_i is a free \mathfrak{R}_i -module, \mathfrak{R}_i being the group ring of $\pi_1(K_i)$ with integral coefficients.

Another obvious corollary of Theorem 6, which is also easy to prove directly, is:

COROLLARY 2. *If K_1^n and K_2^n are of the same n -type, so are their universal covering complexes, \tilde{K}_1^n and \tilde{K}_2^n .*

Hence, using $G_n(X)$ to mean the same as before, with $X = \tilde{K}_i^n$, we have:

COROLLARY 3. *If $K_1^n \equiv_{n-1} K_2^n$ then $G_n(\tilde{K}_1^n) \approx G_n(\tilde{K}_2^n)$.*

3. J_m -complexes. Let K be a (connected) CW-complex and let a 0-cell, e^0 , be taken as base point for all the groups $\pi_n(K^n, K^{n-1})$, $\pi_r(K^r)$ ($n \geq 2, r \geq 1$). Let $\rho_n = \pi_n(K^n, K^{n-1})$ and let $j_n: \pi_n(K^n) \rightarrow \rho_n$ be the homomorphism induced by the identical map $(K^n, e^0) \rightarrow (K^n, K^{n-1})$. We shall describe K as a J_m -complex if, and only if, j_n is an isomorphism into ρ_n for each $n = 2, \dots, m$. Obviously K is a J_m -complex if, and only if, K^m is a J_m -complex. We shall prove that the property of being a J_m -complex is an invariant of the m -type. That is to say, if $K^m \equiv_{m-1} L^m$ and if K is a J_m -complex so is L .

Let $\beta_n: \rho_n \rightarrow \pi_{n-1}(K^{n-1})$ ($n \geq 2$) be the (homotopy) boundary homomorphism and let

$$d_n = j_{n-1}\beta_n: \rho_n \rightarrow \rho_{n-1},$$

where $\rho_1 = \pi_1(K^1)$ and $j_1 = 1$. Then $d_n j_n = 0$ and $d_n d_{n+1} = 0$ since $\beta_n j_n = 0$. Let $i_n: \pi_n(K^{n-1}) \rightarrow \pi_n(K^n)$ be the homomorphism induced by the identical map $K^{n-1} \rightarrow K^n$.

We now prove two theorems which are closely related to Theorem 1 in [25].

THEOREM 7. *The homomorphism $j_n: \pi_n(K^n) \rightarrow \rho_n$ is into $d_n^{-1}(0)$ ($n \geq 2$).*

¹² In each of Corollaries 1 and 3 the symbol \approx means that there is an operator isomorphism (onto), which is defined in terms of the isomorphism, $\pi_1(K_1^n) \rightarrow \pi_1(K_2^n)$, induced by an $(n-1)$ -homotopy equivalence $K_1^n \rightarrow K_2^n$. Compare Corollary 3 with Theorem 28.1 on p. 414 of [13].

It is:

- (a) an isomorphism (into) if, and only if, $i_n\pi_n(K^{n-1})=0$ and is
- (b) onto $d_n^{-1}(0)$ if $i_{n-1}\pi_{n-1}(K^{n-2})=0$.

The fact that $j_n\pi_n(K^n) \subset d_n^{-1}(0)$ follows from the relation $d_nj_n=0$. The assertion (a) follows at once from the exactness of the homotopy sequence

$$(3.1) \quad \pi_r(K^{r-1}) \xrightarrow{i} \pi_r(K^r) \xrightarrow{j} \pi_r(K^r, K^{r-1}) \xrightarrow{\beta} \pi_{r-1}(K^{r-1}),$$

with $r=n$. Let $a \in d_n^{-1}(0)$ be given ($n \geq 2$). Since $d_na = j_{n-1}\beta_na = 0$ it follows from the exactness of (3.1) with $r=n-1$ and $\pi_1(K^0) = \pi_0(K^0) = 0$ in case $n=2$, that $\beta_na \in i_{n-1}\pi_{n-1}(K^{n-2})$. Therefore $\beta_na = 0$ if $i_{n-1}\pi_{n-1}(K^{n-2}) = 0$ and it follows from the exactness of (3.1), with $r=n$, that $a \in j_n\pi_n(K^n)$. This completes the proof.

Since $\pi_2(K^1) = 0$ it follows from Theorem 7 that any complex is a J_2 -complex. Also j_2 is onto $d_2^{-1}(0) = \beta_2^{-1}(0)$ by the exactness of (3.1) with $r=2$. In general we have:

COROLLARY. A complex, K , is a J_m -complex if, and only if, $i_n\pi_n(K^{n-1}) = 0$ for $n = 2, \dots, m$. In this case j_r is onto $d_r^{-1}(0)$ for $r = 2, \dots, m+1$.

Let $\pi_1(K) = 1$. Then the groups ρ_n ($n > 2$) and ρ_1, ρ_2 made Abelian may be taken as chain-groups.¹³ The homology group, $H_n(K)$, may be defined as the residue group

$$(3.2) \quad H_n(K) = d_n^{-1}(0) - d_{n+1}\rho_{n+1} = d_n^{-1}(0) - j_n\beta_{n+1}\rho_{n+1}.$$

It follows from (L), in §5 below, that we may take

$$(3.3) \quad \pi_n(K) = \pi_n(K^{n+1}) = \pi_n(K^n) - \beta_{n+1}\rho_{n+1}.$$

Therefore j_n determines a homomorphism $j_n^* : \pi_n(K) \rightarrow H_n(K)$.

THEOREM 8. If $\pi_1(K) = 1$ and if K is a J_m -complex, then $j_n^* : \pi_n(K) \rightarrow H_n(K)$ is an isomorphism onto for $n = 1, \dots, m$ and j_{m+1}^* is onto.

This is an immediate consequence of (3.2), (3.3) and Theorem 7 and its corollary.

We shall say that a space Y , n -dominates a space X if, and only if, there is a map, $f : X \rightarrow Y$, which has a left n -homotopy inverse, meaning a map $g : Y \rightarrow X$ such that $gf \simeq_n 1$.

THEOREM 9. If a CW-complex, K , is $(m-1)$ -dominated by a J_m -complex, L , then K is a J_m -complex.

¹³ Cf. [6], [5]. For the definition of $H_n(K)$ in terms of the relative homology groups $H_n(K^n, K^{n-1})$, treated as chain groups, see [14].

Let $f:K \rightarrow L$ and $g:L \rightarrow K$ be cellular maps such that $gf \simeq_{m-1} 1$. Since f, g are cellular they induce homomorphisms

$$\begin{aligned} \phi_n &: i_n \pi_n(K^{n-1}) \rightarrow i_n \pi_n(L^{n-1}), \\ \psi_n &: i_n \pi_n(L^{n-1}) \rightarrow i_n \pi_n(K^{n-1}). \end{aligned}$$

Let $\xi_t: K^{m-1} \rightarrow K^m$ be a cellular homotopy such that $\xi_0 = gf|_{K^{m-1}}, \xi_1 = 1$. Let $2 \leq n \leq m$. Then $\xi_t K^{n-1} \subset K^n$ and it follows that $\psi_n \phi_n = \alpha_n$, where α_n is an automorphism of $i_n \pi_n(K^{n-1})$. Therefore ψ_n is onto. But $i_n \pi_n(L^{n-1}) = 0$, by the corollary to Theorem 7. Therefore $i_n \pi_n(K^{n-1}) = 0$, whence K is a J_m -complex and Theorem 9 is proved.

If $K^m \equiv_{m-1} L^m$ then K^m and L^m ($m-1$)-dominate each other. Therefore we have the corollary:

COROLLARY. *The property of being a J_m -complex is an invariant of the m -type.*

Let K be a CW-complex such that $\pi_n(K) = 0$ for $n = 1, \dots, m-1$. Then it follows from Theorem 2 that the map $K^m \rightarrow L^0$ is an $(m-1)$ -homotopy equivalence, where L^0 is a single point. Since L^0 is (obviously) a J_m -complex it follows from Theorem 9 that K^m and hence K is a J_m -complex. Therefore Theorem 8 includes Hurewicz's result¹⁴ that $\pi_m(K) \approx H_m(K)$ if $\pi_n(K) = 0$ for $n = 1, \dots, m-1$. The following example of a simply connected J_4 -complex, K , with $\pi_3(K) \neq 0$, shows that Theorem 8 is more general than Hurewicz's theorem, restricted to CW-complexes. Let $K = e^0 \cup e^3 \cup e^4$, where e^0 is a 0-cell, e^3 is a 3-cell whose closure is a 3-sphere, $S^3 = e^0 \cup e^3$, and e^4 is attached¹⁵ to S^3 by a map, $f: \partial E^4 \rightarrow S^3$, of degree $(2r+1)(r > 0)$. Then $\pi_3(K)$ is cyclic of order $2r+1$. If $1 \leq n \leq 3$ then $K^{n-1} = K^0$, whence $\pi_n(K^{n-1}) = 0$. Let $g: S^4 \rightarrow \partial E^4$ be an essential map. Then $fg: S^4 \rightarrow S^3$ is essential [4, Theorem 9, p. 268] and hence represents the nonzero element of $\pi_4(S^3)$. Therefore [2, Lemma 4, p. 418] $i_4 \pi_4(K^3) = 0$, whence K is a J_4 -complex.

According to (M) and (N) in §5 below, a CW-complex, K , is locally contractible and its universal covering complex, \tilde{K} , is also a CW-complex.

THEOREM 10. *K is a J_m -complex if, and only if, \tilde{K} is a J_m -complex.*

This is an obvious consequence of the corollary to Theorem 7.

¹⁴ [15, p. 522]. See also [19, p. 314]. Hurewicz's theorem applies to a larger class of spaces than CW-complexes.

¹⁵ See §8 below. We use E^n to denote an n -element (i.e. a homeomorph of an n -simplex) and ∂E^n to denote its boundary. By a simplex we mean a simplex with boundary.

Let \tilde{K} be the universal covering complex of a CW-complex K . Since $\pi_1(\tilde{K}) = 1$, $\pi_n(\tilde{K}) \approx \pi_n(K)$ if $n > 1$ and since L is a J_m -complex if $\pi_r(L) = 0$ for $r = 1, \dots, m-1$ it follows from Theorem 10 that K is a J_m -complex if $\pi_n(K) = 0$ for $n = 2, \dots, m-1$. In particular K is a J_m -complex if its universal covering space is an m -sphere or any Euclidean space.

In Paper II we shall consider "Reidemeister" systems of chain groups, $C(K)$, $C(L)$ and chain mappings $C(K) \rightarrow C(L)$, where K , L , are CW-complexes. We shall prove that, if L is a J_m -complex and if $\dim K \leq m+1$, then any chain mapping, $C(K) \rightarrow C(L)$, can be realized by some (cellular) map, $K \rightarrow L$. Hence it follows from Theorem 3 that $K \equiv L$ if there is a chain equivalence $C(K) \rightarrow C(L)$. If, in addition [5, Lemma 4], K and L are finite, simply-connected complexes, then $K \equiv L$ if $H_n(K) \approx H_n(L)$ for all values of n .

4. Cell complexes.¹⁶ By a *cell complex*, K , or simply a *complex*, we mean a Hausdorff space, which is the union of disjoint (open) cells, to be denoted by e , e^n , e_i^n , etc., subject to the following condition. The closure, \bar{e}^n , of each n -cell, $e^n \in K$, shall be the image of a fixed n -simplex, σ^n , in a map, $f: \sigma^n \rightarrow \bar{e}^n$, such that

(4.1) (a) $f|_{\sigma^n - \partial\sigma^n}$ is a homeomorphism onto e^n ,

(b) $\partial e^n \subset K^{n-1}$, where $\partial e^n = f\partial\sigma^n = \bar{e}^n - e^n$ and K^{n-1} is the $(n-1)$ -section¹⁷ of K , consisting of all the cells whose dimensionalities do not exceed $n-1$.

Such a map will be called a *characteristic map* for the cell e^n . If $f: \sigma^n \rightarrow \bar{e}^n$ is a characteristic map for e^n , so obviously is $fh: \sigma^n \rightarrow \bar{e}^n$, where $h: (\sigma^n, \partial\sigma^n) \rightarrow (\sigma^n, \partial\sigma^n)$ is any map such that $h|_{\sigma^n - \partial\sigma^n}$ is a homeomorphism of $\sigma^n - \partial\sigma^n$ onto itself. No restriction other than $\partial e^n \subset K^{n-1}$ is placed on $f|_{\partial\sigma^n}$. Therefore \bar{e}^n need not coincide, as a point set, with a subcomplex of K . Since K , and hence \bar{e}^n , is a Hausdorff space and since σ^n is compact it follows that \bar{e}^n has the identification topology determined¹⁸ by f . A complex is defined as a topological space with a certain cell structure. Therefore we shall not need a separate letter to denote a complex and the space on which it lies.¹⁹ Notice that, in the absence of further restrictions, any (Hausdorff) space may be re-

¹⁶ The use of these complexes was suggested in [3, p. 1235]. They are now called cell complexes, rather than membrane complexes, in conformity with [14].

¹⁷ K^n is defined for every value of n . If there are no m -cells in K for $m > n$ then $K^n = K$.

¹⁸ I.e., $Y\bar{C}\bar{e}^n$ is closed if, and only if, $f^{-1}Y$ is closed. In other words the closed sets in \bar{e}^n are precisely the sets fX for every closed set, $X \subset \sigma^n$, which is saturated with respect to f , meaning that $f^{-1}fX = X$ (cf. [23, pp. 61, 95] and [24, p. 52]).

¹⁹ N.B. $e \in K$ will mean that e is a cell of the complex K and $e \subset K$, $\bar{e} \subset K$, etc., will mean that the sets of points e , \bar{e} , etc., are subsets of the space K .

garded as a complex. For example, we may take it to be the complex $K = K^0$, which consists entirely of 0-cells, each point in K being a 0-cell.

A subcomplex, $L \subset K$, is the union of a subset of the cells of K , which are the cells of L , such that, if $e \subset L$ then $\bar{e} \subset L$. Clearly L is a subcomplex if it is the union of a subset of the cells in K , which is a closed set of points in K . However the above example shows that a subcomplex need not be a closed set of points. Clearly K^n is a subcomplex, for each $n \geq 0$, and we admit the empty set as the subcomplex K^{-1} . Also the union and intersection of any set of subcomplexes, finite or infinite, are obviously subcomplexes. If $X \subset K$ is an arbitrary set of points we shall use $K(X)$ to stand for the intersection of all the subcomplexes of K , which contain X . Obviously $K(p) = K(e) = K(\bar{e})$, where p is any point in K and $e \in K$ is the cell which contains p . A finite subcomplex, L (i.e. one which contains but a finite number of cells) is a closed, and indeed a compact subset of K . For it is the union of the finite aggregate of compact sets, \bar{e} , for each cell $e \in L$.

The topological product, $K_1 \times K_2$, of complexes K_1, K_2 is a complex, whose cells are the products, $e^{n_1+n_2} = e^{n_1} \times e^{n_2}$, of all pairs of cells $e^{n_1} \in K_1, e^{n_2} \in K_2$. For let $f_i: \sigma^{n_i} \rightarrow \bar{e}^{n_i}$ ($i = 1, 2$) be a characteristic map for e^{n_i} , let $g: \sigma^{n_1} \times \sigma^{n_2} \rightarrow \bar{e}^{n_1+n_2}$ be given by $g(p_1, p_2) = (f_1 p_1, f_2 p_2)$ and let $h: \sigma^{n_1+n_2} \rightarrow \sigma^{n_1} \times \sigma^{n_2}$ be a homeomorphism (onto). Then $gh: \sigma^{n_1+n_2} \rightarrow \bar{e}^{n_1+n_2}$ obviously satisfies the conditions (4.1). Therefore $K_1 \times K_2$ is a complex, with this cell structure. In particular $K \times I$ is a complex, which consists of the cells $e \times 0, e \times 1, e \times (0, 1)$, for each cell $e \in K$, where $(0, 1)$ is the open interval $0 < t < 1$.

Let K be a locally connected complex, let \tilde{K} be a (locally connected) covering space of K and let $p: \tilde{K} \rightarrow K$ be the covering map. That is to say there is a basis, $\{U\}$, for the open sets in K such that, if $U \in \{U\}$ then p maps each component of $p^{-1}U$ homeomorphically onto U (cf. [20, p. 40]). Let $\bar{x} \in \tilde{K}$ be a given point and let $e^n \in K$ be the cell which contains $x = p\bar{x}$. Then a characteristic map, $f: \sigma^n \rightarrow \bar{e}^n$, can be "lifted" into a unique map,²⁰ $\tilde{f}: \sigma^n \rightarrow \tilde{K}$, such that $f = p\tilde{f}$ and $f(f^{-1}x) = \bar{x}$. Let $\bar{e}^n = \tilde{f}(\sigma^n - \partial\sigma^n)$ and let $p_0 = p|_{\bar{e}^n}$. Then $f|_{\sigma^n - \partial\sigma^n} = p_0(\tilde{f}|_{\sigma^n - \partial\sigma^n})$ and since $\tilde{f}|_{\sigma^n - \partial\sigma^n}$ is a (1-1) map onto \bar{e}^n it follows that p_0 is (1-1) and is onto e_n . Since p , and hence p_0 , is an open mapping it follows that p_0 is a homeomorphism. Since

$$\tilde{f}|_{\sigma^n - \partial\sigma^n} = p_0^{-1}(f|_{\sigma^n - \partial\sigma^n})$$

²⁰ See [21, Theorem 2, p. 40] or [22]. We shall sometimes use the same symbol, f or g , to denote two maps, $f: A \rightarrow B, g: A \rightarrow C \subset B$, such that $fa = ga$ for each point $a \in A$, even though $B \neq C$.

it follows that $\bar{f}|_{\sigma^n - \partial\sigma^n}$ is a homeomorphism, which, according to the definition of \bar{e}^n , is onto \bar{e}^n . Also $\bar{f}\partial\sigma^n \subset \bar{K}^{n-1} = \mathbf{p}^{-1}K^{n-1}$. Therefore \bar{f} satisfies the conditions (4.1). It follows that \bar{K} is a complex, each of whose cells is mapped by \mathbf{p} homeomorphically onto a cell of K .

Let Q be a subcomplex of \bar{K} and let e be a given cell in Q . Then $\mathbf{p}\bar{e}$ is closed, since \bar{e} is compact, and $\mathbf{p}\bar{e} \subset \mathbf{p}Q$. Therefore $\overline{\mathbf{p}\bar{e}} = \mathbf{p}\bar{e} \subset \mathbf{p}Q$. Therefore $\mathbf{p}Q$ is a subcomplex of K , which consists of the cells $\mathbf{p}e$ for each cell $e \in Q$.

5. CW-complexes. We shall describe a complex, K , as *closure finite* if, and only if, $K(e)$ is a finite subcomplex, for every cell $e \in K$. Since $K(\mathbf{p}) = K(e)$ if $\mathbf{p} \in e$ this is equivalent to the condition that $K(\mathbf{p})$ is finite for each point $\mathbf{p} \in K$. If $L \subset K$ is a subcomplex and $e \in L$ then obviously $L(e) = K(e)$. Therefore any subcomplex of a closure finite complex is closure finite.

We shall say that K has the *weak topology* (cf. [1, pp. 316, 317]) if, and only if, a subset $X \subset K$ is closed (open) provided $X \cap \bar{e}$ is closed (relatively open) for each cell $e \in K$. If K is closure finite this is equivalent to the condition that X is closed provided $X \cap L$ is closed for every finite subcomplex $L \subset K$. For $X \cap L$ is the union of the finite number of sets $X \cap \bar{e}$ ($e \in L$). Therefore $X \cap L$ is closed if each set $X \cap \bar{e}$ is closed. Conversely, if $X \cap L$ is closed for each finite subcomplex, L , and if $K(\bar{e})$ is finite, then $X \cap \bar{e}$ is closed, since $X \cap \bar{e} = X \cap K(\bar{e}) \cap \bar{e}$.

By a *CW-complex* we mean one which is closure finite and has the weak topology. Any finite complex, K , is obviously closure finite and it has the weak topology since $X \subset K$ is the union of the finite number of sets $X \cap \bar{e}$ ($e \in K$). Therefore any finite complex is a CW-complex. Also a complex, K , is a CW-complex if it is *locally finite*, meaning that each point $\mathbf{p} \in K$ is an inner point of some finite subcomplex of K . For let K be locally finite. Then $K(\mathbf{p})$ is finite, for each point $\mathbf{p} \in K$. Therefore K is closure finite. Let $X \subset K$ be such that $X \cap L$ is closed for each finite subcomplex $L \subset K$. Let L be a finite subcomplex of which a given point $\mathbf{p} \in K - X$ is an inner point. Since $X \cap L$ is closed, \mathbf{p} is an inner point of $L - X = L - (X \cap L)$. Therefore X is closed and K has the weak topology. It may be verified that the number of cells, and hence the number of finite subcomplexes of a connected, locally finite complex, K , is countable. Hence, and from (G) below, it may be proved that K is a separable metric space.

If the cells in a CW-complex, K , have a maximum dimensionality we call this the dimensionality, $\dim K$, of K . If there is no such maximum we write $\dim K = \infty$.

Examples of complexes which are not CW-complexes are:

(1) $\partial\sigma^n$ ($n > 1$) regarded as a "0-dimensional" complex, K^0 , whose cells are the points of $\partial\sigma^n$. This is closure finite but does not have the weak topology.

(2) σ^n ($n > 1$), regarded as a complex $K^n = K^0 \cup e^n$, where $e^n = \sigma^n - \partial\sigma^n$ and $K^0 = \partial\sigma^n$, as in (1). This has the weak topology, since $\bar{e}^n = K^n$, but is not closure finite.

(3) a simplicial complex, which has a metric topology but which is not locally finite (e.g. a complex covering the coordinate axes in Hilbert space). The weak topology in such a complex cannot be metricized (cf. [1, pp. 316, 317]).

Let K be a CW-complex. We establish some properties of K .

(A) *A map, $f: X \rightarrow Y$, of a closed (open) subset, $X \subset K$, in any space, Y , is continuous provided $f|X \cap \bar{e}$ is continuous for each cell $e \in K$.*

Let $f_e = f|X \cap \bar{e}$ be continuous, for each cell $e \in K$. Let Y_0 be any closed (open) subset of Y . Obviously $\bar{e} \cap f^{-1}Y_0 = f_e^{-1}Y_0$ and it follows from the continuity of f_e that $\bar{e} \cap f^{-1}Y_0$ is a relatively closed (open) subset of $X \cap \bar{e}$. But $X \cap \bar{e}$ is a closed (relatively open) subset of \bar{e} , whence $\bar{e} \cap f^{-1}Y_0$ is closed (relatively open) in \bar{e} . Therefore $f^{-1}Y_0$ is closed (open) in K , and a fortiori in X . Therefore f is continuous.

(B) *A subcomplex, $L \subset K$, is a closed subspace of K and the topology induced by K is the weak topology in L .*

Let $Y \subset L$ be such that $Y \cap L_0$ is closed, and hence compact, for each finite subcomplex $L_0 \subset L$. Since $Y \cap L_0$ is compact it is a closed subset of K . Let K_0 be any finite subcomplex of K . Then $L_0 = L \cap K_0$ is a finite subcomplex of L and

$$Y \cap K_0 = Y \cap L \cap K_0 = Y \cap L_0.$$

Therefore $Y \cap K_0$ is closed, whence Y is closed in K , and a fortiori in L . Therefore L has the weak topology. Also, taking $Y = L$, it follows that L is closed, which establishes (B).

(C) *If K is connected so is K^n for each $n > 0$.*

Let $n > 0$ and let K^n be the union of disjoint, nonvacuous closed sets K_1^n, K_2^n . Since the closure of a cell $e \in K$ is connected it follows that $\bar{e} \subset K_i^n$ if $e \cap K_i^n \neq \emptyset$ ($i = 1, 2$). Therefore K_i^n is a subcomplex of K . Clearly ∂e^{n+1} is connected ($e^{n+1} \in K$), whence it lies either in K_1^n or in K_2^n . Therefore K^{n+1} is the union of disjoint subcomplexes, K_1^{n+1}, K_2^{n+1} , where $K_i^n \subset K_i^{n+1}$ and $e^{n+1} \in K_i^{n+1}$ if $\partial e^{n+1} \subset K_i^n$. A similar (inductive) argument shows that K^m is the union of disjoint subcomplexes, K_1^m, K_2^m , such that $K_i^m \subset K_i^{m+1}$ ($m = n, n+1, \dots$). Let K_i be the union of the K_i^m for $m = n, n+1, \dots$. Then $K_i \cap K^m = K_i^m$ and

$$K_1 \cap K_2 = \bigcup_m (K_1 \cap K_2) \cap K^m = \bigcup_m K_1^m \cap K_2^m = \emptyset.$$

Also $K_i \neq 0$, since $K_i^n \subset K_i$, and K_i is a closed subset of K , according to (B). Therefore K is not connected, which establishes (C).

(D) *If $X \subset K$ is compact, then $K(X)$ is a finite complex.*

If X meets but a finite number of cells, $e_1, \dots, e_k \subset K$, it is contained in the finite union of the (finite) subcomplexes $K(e_1), \dots, K(e_k)$. Assume that there is an infinite set of cells, $\{e_i\}$, each of which meets X and let $p_i \in X \cap e_i$. Then a finite subcomplex, $L \subset K$, contains but a finite set of the cells in $\{e_i\}$ and $e_i \cap L = 0$ unless $e_i \in L$. Therefore L contains but a finite number of points in the set $P = \{p_i\}$, whence P is closed. Similarly any subset of P is closed, whence P is discrete. But this is absurd, since P is compact, being a closed subset of X . Therefore (D) is established.

(E) *If a complex L , and also L^n for each $n \geq 0$, all have the weak topology, then L is a CW-complex.*

Certainly L^0 is closure finite. Assume that L^{n-1} is closure finite, and hence a CW-complex, for some $n > 0$. Let e^n be a given n -cell in L^n . Since ∂e^n is compact it follows from (D) that $L(\partial e^n)$ is finite. But obviously $L(e^n) = L(\partial e^n) \cup e^n$ and it follows from induction on n that L is closure finite, which establishes (E).

Let $f: K \rightarrow L$ be a map of K onto a closure finite complex L , which has the identification topology¹⁸ determined by f . Further let the subcomplex $L(f\bar{e})$ be finite for each cell $e \in K$.

(F) *Subject to these conditions L is a CW-complex.*

Let $Y \subset L$ be such that $Y \cap L_0$ is closed for each finite subcomplex $L_0 \subset L$. Let $L_0\bar{e} = L(f\bar{e})$ for a given cell $e \in K$. Then $\bar{e} \subset f^{-1}L_0$ and

$$f^{-1}Y \cap \bar{e} = f^{-1}Y \cap \bar{e} \cap \bar{e} \subset (f^{-1}Y \cap f^{-1}L_0) \cap \bar{e} \subset f^{-1}(Y \cap L_0) \cap \bar{e},$$

since $f^{-1}A \cap f^{-1}B \subset f^{-1}(A \cap B)$ for any sets $A, B \subset L$. But $f^{-1}(Y \cap L_0) \subset f^{-1}Y$. Therefore

$$f^{-1}Y \cap \bar{e} = f^{-1}(Y \cap L_0) \cap \bar{e}.$$

Since $Y \cap L_0$ is closed it follows that $f^{-1}Y \cap \bar{e}$ is closed. Therefore $f^{-1}Y$ is closed, since K has the weak topology. Since L has the identification topology determined by f it follows that Y is closed. Therefore L has the weak topology. Since L is closure finite by hypothesis this proves (F).

(G) *K is a normal space.*

Let $X_1, X_2 \subset K$ be disjoint, closed subsets and let $X_i^r = X_i \cap K^r$ ($i=1, 2; r \geq 0$). Clearly K^0 is a discrete set, and hence normal. Let $n > 0$ and assume that there are disjoint, relatively open subsets, $U_1^{n-1}, U_2^{n-1} \subset K^{n-1}$, such that $X_i^{n-1} \subset U_i^{n-1}$. Then $X_i \cap \bar{U}_j^{n-1} = 0$ ($i, j=1, 2; i \neq j$). If $K^n = K^{n-1}$ we define $U_i^n = U_i^{n-1}$. Otherwise let

$f: \sigma^n \rightarrow \bar{e}^n$ be a characteristic map for a given n -cell $e^n \in K$ and let

$$V_i = f^{-1}U_i^{n-1} \subset \partial\sigma^n, \quad Y_i = f^{-1}X_i \subset \sigma^n.$$

Since $X_1 \cap X_2 = 0, X_i \cap \bar{U}_j^{n-1} = 0$ we have $Y_1 \cap Y_2 = 0, Y_i \cap \bar{V}_j = 0$. Let p_0 be the centroid of σ^n and let r, p be polar coordinates for σ^n ($r \in I, p \in \partial\sigma^n$) such that (r, p) is the point which divides the rectilinear segment p_0p in the ratio $r:1-r$. Let $V'_i \subset \sigma^n$ be the (open) subset, which consists of all points (r, p) with $p \in V_i$ and $1-\epsilon < r \leq 1$, where $0 < \epsilon < 1$. Since $Y_i \cap \bar{V}_j = 0$ it follows that, if ϵ is sufficiently small, then $Y_i \cap \bar{V}'_j = 0$, which we assume to be the case. Since $fY_i \subset X_i, f\partial\sigma^n \subset K^{n-1}$ and $X_i \cap K^{n-1} \subset U_i^{n-1}$ it follows that $Y_i \cap \partial\sigma^n \subset V_i$. Let V''_i be an η -neighborhood of Y_i , defined in terms of a metric for σ^n , where η is so small that $V''_i \cap V''_j = 0, V''_i \cap \bar{V}'_j = 0$ and $V''_i \cap (\partial\sigma^n - V_i) = 0$. Then $V''_i \cap \partial\sigma^n \subset V_i$. Let

$$W_i = V'_i \cup V''_i.$$

Then $Y_i \subset W_i$ and $W_1 \cap W_2 = 0$. Obviously $V'_i \cap \partial\sigma^n = V_i$, whence

$$(5.1) \quad W_i \cap \partial\sigma^n = V_i = f^{-1}U_i^{n-1}.$$

Since $f: \sigma^n - \partial\sigma^n$ is a (1-1) map onto e^n and $f\partial\sigma^n \cap e^n = 0$ it follows that W_i is saturated¹⁸ with respect to f . Therefore fW_i is a relatively open subset of \bar{e}^n . From (5.1) we have

$$(5.2) \quad fW_i \cap K^{n-1} = U_i^{n-1} \cap \partial e^n$$

and it follows that $fW_1 \cap fW_2 = 0$.

Let us write $W_i = W_i(e^n)$ and let

$$U_i^n = U_i^{n-1} \cup \bigcup_{e^n \in K} fW_i(e^n).$$

Then it follows from (5.2) that $U_i^n \cap K^{n-1} = U_i^{n-1}$ and that

$$U_i^n \cap \partial e^n = U_i^{n-1} \cap \partial e^n = fW_i(e^n) \cap K^{n-1}.$$

Also $fW_i(e^n) \subset K^{n-1} \cup e^n$ and $U_i^n \cap e^n = fW_i(e^n) \cap e^n$. Therefore

$$\begin{aligned} U_i^n \cap \bar{e}^n &= (U_i^n \cap \partial e^n) \cup (U_i^n \cap e^n) \\ &= fW_i(e^n) \cap (K^{n-1} \cup e^n) \\ &= fW_i(e^n). \end{aligned}$$

Therefore U_i^n is a relatively open subset of K^n . Obviously $X_i^n \subset U_i^n$ and $U_1^n \cap U_2^n = 0$. Therefore such sets, U_i^n , may be defined inductively for every value of n . Let them be so defined and let

$$U_i = \bigcup_n U_i^n.$$

Since $U_i^{n+1} \cap K^n = U_i^n$ it follows by induction on $m > n$ that

$$U_i^m \cap K^n = U_i^m \cap K^{m-1} \cap K^n = U_i^{m-1} \cap K^n = U_i^n$$

and hence that $U_i \cap K^n = U_i^n$. Therefore it follows, first that U_i is an open subset of K and second that $U_1 \cap U_2 = 0$. Obviously $X_i \subset U_i$, which completes the proof of (G).

(H) *If L is a locally finite²¹ complex then $K \times L$ is a CW-complex.*

If $e \in K, e' \in L$ are cells in K and L respectively, then the cell $e \times e' \in K \times L$ is contained in the finite subcomplex $K(e) \times L(e') \subset K \times L$. Therefore $K \times L$ is closure finite.

Let the cells in K be indexed and with each m -cell, $e_i^m \in K, (m = 0, 1, \dots)$ let us associate an m -element, E_i^m , as follows. The points in E_i^m shall be the pairs (x, e_i^m) , for every point $x \in \sigma^m$, and E_i^m shall have the topology which makes the map $x \rightarrow (x, e_i^m)$ a homeomorphism. No two of these elements have a point in common and we unite them into a topological space,

$$P = \bigcup_{m,i} E_i^m,$$

in which each E_i^m , with its own topology, is both open and closed. Let $f_i^m: \sigma^m \rightarrow \bar{e}_i^m$ be a characteristic map for e_i^m and let $\phi: P \rightarrow K$ be the map which is given by $\phi(x, e_i^m) = f_i^m x$, for each point $(x, e_i^m) \in P$. Since \bar{e}_i^m has the identification topology determined by f_i^m it follows that the weak topology in K is the identification topology determined by ϕ .

Let a space,

$$Q = \bigcup_{n,j} E_j^n,$$

and a map, $\psi: Q \rightarrow L$, be similarly associated with L . Then $K \times L = \theta(P \times Q)$, where $\theta: P \times Q \rightarrow K \times L$ is given by $\theta(p, q) = (\phi p, \psi q)$ ($p \in P, q \in Q$). Also $P \times Q$ is the union of the $(m+n)$ -elements $E_i^m \times E_j^n$, and $\theta(E_i^m \times E_j^n) = \bar{e}_{ij}^{m+n}$, where $\bar{e}_{ij}^{m+n} = e_i^m \times e_j^n$. Therefore the weak topology in $K \times L$ is obviously the same as the identification topology determined by θ .

Let $V \subset L$ be an open subset and $y \in V$ an arbitrary point in V . Since y is an inner point of a finite subcomplex, $L_0 \subset L$, it is contained in a subset, $V_0 \subset V \cap L_0$, which is open in L . Since L is normal there is a neighborhood, W , of y such that $\bar{W} \subset V_0$. Since $\bar{W} \subset V_0 \subset L_0$ and

²¹ I do not know if this restriction on L is necessary.

since V_0 is open in L , it follows that $\overline{W} \cap \bar{e} = 0$ for any cell $e \in L - L_0$. Therefore there are only a finite number of cells in L , whose closures meet \overline{W} . Therefore $\psi^{-1}\overline{W}$ is contained in the union of a finite subset of the components $E_i^n \subset Q$. Therefore $\psi^{-1}\overline{W}$ is compact and (H) follows from Lemma 4 in [7].

(I) *A homotopy, $f_t: X \rightarrow Y$, of a closed (open) subset, $X \subset K$, in an arbitrary space, Y , is continuous provided $f_t|X \cap \bar{e}$ is continuous for each cell $e \in K$.*

This follows from (H), with $L = I$, and (A), applied to the subset $X \times I \subset K \times I$ and the map $f: X \times I \rightarrow Y$, which is given by $f(x, t) = f_t x$.

(J) *(Homotopy extension.) Let $f_0: K \rightarrow X$ be a given map of K in an arbitrary space X . Let $g_t: L \rightarrow X$ be a homotopy of $g_0 = f_0|L$, where L is a subcomplex of K . Then there is a homotopy, $f_t: K \rightarrow X$, such that $f_t|L = g_t$.*

Let $K_r = L \cup K^r$ ($r \geq -1$; $K_{-1} = L$) and assume that g_t has been extended to a homotopy, $f_t^{n-1}: K_{n-1} \rightarrow X$, such that $f_0^{n-1} = f_0|K_{n-1}$, $f_t^{n-1}|L = g_t$ ($n \geq 0$). The homotopy f_t^{n-1} can be extended throughout $K_{n-1} \cup e^n$, for each n -cell²² $e^n \in K_n - L$, and hence, by (I), to a (continuous) homotopy $f_t^n: K_n \rightarrow X$. Starting with $f_t^{-1} = g_t$ it follows by induction on n that there is a sequence of homotopies, $f_t^n: K_n \rightarrow X$ ($n = 0, 1, \dots$), such that $f_0^n = f_0|K_n$, $f_t^n|K_{n-1} = f_t^{n-1}$. It follows from (I) that a homotopy, $f_t: K \rightarrow X$, which satisfies the requirements of (J), is given by $f_t|K_n = f_t^n$.

Let $X_0 \subset X_1 \subset \dots$ be a sequence of subspaces of a given space, X , such that any map, $(\sigma^n, \partial\sigma^n) \rightarrow (X, X_{n-1})$, is homotopic, rel. $\partial\sigma^n$, to a map²³ $\sigma^n \rightarrow X_n$ ($n = 0, 1, \dots$). Let $L \subset K$ be a given subcomplex, which may be empty, and let $f_0: K \rightarrow X$ be a map such that $f_0 L^n \subset X_n$, for each $n = 0, 1, \dots$.

(K) *There is a homotopy, $f_t: K \rightarrow X$, rel. L , such that $f_t K^n \subset X_n$ for each $n = 0, 1, \dots$.*

Since each point in X is joined by an arc to some point in X_0 there is a homotopy, $f_t^0: K^0 \rightarrow X$, rel. L^0 , such that $f_0^0 = f_0|K^0$ and $f_1^0 K^0 \subset X_0$. Let $n > 0$ and assume that there is a homotopy $f_t^{n-1}: K^{n-1} \rightarrow X$, rel. L^{n-1} , such that $f_0^{n-1} = f_0|K^{n-1}$, $f_1^{n-1} K^{n-1} \subset X_{n-1}$. It follows from (J) that f_t^{n-1} can be extended, first throughout L^n by writing $f_t^{n-1}|L^n = f_0|L^n$, and then to a homotopy, $\xi_t: K^n \rightarrow X$, rel. L^n ($\xi_0 = f_0|K^n$). Since $\xi_1 K^{n-1} \subset X_{n-1}$ it follows from a standard argument (see [6, §8]), and the condition on X_0, X_1, \dots , that there is a homotopy, $\eta_t: K^n \rightarrow X$, rel. $(K^{n-1} \cup L^n)$, such that $\eta_0 = \xi_1$, $\eta_1 K^n \subset X_n$. If $\dim K < \infty$ we

²² See [5, Lemma 10 in §16].

²³ If $n = 0$ this simply means that each point in X is joined by an arc to some point in X_0 .

define $f_t^n: K^n \rightarrow X$ as the resultant of ξ_t followed by η_t . Then f_t^n may be defined inductively for every $n \geq 0$ and we take $f_t = f_t^m$, where $m = \dim K$. But if $\dim K = \infty$ this method fails and we shall define f_t^n as an extension of f_t^{n-1} , not as the resultant of ξ_t followed by η_t .

If $K^n = K^{n-1}$ we define $f_t^n = f_t^{n-1}$. Otherwise let $g: \sigma^n \rightarrow \bar{e}^n$ be a characteristic map for a given n -cell $e^n \in K$. Let r, ρ be polar coordinates for σ^n , defined as in (G), and let $\rho_t: \bar{e}^n \rightarrow X$ be defined by

$$(5.3) \quad \begin{aligned} \rho_t g(r, \rho) &= \xi_{2t/(1+r)} g(r, \rho) && (\text{if } 0 \leq 2t \leq 1+r) \\ &= \eta_{(2t-1-r)/(1-r)} g(r, \rho) && (\text{if } 1+r < 2t \leq 2). \end{aligned}$$

Since $\eta_0 = \xi_1$ and $g^{-1}|e^n$ is a homeomorphism onto $\sigma^n - \partial\sigma^n$ it follows that $\rho_t|e^n$ is single-valued and continuous. Since $\rho_t x = \xi_t x$ for any point $x = g(1, \rho) \in \partial e^n$ it follows that ρ_t is single-valued. Also ρ_t is continuous at $\{g(r, \rho), t\}$ if $r < 1$ and, obviously, if $t < 1$. I say that it is continuous at $\{g(1, \rho), 1\} = (gp, 1)$. For $gp \in K^{n-1}$ and $\eta_t|K^{n-1} = \eta_0|K^{n-1} = \xi_1|K^{n-1}$. Therefore, given a neighborhood, $U \subset X$, of $\xi_1 gp = \eta_1 gp$, it follows from the compactness of I that there is a neighborhood, $V \subset \bar{e}^n$, of gp such that $\eta_t x \in U$ for every $t \in I$, provided $x \in V$. There is also a neighborhood, $V' \subset \bar{e}^n$, of gp , and a $\delta > 0$ such that $\xi_t x \in U$ if $x \in V'$, $1 - 2\delta < t \leq 1$. Since $(2 - 2\delta)/(1+r) > 1 - 2\delta$ it follows that $\rho_t x \in U$ if $x \in V \cap V'$, $1 - \delta < t \leq 1$. Therefore ρ_t is continuous. Also

$$\begin{aligned} \rho_t g(1, \rho) &= \xi_t g(1, \rho) = f_t^{n-1} g(1, \rho), \\ \rho_0 g(r, \rho) &= \xi_0 g(r, \rho) = f_0 g(r, \rho), \\ \rho_1 g(r, \rho) &= \eta_1 g(r, \rho) \in X_n. \end{aligned}$$

Therefore a homotopy, $f_t^n: K^n \rightarrow X$, rel. L^n , such that

$$f_t^n|K^{n-1} = f_t^{n-1}, \quad f_0^n = f_0|K^n, \quad f_1 K^n \subset X_n,$$

is defined by $f_t^n|K^{n-1} = f_t^{n-1}$, $f_t^n|\bar{e}^n = \rho_t$, for each n -cell $e^n \in K^n$. It follows from induction on n that such a homotopy is defined for each $n \geq 0$ and a homotopy, $f_t: K \rightarrow X$, which satisfies the requirements of (K), is defined by $f_t|K^n = f_t^n$.

Let $f_0: K \rightarrow P$ be a map of K into a CW-complex, P , such that $f_0|L$ is cellular, where $L \subset K$ is a subcomplex. Also let $g_t: K \rightarrow P$ be a homotopy such that the maps g_0, g_1 and the homotopy $g_t|L$ are cellular.

(L) *There is a homotopy, $f_t|K \rightarrow P$, rel. L , of f_0 into a cellular map f_1 . There is a cellular homotopy, $g'_t: K \rightarrow P$, such that $g'_0 = g_0, g'_1 = g_1, g'_t|L = g_t|L$.*

Since any continuous image of σ^n in P is compact it is contained in

a finite subcomplex $Q \subset P$, according to (D). Any map $(\sigma^n, \partial\sigma^n) \rightarrow (Q, Q^{n-1})$ is homotopic,²⁴ rel. $\partial\sigma^n$, in Q to a map $\sigma^n \rightarrow Q^n$. Therefore the first part follows from (K). The second part follows from the first part with K, L, f_0 replaced by $K \times I, (K \times 0) \cup (L \times I) \cup (K \times 1), g: K \times I \rightarrow P$, where $g(p, t) = g_t p$.

(M) K is locally contractible.

Let $a_0 \in K$ be a given point, let $U \subset K$ be a given neighborhood of a_0 and let $e^r \in K$ be the cell which contains a_0 . Let $E^r \subset U \cap e^r$ be an r -element, which contains a_0 in its interior, $V^r = E^r - \partial E^r$, and let $f_t^r: V^r \rightarrow V^r$ be a homotopy such that $f_0^r = 1, f_1^r V^r = a_0$. Using induction on n we shall define sequences of relatively open subsets, $V^n \subset K^n$ ($n = r, r+1, \dots$), such that $V^{n+1} \cap K^n = V^n, \bar{V}^n \subset U$, and of homotopies $f_t^n: V^n \rightarrow V^n$, such that $f_t^{n+1}|_{V^n} = f_t^n, f_0^n = 1, f_1^n V^n = a_0$. Assuming that this has been done, let

$$V = \bigcup_n V^n$$

and let $f_t: V \rightarrow V$ be defined by $f_t|_{V^n} = f_t^n$. Then it follows from the definition of the weak topology and from (I) that V is open in K and f_t continuous. Obviously $V \subset U, f_0 = 1, f_1 V = a_0$ and (M) follows.

Assume that V^{n-1} and f_t^{n-1} satisfy the above conditions for some $n > r$. Let $g: \sigma^n \rightarrow e^n$ be a characteristic map for a given n -cell, $e^n \in K$, and let polar coordinates, r, p , for σ^n be defined as in (G). If $g\partial\sigma^n \cap V^{n-1} = 0$, let $W \subset \sigma^n$ be the empty set. Otherwise let $W \subset \sigma^n$ be the (open) subset, which consists of all points, (r, p) , such that

$$1 - \epsilon < r \leq 1, \quad p \in g^{-1}V^{n-1},$$

where $0 < \epsilon < 1$. Since $\bar{V}^{n-1} \subset U$, whence $g^{-1}\bar{V}^{n-1} \subset g^{-1}U$, it follows that $\bar{W} \subset g^{-1}U$ if ϵ is sufficiently small, which we assume to be the case. Let $\xi_t: W \rightarrow W$ be the "radial projection," which is defined by

$$\xi_t(r, p) = (r + t - rt, p),$$

and let $\theta_t g: W \rightarrow V^{n-1} \cup gW$ be given by

$$\begin{aligned} \theta_t g(r, p) &= g\xi_{2t/(1-r)}(r, p) && \text{(if } 0 \leq 2t < 1 - r) \\ &= f_{(2t-1+r)/(1+r)}^{n-1} g(1, p) && \text{(if } 1 - r \leq 2t \leq 2). \end{aligned}$$

Since $\xi_t(1, p) = (1, p)$ and $g\xi_1(r, p) = g(1, p) = f_0^{n-1} g(1, p)$ it follows from an argument similar to the one which comes after (5.3) that θ_t is single-valued and continuous. Also

²⁴ [5, §16, Theorem 6]. It follows from Theorem 6 in [5] that the condition $f_t x \in P(f_0 x) (x \in K)$ may be imposed on the homotopy f_t in (L).

$$\begin{aligned}
 (5.4) \quad & \theta_i g(1, p) = f_i^{n-1} g(1, p), \\
 & \theta_0 g(r, p) = g\xi_0(r, p) = g(r, p), \\
 & \theta_1 g(r, p) = f_1^{n-1} g(1, p) = a_0.
 \end{aligned}$$

Let V^n be the union of V^{n-1} and the sets gW , which are thus defined for all the n -cells in K . Arguments used in (G) show that V^n is a relatively open subset of K^n and that $V^n \cap K^{n-1} = V^{n-1}$. Also it follows from the definition of W that $\partial\sigma^n \cap \overline{W} \subset g^{-1}\overline{V}^{n-1}$, whence

$$K^{n-1} \cap g\overline{W} \subset \overline{V}^{n-1}.$$

Hence it follows from the definition of the weak topology that \overline{V}^n is the union of \overline{V}^{n-1} and the sets $g\overline{W}$, which are closed since \overline{W} is compact. Since $\overline{V}^{n-1} \subset U$, $\overline{W} \subset g^{-1}U$ it follows that $\overline{V}^n \subset U$. Finally define $f_i^n: V^n \rightarrow V^n$ by $f_i^n|_{V^{n-1}} = f_i^{n-1}$, $f_i^n|_{gW} = \theta_i$. It follows from (5.4) and from (I) that f_i^n is single-valued and continuous and that $f_0^n = 1$, $f_1^n V^n = a_0$. Therefore (M) follows by induction on n .

(N) *Any covering complex, \tilde{K} , of K is a CW-complex.*

Since \tilde{K} is locally connected, by the definition of a covering space, each of its components is both open and closed and is a covering complex of a component of K . A locally connected complex is obviously a CW-complex if, and only if, each of its components is a CW-complex. Therefore (N) will follow when we have proved it in case K and \tilde{K} are connected. We assume that this is so and also, to begin with, that \tilde{K} is a regular covering complex of K . That is to say the group, G , of covering transformations²⁵ in \tilde{K} operates transitively on the set $\mathbf{p}^{-1}q$, for any point $q \in K$, where $\mathbf{p}: \tilde{K} \rightarrow K$ is the covering map. We shall describe an open set, $U \subset K$, as an *elementary neighborhood* if, and only if, each component of $\mathbf{p}^{-1}U$ is mapped by \mathbf{p} topologically onto U . We shall describe an elementary neighborhood in K as a *basic neighborhood* if, and only if, its closure is contained in an elementary neighborhood. We shall describe a subset of \tilde{K} as a *basic neighborhood* if and only if it is a component of $\mathbf{p}^{-1}U$, where U is a basic neighborhood in K . If $\tilde{U} \subset \tilde{K}$ is a basic neighborhood the component of $\mathbf{p}^{-1}(\mathbf{p}\tilde{U})$ are the sets $T\tilde{U}$ for every $T \in G$. It follows from the definition of \tilde{K} and the normality of K that the basic neighborhoods constitute a basis for the open sets, both in K and in \tilde{K} .

Let $U \subset K$ be a basic neighborhood and let V be an elementary neighborhood such that $\overline{U} \subset V$. Then the components of $\mathbf{p}^{-1}V$ are disjoint open sets in \tilde{K} , each of which contains exactly one component of $\mathbf{p}^{-1}\overline{U}$. Let $Q \subset \mathbf{p}^{-1}\overline{U}$ be a set of points, of which at most

²⁵ I.e., the group of homeomorphisms, $T: \tilde{K} \rightarrow \tilde{K}$, such that $\mathbf{p}T = \mathbf{p}$.

one lies in each component of $p^{-1}\bar{U}$. Then Q is a closed discrete set. For if Q has a limit point, \bar{q} , then $p\bar{q} \in \bar{U} \subset V$, whence \bar{q} lies in one of the components, \bar{V} , of $p^{-1}V$. But this is absurd, since \bar{V} contains at most one point of Q . Therefore Q is closed and discrete.

Let $\bar{U} \subset \bar{K}$ be a basic neighborhood, let U^* be its closure²⁶ and let $C \subset \bar{K}$ be compact. I say that only a finite number of the sets TC meet U^* , where $T \in G$. For if TC meets U^* then C meets $T^{-1}U^*$. Let $q_T \in C \cap T^{-1}U^*$. Since $T'U^* \cap T''U^* = 0$ if $T' \neq T''$ it follows from the preceding paragraph that the aggregate of points q_T , for every T such that $U^* \cap TC \neq 0$, is a discrete, closed subset of C . Since C is compact the set $\{q_T\}$ is finite, which proves our assertion.

We now prove that \bar{K} has the weak topology. Let $\bar{X} \subset \bar{K}$ be a subset such that $\bar{X} \cap e^*$ is closed, for every cell $\bar{e} \in \bar{K}$. In order to prove that \bar{X} is closed it is enough to prove that $\bar{X} \cap U^*$ is closed, where U^* is the closure of an arbitrary basic neighborhood $\bar{U} \subset \bar{K}$. For this implies that $\bar{U} - \bar{X} = \bar{U} - (\bar{X} \cap U^*)$ is open, whence it follows that $\bar{K} - \bar{X}$ is open. Therefore, to simplify the notation, we assume that $\bar{X} \subset U^*$, where \bar{U} is a basic neighborhood in \bar{K} . Let $X = p\bar{X}$ and let e be a given cell in K . Then²⁷

$$X \cap \bar{e} = p(\bar{X} \cap p^{-1}\bar{e}).$$

Let $\bar{e} \in \bar{K}$ be a cell which covers e . Then $p^{-1}\bar{e}$ consists of the sets Te^* for every $T \in G$, and Te^* is the closure of the cell $T\bar{e} \in \bar{K}$. Since e^* is compact it follows from the preceding paragraph that only a finite number of the sets Te^* , say T_1e^*, \dots, T_ke^* , meet U^* . Let $P_i = \bar{X} \cap T_ie^*$ ($i = 1, \dots, k$). Then

$$X \cap \bar{e} = p(\bar{X} \cap p^{-1}\bar{e}) = p(P_1 \cup \dots \cup P_k).$$

But P_i is closed, by the hypothesis concerning \bar{X} , and hence compact, since T_ie^* is compact. Therefore, $P_1 \cup \dots \cup P_k$ and hence $X \cap \bar{e}$ are compact. Since the cell $e \in K$ is arbitrary it follows that X is closed. Therefore $p^{-1}X$ is closed. Since $U^* \cap TU^* = 0$ if $T \neq 1$ it follows that

$$\bar{X} = U^* \cap \bigcup_T T\bar{X} = U^* \cap p^{-1}X.$$

Therefore \bar{X} is closed and it follows that \bar{K} has the weak topology.

Since K^0 is discrete it follows that $\bar{K}^0 = p^{-1}K^0$ is a discrete set of points. That is to say, \bar{K}^0 has the weak topology. If $n > 0$ then K^n is connected, according to (C), and \bar{K}^n is obviously a covering complex of K^n . It follows from (L) that the injection homomorphism, $\pi_1(K^n)$

²⁶ We shall denote the closure of a set $\bar{P} \subset \bar{K}$ by P^* .

²⁷ If $f: P \rightarrow Q$ is any map and $A \subset P, B \subset Q$, then $f(A \cap f^{-1}B) = (fA) \cap B$.

$\rightarrow \pi_1(K)$, is onto, whence \tilde{K}^n is connected. Obviously $T\tilde{K}^n = \tilde{K}^n$ for any $T \in G$ and it follows that \tilde{K}^n is a regular covering complex of K^n . Therefore \tilde{K}^n has the weak topology, according to what we have just proved. It follows from (E) that \tilde{K} is a CW-complex.

Now let \tilde{K} be a (connected) covering complex of K , which is not regular. Then a universal covering complex, \hat{K} of \tilde{K} is a universal covering complex of K . Therefore \hat{K} is a CW-complex. Let $p: \hat{K} \rightarrow \tilde{K}$ be the covering map. Since p is an open map it follows that \tilde{K} has the identification topology determined by p . It follows from the final paragraph in §4 that \tilde{K} is closure finite and that the remaining condition of (F) is satisfied. Therefore it follows from (F) that \tilde{K} is a CW-complex, which completes the proof of (N).

6. Proof of Theorems 1-4. Let X, Y be spaces in the class α and let $f: X \rightarrow Y$ be a map such that the induced homomorphism, $f_n: \pi_n(X) \rightarrow \pi_n(Y)$, is an isomorphism onto if $1 \leq n < N+1$, where $N = \max(\Delta X, \Delta Y)$. Let P and Q be CW-complexes, of dimensionalities ΔX and ΔY , which dominate X and Y respectively. Let $\lambda: X \rightarrow P$, $\lambda': P \rightarrow X$ and $\mu: Y \rightarrow Q$, $\mu': Q \rightarrow Y$ be maps such that $\lambda'\lambda \simeq 1$, $\mu'\mu \simeq 1$. Let Z be the mapping cylinder²⁸ of f and R the mapping cylinder of the map $\mu f \lambda': P \rightarrow Q$, assuming that no two of the spaces $X, Y, X \times I$, etc., have a point in common.

Assume that there is a homotopy, $h_t: (Z, X) \rightarrow (Z, X)$, such that $h_0 = 1$, $h_1 Z \subset X$. Let $h: Z \rightarrow X$ be given by $hz = h_1 z$ ($z \in Z$) and let $i: X \rightarrow Z$ be the identical map. Then $ih = h_1: Z \rightarrow Z$, whence $ih \simeq 1$. Also $hi \simeq 1$, since $k_t: X \rightarrow X$, given by $k_t x = h_t x$ ($x \in X$), is a deformation of $k_0 = 1$ into $k_1 = hi$. Therefore i is a homotopy equivalence. As in [6] it follows that $f: X \rightarrow Y$ is a homotopy equivalence.

We now prove the existence of the homotopy $h_t: (Z, X) \rightarrow (Z, X)$. The argument in [6], which refers to the homotopy sequence

$$(6.1) \quad \pi_n(X) \rightarrow \pi_n(Z) \rightarrow \pi_n(Z, X) \rightarrow \pi_{n-1}(X) \rightarrow \pi_{n-1}(Z),$$

shows that $\pi_n(Z, X) = 0$ if $1 \leq n < N+1$. Since $\dim Q \leq N$ it follows from (K), in §5 above, that there is a homotopy, $\rho_t: Q \rightarrow Z$, such that $\rho_0 = j\mu'$, $\rho_1 Q \subset X$, where j is the identical map $j: Y \rightarrow Z$. Therefore $j \simeq j\mu'\mu \simeq \rho_1\mu$. Since $\rho_1\mu Y \subset \rho_1 Q \subset X$, the argument used in [6] shows that $\pi_{N+1}(Z, X) = 0$, in case $N < \infty$.

Let $K = (P \times I) \cup Q$ and let R be formed from the CW-complex K by identifying²⁸ $(p, 0)$ with p and $(p, 1)$ with $\mu f \lambda' p \in Q$ for each point $p \in P$. Let $\phi: K \rightarrow R$ be given by $\phi(p, 0) = p$, $\phi(p, 1) = \mu f \lambda' p$, $\phi(p, t) = (p, t)$ for $0 < t < 1$, $\phi|_Q = 1$. The arguments in [6, §2] show that

²⁸ See §7 below.

there are maps

$$\nu: (Z, X) \rightarrow (R, P), \quad \nu': (R, P) \rightarrow (Z, X),$$

with $\nu'|Q = j\mu'$, and a homotopy, $\zeta_i: (Z, X) \rightarrow (Z, X)$, such that $\zeta_0 = \nu'\nu$, $\zeta_1 = 1$. Let $L = (P \times 0) \cup (P \times 1) \cup Q \subset K$, let $\rho_i: Q \rightarrow Z$ mean the same as before and let $\rho'_i: L \rightarrow Z$ be given by

$$\begin{aligned} \rho'_i(p, 0) &= \nu'\phi(p, 0), \\ \rho'_i(p, 1) &= \rho_i\phi(p, 1), \\ \rho'_i q &= \rho_i\phi q \quad (p \in P, q \in Q). \end{aligned}$$

Then $\rho'_0 = \nu'\phi|L$, since $\rho_0 = j\mu' = \nu'|Q$, and $\rho'_1 L \subset X$. Let $\psi_0 = \nu'\phi: K \rightarrow Z$. Then it follows from (J) that ρ'_i has an extension $\psi_i: K \rightarrow Z$. Since $\psi_1 L = \rho'_1 L \subset X$ and since $\dim K \leq N+1$ and $\pi_n(Z, X) = 0$ if $1 \leq n < N+2$ it follows from (K) that there is a homotopy, $\psi'_i: K \rightarrow Z$, rel. L , such that $\psi'_0 = \psi_0$, $\psi'_1 K \subset X$. Let $\theta_i: K \rightarrow Z$ be the resultant of ψ_i followed by ψ'_i . Then it is easily verified that $\chi_i = \theta_i\phi^{-1}: R \rightarrow Z$ is single-valued, and hence continuous [7, §5]. Moreover $\chi_0 = \psi_0\phi^{-1} = \nu'\phi\phi^{-1} = \nu'$ and $\chi_1 R = \psi'_1\phi^{-1} R \subset X$. Therefore the required homotopy, $h_i: (Z, X) \rightarrow (Z, X)$, may be defined as the resultant of ζ_{1-i} followed by $\chi_i\nu$. This proves Theorem 1.

The proofs of Theorems 2, 3, 4 are the same as those of the corresponding theorems in [6].

7. Note on n -homotopy. Let us return to Fox's original definition of n -homotopy, in terms of finite polyhedra. It is not certain, for reasons indicated in [6], that an n -homotopy equivalence, $f: X \rightarrow Y$, induces an isomorphism of $\pi_r(X)$ onto $\pi_r(Y)$ ($r \leq n$), where X, Y are arbitrary, arcwise connected spaces. This question reduces to the following. Let $h_r: \pi_r(X) \rightarrow \pi_r(X)$ be the endomorphism induced by a map, $h: X \rightarrow X$, such that $h \simeq_n 1$. Is h_r an automorphism if $1 \leq r \leq n$? Consider the case $r = 1$. It is easily proved that, given any finite set of elements, $a_1, \dots, a_k \subset \pi_1(X)$, there is an element, $\eta \in \pi_1(X)$, such that $h_1 a_i = \eta a_i \eta^{-1}$ ($i = 1, \dots, k$). It follows that h_1 is an isomorphism into. Also it is an inner automorphism if $\pi_1(X)$ has a finite set of generators. Is h_1 an automorphism if $\pi_1(X)$ is arbitrary?

Meanwhile we show that these questions can be answered with our definition. Let X be an arbitrary arcwise connected space and let a point $x_0 \in X$ be chosen as base point for all the groups $\pi_r(X)$. We construct a CW-complex, $K = K(X, x_0)$, which is somewhat analogous to the singular complex of homology theory. Let K^0 consist of a single 0-cell e^0 . With each map, $\theta: (\sigma^r, \partial\sigma^r) \rightarrow (X, x_0)$ ($r = 1, 2, \dots$), we associate a cell, $e^r_\theta \in K$, such that $\partial e^r_\theta = e^0$. The points of e^r_θ are the

pairs (p, θ) , for each point $p \in \sigma^r - \partial\sigma^r$, and $g: \sigma^r \rightarrow \bar{e}'_\theta$, given by $gp = (p, \theta)(p \in \sigma^r - \partial\sigma^r)$, $g\partial\sigma^r = e^0$, is a characteristic map for e'_θ . The complex thus defined is obviously closure finite and we give it the weak topology.

Let $f, f': X \rightarrow Y$ be maps of X in any arcwise connected space Y , and let $f_r, f'_r: \pi_r(X) \rightarrow \pi_r(Y)$ be the homomorphisms induced by f, f' , when the base-point $y_0 \in Y$ is joined to $fx_0, f'x_0$ by paths (see [12] and [1, p. 279]) $\lambda, \lambda': (I, 0) \rightarrow (Y, y_0)$. If $\eta \in \pi_1(Y)$, $b \in \pi_r(Y, y_0)$ ($r \geq 1$) we shall use $\eta \cdot b$ to denote the image of b in the automorphism of $\pi_r(Y, y_0)$, which is determined by η . Thus $\eta \cdot b = \eta b \eta^{-1}$ if $r = 1$. Using our definition of n -homotopy we have:

THEOREM 11. *If $f \simeq_n f'$ there is a fixed element, $\eta \in \pi_1(Y)$, such that $f'_r a = \eta \cdot f_r a$ for every $a \in \pi_r(X)$ and every $r = 1, \dots, n$.*

Let $K = K(X, x_0)$ mean the same as before and let $\phi: K \rightarrow X$ be given by $\phi e^0 = x_0$, $\phi(p, \theta) = \theta p$, for each point $(p, \theta) \in K - e^0$. It follows from (A) in §5 that ϕ is continuous. The homomorphism, $\phi_r: \pi_r(K) \rightarrow \pi_r(X)$, induced by ϕ is obviously onto. Since $f \simeq_n f'$ there is a homotopy, $\xi_t: K^n \rightarrow Y$, such that $\xi_0 = f\phi|K^n$, $\xi_1 = f'\phi|K^n$. Let $\mu: I \rightarrow Y$ be given by $\mu(t) = \xi_t e^0$. Then $\mu(0) = \lambda(1)$, $\mu(1) = \lambda'(1)$. Let $\eta \in \pi_1(Y)$ be the element represented by the circuit, which consists of λ' , followed by μ^{-1} , given by $\mu^{-1}(t) = \mu(1-t)$, followed by λ^{-1} . Then it is easily verified that $f'_r \phi_r a = \eta \cdot f_r \phi_r a$, where $a \in \pi_r(K)$ and $1 \leq r \leq n$. Since ϕ_r is onto this proves the theorem.

8. A process of identification. We shall prove a theorem, which is analogous to Theorem 2 in [7]. For this we shall need a lemma concerning a process of identification, which is often used (e.g., in constructing mapping cylinders, attaching cells k to complexes, etc.) but which, as far as I know, has never been described in full generality and detail.

Let X and Y_0 be given topological spaces and let $f_0: X_0 \rightarrow Y_0$ be a map of a closed subset, $X_0 \subset X$, into Y_0 . We allow X and Y_0 to have common points, but only if $X \cap Y_0 \subset X_0$, $X \cap Y_0$ being a closed subset, both of X_0 and of Y_0 , and $f_0|X_0 \cap Y_0 = 1$. Let Y be the space which consists of the points in $X - X_0$ and in Y_0 , with the identification topology determined by the map²⁹ $\phi: X \cup Y_0 \rightarrow Y$, where $\phi|X - X_0 = 1$, $\phi|Y_0 = 1$, $\phi|X_0 = f_0$. Notice that a set $B_0 \subset Y_0$ is closed in Y if, and

²⁹ $X \cup Y_0$ denotes the space, which consists of the points in X and in Y_0 and in which a subset, A , is closed if, and only if, $A \cap X$ and $A \cap Y_0$ are closed subsets of the spaces X and Y_0 . Since $X \cap Y_0$ is closed in both X and Y_0 it follows that X and Y_0 keep their own topologies in $X \cup Y_0$.

only if, $Y_0 \cap \phi^{-1}B_0 (= B_0)$ and $X \cap \phi^{-1}B_0 (= f_0^{-1}B_0)$ are both closed in $X \cup Y_0$. Since X_0 is closed it follows from the continuity of f_0 that $f_0^{-1}B_0$ is closed in X if B_0 is closed in Y_0 . Therefore Y_0 keeps its own topology and is a closed subset of Y . Similarly $X - X_0$ keeps its own topology. We shall say that Y is formed by *attaching* X , or $X - X_0$ to Y_0 by means of the map f_0 . We shall also say that Y is formed by *identifying* each point $x \in X_0$ with $f_0x \in Y_0$.

In some cases we may be given the spaces X , Y and a map, $f: (X, X_0) \rightarrow (Y, Y_0)$, where $X_0 \subset X$ and $Y_0 \subset Y$ are closed subsets, subject to the conditions:

- (8.1) (a) $f|X - X_0$ is a homeomorphism onto $Y - Y_0$;
- (b) a subset $B \subset Y$ is closed (open) if, and only if, $B \cap Y_0$ is closed (relatively open) and if $f^{-1}B$ is closed (open).

In this case we may always regard Y as formed by attaching a homeomorph of X to Y_0 . For let $h_0: X_0 \rightarrow X'_0$ be a homeomorphism of X_0 onto a new space, X'_0 , which does not meet Y . Let X' be the space consisting of the points in X'_0 and in $Y - Y_0$ with the topology which makes $h: (X, X_0) \rightarrow (X', X'_0)$ a homeomorphism, where $h|X_0 = h_0$, $h|X - X_0 = f|X - X_0$. The conditions (8.1) are satisfied by X' , Y and the map $fh^{-1}: (X', X'_0) \rightarrow (Y, Y_0)$. Since $X' - X'_0 = Y - Y_0$ and $fh^{-1}|X' - X'_0 = 1$ it follows that Y is the result of attaching X' to Y_0 by means of the map $fh^{-1}|X'_0$.

A particularly important application of this construction is that of the mapping cylinder of a map, $g: A \rightarrow B$, where A, B are disjoint spaces. After replacing $A \times I$ by a homeomorph, if necessary, we assume that it has no point in common with A or B . Then the mapping cylinder is defined as above, with $Y_0 = A \cup B$, $X = A \times I$, $X_0 = (A \times 0) \cup (A \times 1)$ and $f_0(a, 0) = a$, $f_0(a, 1) = ga$.

Let X, Y and $f: (X, X_0) \rightarrow (Y, Y_0)$ satisfy (8.1) and let $X - X_0 = Y - Y_0$, $f|X - X_0 = 1$. Let X be a normal, Hausdorff space and let Y_0 satisfy the following "weak regularity" condition: distinct points $y_1, y_2 \subset Y_0$ have neighborhoods $V_1, V_2 \subset Y_0$ such that $\bar{V}_1 \cap \bar{V}_2 = \emptyset$.

LEMMA 1. *Under these conditions Y is a Hausdorff space.*

Let y_1, y_2 be distinct points in Y . First assume that at least one of them, say y_1 , is in $Y - Y_0 = X - X_0$. Since Y_0 and X are Hausdorff spaces, the sets y_1, y_2 and $f^{-1}y_2$ are closed. Since X is normal there is therefore a neighborhood, $U \subset X$, of y_1 , such that

$$\bar{U} \subset X - (X_0 \cup f^{-1}y_2).$$

Then $f\bar{U} = \bar{U} \subset Y - Y_0$ and $y_1 \in U, y_2 \in Y - \bar{U}$. Also U is open and \bar{U} closed in Y . Therefore $U, Y - \bar{U}$ are disjoint, open subsets of Y ,

which contain y_1, y_2 respectively.

Let $y_1, y_2 \subset Y_0$ and let $V_1^0, V_2^0 \subset Y_0$ be neighborhoods of y_1, y_2 such that $\bar{V}_1^0 \cap \bar{V}_2^0 = 0$. Let $U_i^0 = f^{-1}V_i^0 \subset X_0$ ($i = 1, 2$). Then $\bar{U}_i^0 \subset f^{-1}\bar{V}_i^0$, whence $\bar{U}_1^0 \cap \bar{U}_2^0 = 0$. Since X is normal there are disjoint open sets, $U_1', U_2' \subset X$, such that $\bar{U}_i^0 \subset U_i'$. Let

$$U_i = U_i' - (X_0 - U_i^0).$$

Then U_i is open, since $X_0 - U_i^0$ is closed, and $U_1 \cap U_2 = 0, U_i \cap X_0 = U_i^0$. Let

$$V_i = (U_i - U_i^0) \cup V_i^0 \subset Y.$$

Since $U_i - U_i^0 \subset Y - Y_0$ it follows that $Y_0 \cap V_i = V_i^0$, which is open in Y_0 . Also

$$f^{-1}V_i = f^{-1}(U_i - U_i^0) \cup f^{-1}V_i^0 = (U_i - U_i^0) \cup U_i^0 = U_i.$$

Therefore V_i is open in Y . Also $y_i \in V_i$ and, obviously, $V_1 \cap V_2 = 0$. This completes the proof.

Now let X and Y_0 be disjoint CW-complexes and let X_0 be a subcomplex of X . Since X, Y_0 are normal, Hausdorff spaces it follows from Lemma 1 that Y is a Hausdorff space. Let $g: \sigma^n \rightarrow \bar{e}^n$ be a characteristic map for a given cell $e^n \in X - X_0$. Then $fg|_{\sigma^n} - \partial\sigma^n$ is a homeomorphism onto fe^n , where $f: X \rightarrow Y$ means the same as before. It follows that Y is a complex, consisting of the cells in Y_0 and the cells $fe(e \in X - X_0)$, provided³⁰ $f(X_0 \cap \partial e^n) \subset Y_0^{n-1}$ for each n -cell $e^n \in X - X_0$ ($n = 0, 1, \dots$). This being so, I say that Y is closure finite. For any cell $e \in Y_0$ is contained in the finite subcomplex $Y_0(e)$. Let e be a given cell in $X - X_0$, let $K = X(e)$ and let $K_0 = K \cap X_0$. Then K_0 , and hence fK_0 , is compact, and it follows from (D) in §5 that $L_0 = Y_0(fK_0)$ is finite. The subset $L_0 \cup fK$ is compact and is the union of the cells in L_0 and in $f(K - K_0)$. It is therefore a subcomplex. Moreover it is finite and contains fe . Therefore Y is closure finite. Also Y has the identification topology, which is determined by the map $\phi: X \cup Y_0 \rightarrow Y$, where $\phi|_X = f, \phi|_{Y_0} = 1$. Obviously $X \cup Y_0$ is a CW-complex, consisting of the cells in X and in Y_0 , and the cells in Y are the cells ϕe for each cell $e \in (X - X_0) \cup Y_0$. Since Y is closure finite it follows from this and from (D) in §5 that $Y(\phi\bar{e})$ is finite for any $e \in X \cup Y_0$. Therefore Y is a CW-complex, according to (F) in §5. Under these conditions³¹ we say that $Y - Y_0$ is *isomorphic* to $X - X_0$ and that $f|_{X - X_0}$ is an *isomorphism* of $X - X_0$ onto $Y - Y_0$.

³⁰ E.g., if $f_0: X_0 \rightarrow Y_0$ is cellular or if $X^{n-1} \subset X_0, fX_0 \subset Y_0^{n-1}$.

³¹ In this definition we allow X_0 and Y_0 to be empty.

We mention two applications of the preceding paragraph. The first is that the mapping cylinder of a cellular map, $f:K \rightarrow L$, where K, L are CW-complexes, is itself a CW-complex. Secondly, let X be the union of a set of n -elements, $\{E_i^n\}$, which are disjoint from each other and from Y_0 , each E_i^n being a subcomplex, $E_i^n = e_i^0 \cup e_i^{n-1} \cup \partial E_i^n$. Let X_0 be the union of the boundaries, $\partial E_i^n = e_i^0 \cup e_i^{n-1}$, and let f_0 be any map of the form $f_0: X_0 \rightarrow Y_0^{n-1}$. Then we say that Y is formed by attaching the set of cells $\{e_i^n\}$ to Y_0 by means of the maps $f_0|_{\partial E_i^n}$.

Let X, Y and $f:(X, X_0) \rightarrow (Y, Y_0)$ satisfy (8.1), where X, Y are CW-complexes, X_0, Y_0 are subcomplexes and the map f is cellular. Let $f_0: X_0 \rightarrow Y_0$ be the map given by $f_0x = fx$ ($x \in X_0$). Then we have:

THEOREM 12. *If $f_0: X_0 \rightarrow Y_0$ is a homotopy equivalence, so is $f: X \rightarrow Y$.*

The proof is the same³² as that of Theorem 2 in [7].

If Y_0 is a single point we describe the above process of identification as shrinking X_0 into the point Y_0 . It follows from Theorems 1 and 12 that a simply connected, aspherical subcomplex, $X_0 \subset X$, may be shrunk into a point without altering the homotopy type of X .

Let $X_0 = X^n$ and let $Y_0 = Y_0^n$. Then $Y^n = Y_0^n$ and we have:

COROLLARY 1. *Given a CW-complex, $Y_0^n \equiv X^n$, there is a CW-complex, Y , such that $Y^n = Y_0^n, X \equiv Y$ and $Y - Y^n$ is isomorphic to $X - X^n$.*

Let X and Y_0 be connected, let $Y_0 = Y_0^n$ and let $Y_0^n \equiv_{n-1} X^n$ ($n \geq 2$). Then

$$X^n \cup \Sigma_1^n \equiv Y_0^n \cup \Sigma_2^n,$$

where Σ_1^n, Σ_2^n mean the same as in Theorem 6. Let us "fill up" the closure, S^n , of each cell $e^n \in \Sigma_1^n$ with an $(n+1)$ -cell, e^{n+1} , in such a way that \bar{e}^{n+1} is an $(n+1)$ -element bounded by S^n . The result is a CW-complex,³³ $K = X \cup \Sigma_1^n \cup \{e^{n+1}\}$. There is obviously a retracting deformation of K into X in which each $(n+1)$ -element, \bar{e}^{n+1} , is retracted into the point $X \cap \bar{e}^{n+1}$. Therefore $K \equiv X$. Also $K^n = X^n \cup \Sigma_1^n, K^{n-1} = X^{n-1}$. Let $L^n = Y_0^n \cup \Sigma_2^n$. Then $L^{n-1} = Y_0^{n-1}$. Therefore, replacing X, X^n, Y_0^n in Corollary 1 by K, K^n, L^n , we have:

COROLLARY 2. *If $Y_0^n \equiv_{n-1} X^n$ there is a CW-complex, Y , such that $X \equiv Y$ and $Y^{n-1} = Y_0^{n-1}$.*

For example, if $\pi_r(X) = 0$ for $r = 1, \dots, n-1$, we may take Y

³² Notice that CW-complexes and subcomplexes have all the homotopy extension properties required by [10].

³³ We assume that the cells in Σ_1^n and the new $(n+1)$ -cells, $\{e^{n+1}\}$, are disjoint from each other and from X .

to be a single point, as observed in §3. Then $X \equiv Y$, where $Y^{n-1} = Y_0^n$ and Y^n consists of the n -spheres, which are the closures of the n -cells in Σ_2^n .

9. Countable complexes. We describe a complex as *countable* if, and only if, the number of its cells is countable. By a locally finite (finite) polyhedron we mean a space which is covered by a locally finite (finite) simplicial complex. The purpose of this section is to prove:

THEOREM 13. *Any countable (finite), p -dimensional CW-complex ($p \leq \infty$) is of the same homotopy type as a locally finite (finite), p -dimensional polyhedron.*

We shall need three lemmas for the proof of this. A CW-complex, K' , will be described as a *subdivision* of a CW-complex K if, and only if, both complexes cover the same space and if each cell of K' is contained in a cell of K . If K' is a simplicial complex we shall describe it as a simplicial subdivision of K . Let K' be a simplicial subdivision of K and let a set of disjoint n -cells, $\{e_i^n\}$, be attached to K by maps, $g_i: \partial E_i^n \rightarrow K^{n-1}$ ($K \cap E_i^n = 0$), each of which is simplicial with respect to K' and some triangulation of ∂E_i^n . Then we have:

LEMMA 2. *The complex $K \cup \{e_i^n\}$ has a simplicial subdivision, of which K' is a subcomplex.*

This will be proved in §10 below.

Let K and $\{e_i^n\}$ mean the same as in Lemma 2, except that K need not have a simplicial subdivision. Let $f: K \rightarrow L$ be a homotopy equivalence of K into a CW-complex L . Let

$$h_i \simeq fg_i: \partial E_i^n \rightarrow L,$$

where $h_i \partial E_i^n \subset L^{n-1}$ but \simeq indicates homotopy in L . Let the cells $\{e_i^n\}$ be attached to L by the maps h_i , thus forming a complex $L \cup \{e_i^n\}$ ($L \cap E_i^n = 0$).

LEMMA 3. *The map $f: K \rightarrow L$ can be extended to a homotopy equivalence $h: K \cup \{e_i^n\} \rightarrow L \cup \{e_i^n\}$.*

If $h_i = fg_i$ this follows from Theorem 12, taking $h|_{e_i^n} = 1$. In general it follows from Theorem 12 and the proof of Lemma 5 in [5].

Let a connected, CW-complex, K , be the union of an "expanding" set of subcomplexes, $K_0 \subset K_1 \subset \dots$, such that if $n > 0$ then K_n is connected and the injection homomorphism

$$i_n: \pi_n(K_{n+1}) \rightarrow \pi_n(K)$$

is an isomorphism onto (e.g., $K_n = K^n$). Let L and $L_0 \subset L_1 \subset \dots \subset L$ satisfy the same conditions and let $\{K_n\}$, $\{L_n\}$ be related by a sequence of homotopy equivalences,

$$f^n : (K_n, K_{n-1}) \rightarrow (L_n, L_{n-1}),$$

such that $f^n|_{K_{n-1}} = f^{n-1}$.

LEMMA 4. *The map, $f: K \rightarrow L$, which is given by $f|_{K_n} = f^n$, is a homotopy equivalence.*

Let i_n mean the same in L as in K and let

$$f_n : \pi_n(K) \rightarrow \pi_n(L), \quad f_n^{n+1} : \pi_n(K_{n+1}) \rightarrow \pi_n(L_{n+1})$$

be the homomorphisms induced by f, f^{n+1} . Then f_n^{n+1} is an isomorphism onto since f^{n+1} is a homotopy equivalence. Also

$$i_n : \pi_n(K_{n+1}) \rightarrow \pi_n(K), \quad i_n : \pi_n(L_{n+1}) \rightarrow \pi_n(L)$$

are isomorphisms onto and obviously $f_n i_n = i_n f_n^{n+1}$. Therefore f_n is an isomorphism onto, for each $n = 1, 2, \dots$, and the lemma follows from Theorem 1.

We are now ready to prove Theorem 13. This will obviously follow when we have proved it for a countable CW-complex, K , which is connected if $p = \dim K > 0$. We first assume K to be finite and use induction on the number of cells in K , starting with the trivial case $K = K^0$. Let $p > 0$ and let $K_0 = K - e^p$, where e^p is any p -cell in K . Assume that there is a finite complex, L_0 , with a simplicial subdivision L'_0 , which is related to K_0 by a homotopy equivalence $f_0: K_0 \rightarrow L_0$. Let e^p be attached to K_0 by a map $g: \partial E^p \rightarrow K_0$, where $e^p = E^p - \partial E^p$. Assuming that $L_0 \cap E^p = 0$ let e^p be attached to L_0 by a map

$$h_0 \simeq f_0 g : \partial E^p \rightarrow L_0,$$

which is simplicial with respect to L'_0 and some triangulation of ∂E^p . The result is a complex, L , which has a simplicial subdivision, according to Lemma 2. The theorem, for K finite, now follows from the induction and Lemma 3.

Now let K be infinite. In order to repeat the above construction an infinite number of times we shall, so to speak, "spread L_0 out" so that L may be locally finite. Assume that there is a sequence of locally finite complexes, $L_0 \subset L_1 \subset \dots$, such that L_n is connected if $n > 0$, and a sequence of subcomplexes, $P_n \subset L_n$, such that:

(9.1) (a) $L_{n-1} - P_{n-1} = U_{n-1}$ is open in L_n and $L_{n-1} \subset U_n$ ($L_{-1} = P_{-1} = 0$),

- (b) P_n is a deformation retract of L_n ,
- (c) if $n > m > 1$ the injection homomorphism $\pi_{m-1}(L_m) \rightarrow \pi_{m-1}(L_n)$ is an isomorphism onto,
- (d) there is a sequence of homotopy equivalences, $f^n: (K^n, K^{n-1}) \rightarrow (L_n, L_{n-1})$, such that $f^n|_{K^{n-1}} = f^{n-1}$,
- (e) L_n has a simplicial subdivision, L'_n , of which L'_{n-1} is a subcomplex.
- (f) $\dim L_n = \dim K^n \leq n$.

Then we define the topological space

$$L = \bigcup_{n=0}^p L_n \quad (p = \dim K \leq \infty),$$

in which a subset $X \subset L$ is closed (open) if, and only if, $X \cap L_n$ is a closed (relatively open) subset of L_n , for each $n \geq 0$. Clearly L_n keeps its own topology in L and since each L_n is a locally finite complex it follows that L is a CW-complex, which consists of the cells in L_0, L_1, \dots . I say that L is locally finite. For $U_{n-1} \subset L_{n-1} \subset U_n$ and U_{n-1} is open in L_n and hence in U_n . Therefore, given $m \geq 0$, it follows by induction on n that U_m is open in U_n and hence in L_n for every $n > m$. Therefore U_m is open in L . Since $L_{n-1} \subset U_n$ it follows that any point $q \in L$ is in U_n for some n . Let $q \in U_n$ and let $Q \subset L_n$ be a finite subcomplex, of which q is an inner point, according to the topology of L_n . Since U_n is open in L it follows that q is an inner point of Q , according to the topology of L . Therefore L is locally finite.

Since L is locally finite it follows from (D), in §5 above, that any compact subset is contained in a finite subcomplex of L . It is therefore contained in L_n , for some $n \geq 0$, and it follows without difficulty from (9.1c) that the injection homomorphism $\pi_{m-1}(L_m) \rightarrow \pi_{m-1}(L)$ ($m > 1$) is an isomorphism onto. Therefore it follows from (9.1d) and Lemma 4 that $K \equiv L$. Clearly

$$L' = \bigcup_n L'_n$$

is a (locally finite) simplicial subdivision of L . Finally $\dim L = p = \dim K$ and the theorem follows.

Now assume that L_n, P_n have been constructed for $n = 1, \dots, r \geq 0$ so as to satisfy (9.1) with $n \leq r$. If $K^{r+1} = K^r$ we define $L_{r+1} = L_r, P_{r+1} = P_r$. Let $K^{r+1} \neq K^r$ and let X_a be the complex covering the half line $\langle a, \infty \rangle$, which has 0-cells at the points $a+i$ ($i=0, 1, \dots$). Assuming that the topological product $P_r \times X_0$ has no point in common with L_r , we identify each point $x \in P_r$ with $(x, 0) \in P_r \times X_0$ so as to form a locally finite complex

$$M_r = L_r \cup (P_r \times X_0).$$

This has the simplicial subdivision $L_r' \cup (P_r \times X_0)'$, where $(P_r \times X_0)'$ is formed by starring the prisms $\sigma \times \langle i, i+1 \rangle$ ($i=0, 1, \dots$), for each simplex, σ , in the subcomplex of L_r' which covers P_r . Clearly L_r is a deformation retract of M_r , whence $g^r: K^r \rightarrow M_r$, given by $g^r x = f^r x$ ($x \in K^r$), is a homotopy equivalence. It obviously follows from (9.1b) that $P_r \times X_0$ is a deformation retract of M_r and hence that $P_r \times k$ is a deformation retract of M_r , for each $k=1, 2, \dots$. Therefore any map $\partial E^{r+1} \rightarrow M_r$ is homotopic to a map in $P_r \times k$. We now repeat the construction used in the finite case, attaching an $(r+1)$ -cell, e_k^{r+1} , to M_r by a suitable map $\partial E_k^{r+1} \rightarrow P_r \times k$ ($k > 0$), where $e_1^{r+1}, e_2^{r+1}, \dots$ are the $(r+1)$ -cells in K . The result is a locally finite complex, L_{r+1} , which satisfies (9.1d, e, f) with $n \leq r+1$. We define

$$P_{r+1} = (P_r \times X_1) \cup \{e_k^{r+1}\}.$$

Clearly $P_r \times X_1$ is a deformation retract of $P_r \times X_0$, and hence of M_r . Therefore P_{r+1} is a deformation retract of L_{r+1} . Therefore (9.1b), and obviously (9.1a), is satisfied with $n \leq r+1$. Since L_r is a deformation retract of M_r and L_{r+1} is formed by attaching $(r+1)$ -cells to M_r , it follows that (9.1c), and hence all the conditions (9.1), is satisfied with $n \leq r+1$. Starting with $L_0 = P_0 = K^0$ the theorem now follows by induction on r .

10. Proof of Lemma 2. An n -element, E^n , in an n -sphere, S^n , will be called a *hemisphere* if, and only if, the closure of $S^n - E^n$ is also an n -element. Let $f: \sigma^n \rightarrow \sigma^k$ be a barycentric map of an n -simplex, σ^n , onto a k -simplex σ^k ($k \leq n$). Assuming that σ^k does not meet σ^n let Γ be the mapping cylinder of f . We shall prove:

LEMMA 5. Γ is an $(n+1)$ -element and σ^n is a hemisphere in $\partial\Gamma$.

Before proving this we deduce a corollary and show that it implies Lemma 2. Let σ_1^n be a face of an $(n+1)$ -simplex, σ^{n+1} . Let $f_1: \sigma_1^n \rightarrow \sigma^k$ be a barycentric map of σ_1^n onto σ^k and, assuming that σ^k does not meet σ_1^{n+1} , let E^{n+1} be the space formed by identifying each point $p \in \sigma_1^n$ with $f_1 p$. Obviously E^{n+1} is homeomorphic to $\sigma_0^{n+1} \cup \Gamma$, where Γ means the same as in Lemma 5 and σ_0^{n+1} is an $(n+1)$ -simplex, of which $\sigma^n = \sigma_0^{n+1} \cap \Gamma$ is a face. Since σ^n is a hemisphere both of $\partial\sigma_0^{n+1}$ and of $\partial\Gamma$ we obviously have the corollary:

COROLLARY. E^{n+1} is an $(n+1)$ -element.

Let an n -cell, e^n , be attached to a complex, K , by means of a map, $\partial E^n \rightarrow K$, which is simplicial with respect to a simplicial subdivision, K' , of K and some triangulation, S , of ∂E^n . Let p_0 be an interior

point of E^n . When e^n is attached to K each simplex $\sigma^{r+1} = p_0\sigma^r$, where σ^r is a simplex of S , is transformed into an $(r+1)$ -element E^{r+1} , according to the above corollary. We triangulate $K' \cup e^n$ by starring all these $(r+1)$ -elements, in order of increasing dimensionality. The result is a simplicial subdivision of $K \cup e^n$, which contains K' as a subcomplex. Lemma 2 follows from this construction, carried out in each of the cells e_i^n .

We now prove Lemma 5, using a method suggested by M. H. A. Newman. Let σ^n be a rectilinear simplex in Euclidean space R^n . Let b_0, \dots, b_k be the vertices of σ^k and let σ_i be the face of σ^n , whose vertices are those which are mapped by f on b_i . Let a_i be the centroid of σ_i , let σ_0^k be the k -simplex $a_0 \dots a_k$ and let $g: \sigma^n \rightarrow \sigma_0^k$ be the barycentric map, which is determined by $g\sigma_i = a_i$. Then $ga_i = a_i$, whence $ggx = gx$ for each $x \in \sigma^n$, or $g^2 = g$. Also $c \in \sigma_0^k$, where c is the centroid of σ^n , since a_i is the centroid of σ_i , and it follows from the definition of a barycentric map that $gc = c$.

We treat R^n as a vector space, taking c to be the origin or null vector, which we also denote by 0. We shall use small Roman letters to denote vectors in R^n and small Greek letters, as in $\lambda x + \mu y$, to denote (real) scalar multipliers. Let $\theta_\lambda: \sigma^n \rightarrow \sigma^n$ be the map which is given by

$$(10.1) \quad \theta_\lambda x = (1 - \lambda)x + \lambda gx/2 \quad (0 \leq \lambda \leq 1; x \in \sigma^n).$$

I say that if $\lambda < 1$ the map θ_λ is nonsingular. For θ_λ is a linear map, such that $\theta_\lambda 0 = 0$, and if it is singular, then $\theta_\lambda x = 0$ for some nonzero vector $x \in \sigma^n$. Since $g^2 = g$ it follows from the linearity of g that

$$(10.2) \quad g\theta_\lambda x = (1 - \lambda)gx + \lambda gx/2 = (1 - \lambda/2)gx.$$

Therefore $\theta_\lambda x = 0$ implies, first, $gx = 0$, and then $(1 - \lambda)x = 0$, or $x = 0$ if $\lambda < 1$, which proves the assertion.

I say that, if $0 \leq \lambda < \mu \leq 1$, then $\theta_\mu \sigma^n$ is in the interior of $\theta_\lambda \sigma^n$. For $\theta_\lambda x$ is the point which divides the segment joining x to $gx/2$ in the ratio $\lambda: (1 - \lambda)$. When we have shown that the simplex $\theta_1 \sigma_0^k$ is in the interior of $\theta_\lambda \sigma^n$ ($\lambda < 1$) it will follow that the vertices of $\theta_\mu \sigma^n$ ($\mu > \lambda$) lie on rectilinear segments, which join the vertices of $\theta_\lambda \sigma^n$ to interior points of $\theta_\lambda \sigma^n$. Hence, and since $\mu > \lambda$, they are interior points of $\theta_\lambda \sigma^n$. Since θ_ρ is a linear map the assertion will follow. Thus we have to prove that $\theta_1 \sigma_0^k$ is inside $\theta_\lambda \sigma^n$, or that any point $ga/2$ ($a \in \sigma^n$) is of the form $\theta_\lambda x$, for some interior point $x \in \sigma^n$. Let $\rho = 1/(2 - \lambda)$. Then

$$\begin{aligned} \theta_\lambda \rho ga &= (1 - \lambda)\rho ga + \lambda \rho g^2 a/2 \\ &= \rho(1 - \lambda/2)ga = ga/2. \end{aligned}$$

But $\rho < 1$ since $\lambda < 1$. Therefore ρga is an interior point of σ^n and the assertion is proved.

The set of values $\lambda \in I$, such that a given point, $x \in \sigma^n$, does not lie in $\theta_\lambda \sigma^n$, is obviously open. Therefore there is a maximum, $\phi(x)$, of the values of λ such that $x \in \theta_\lambda \sigma^n$. It follows without difficulty from the preceding paragraph that ϕ is a continuous, real-valued function of x , such that

$$(10.3) \quad \begin{array}{ll} \text{(a) } \phi(x) > 0 & \text{if } x \in \sigma^n - \partial\sigma^n, \\ \text{(b) } \phi(x) = 0 & \text{if } x \in \partial\sigma^n, \\ \text{(c) } x \in \theta_\lambda \sigma^n & \text{if, and only if, } \lambda \leq \phi(x). \end{array}$$

We now imbed σ^n in $\sigma^n \times I$, in such a way that $x = (x, 0)$ if $x \in \sigma^n$, and take σ^k to be the simplex $\theta_1 \sigma_0^k \times 1$ and $f: \sigma^n \rightarrow \sigma^k$ to be the map given by

$$fx = (\theta_1 x, 1) = (gx/2, 1).$$

If x is fixed and λ varies from 0 to 1 we may regard the locus of the point $(\theta_\lambda x, \lambda)$ as the linear segment in $\sigma^n \times I$, which joins $x \in \sigma^n$ to $fx \in \sigma^k$. If $\theta_\lambda x = \theta_\lambda y$ it follows from (10.2) that $gx = gy$ and from (10.1) that $x = y$ if $\lambda < 1$. Therefore no two of these linear segments have an inner point in common and we may take Γ to be the union of the sets $\theta_\lambda \sigma^n \times \lambda$ for each $\lambda \in I$. It follows from (10.3c) that Γ is the set of all points (x, λ) such that $0 \leq \lambda \leq \phi(x)$.

Let $\psi: \sigma^n \rightarrow I$ be the real-valued function which is defined by $\psi(\rho v) = 1 - \rho$ for each vector $v \in \partial\sigma^n$ and each $\rho \in I$. Then the subset of $\sigma^n \times I$, which consists of all points (x, λ) such that $0 \leq \lambda \leq \psi(x)$, is the join, σ^{n+1} , of σ^n and the point $(0, 1)$. A homeomorphism (onto), $h: \Gamma \rightarrow \sigma^{n+1}$, is defined by

$$h(x, \lambda) = \{x, \rho(x)\lambda\},$$

where $0 \leq \lambda \leq \phi(x)$ and $\rho(x) = \psi(x)/\phi(x)$ if $x \in \sigma^n - \partial\sigma^n$, $\rho(x) = 0$ if $x \in \partial\sigma^n$. Clearly $h\sigma^n = \sigma^n$ and σ^n is a hemisphere of $\partial\sigma^{n+1}$. Therefore Lemma 5 is established.

Let K, L be simplicial complexes and let $f: K \rightarrow L$ be a simplicial map. Let C be the "combinatorial" mapping cylinder of f , defined as in [1], and Γ the "topological" mapping cylinder defined as in §8 above. Let σ be any simplex in K and let $C_\sigma \subset C, \Gamma_\sigma \subset \Gamma$ be the mapping cylinders of the map, $\sigma \rightarrow f\sigma$, which is determined by f . It follows from Lemma 5 that Γ is a polyhedral complex, whose (closed) cells are the simplexes in K and in L and the elements Γ_σ , for each simplex, σ , in K . Since Γ_σ is an element it may be starred from an internal

point. Since C_σ is also a star an easy induction shows that C is combinatorially isomorphic to the triangulation of Γ , which is obtained by starring the elements Γ_σ in order of increasing dimensionality. Hence the conjecture in the footnote on p. 1235 of [3] is easily verified.

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