NONLINEAR NETWORKS. III

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The object of this paper is to show that a certain system of nonlinear differential equations has one and only one periodic solution. These equations are of interest in that they describe the vibrations of a common type of electric network; therefore, the physical origin of the equations will be discussed first.

A linear network is a collection of linear inductors, linear resistors, and linear capacitors arbitrarily interconnected. If a periodic electromotive force is applied to this network, a periodic system of currents can exist, provided that the network has no free vibration of the same period. This, of course, is well known. The main theorem of this paper states that if in such a network the linear resistors are replaced by *quasi-linear* resistors, a periodic system of currents can again exist.

A quasi-linear resistor is a conductor whose differential resistance lies between positive limits. Quasi-linear resistors have extensive practical applications. No other type of nonlinearity except this type of nonlinear damping is considered here.

For example, consider a linear network with one degree of freedom. An inductor of inductance L, a resistor of resistance R, and a capacitor of capacitance S^{-1} are connected in series. The current $i(t)$ flowing in this circuit must satisfy the following differential equation:

$$L\frac{di}{dt} + Ri + S\int idt = g.$$

Here $g(t)$ is the electromotive force impressed in the circuit and is a periodic function of time.

The corresponding nonlinear equation to be studied is obtained by replacing the linear relation Ri by a function $V(i)$ which for all values of i is such that $A^{-1} \leqq V'(i) \leqq A$, where A is a positive constant.

In a general network with m degrees of freedom, if a set of m independent circuits (meshes) is chosen, any distribution of current in the network may be uniquely specified by assigning suitable values to the cyclic currents $i_1, i_2, \cdots, i_m$ flowing in these circuits. Let $g_1, g_2, \cdots, g_m$ be the electromotive forces acting in these circuits. It is convenient to introduce the electric charge variables $y_1, y_2, \cdots, y_m$ such that $i_j = y_j'$. The linear network equations may

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then be written as $Ly''+Ry'+Sy=g$. Here L, R, and S are m-way matrices and y and g are vectors with components y_j and g_j. Well known arguments concerning electric and magnetic energy show that the matrices L, R, and S are symmetric semidefinite. The nonlinear network equations are $Ly''+V(y')+Sy=g$ where V is a vector function.

The central idea of this paper is to consider the network equations as a transformation of Hilbert space; that is, the equations give a transformation from the Hilbert space of electric charge to the Hilbert space of electromotive force. The existence theorem to be proved, then, is that this transformation has an inverse. The novelty of the proof arises when the linear transformation is replaced by a nonlinear transformation. Theorem 4, below, gives conditions which insure that a nonlinear transformation have an inverse. This theorem is analogous to the theorem that closure and completeness are equivalent for linear transformations. The casual reader may turn directly to this theorem and skip the algebraic complications which must be disposed of first.

It should be noted that Levinson, Chevalley, Lefschetz, and others have treated even more general types of nonlinearity in a single second-order differential equation. Therefore, in the case of one degree of freedom, the contribution of this paper is to furnish a method of proof which differs radically from theirs. It would be interesting to know if their techniques could be extended to treat the system of equations discussed here.

In a previous paper the transient solutions of the nonlinear system of equations were discussed. Combining the results of these two papers, it follows that for networks of the type considered, all solutions must approach the periodic solution. No appeal, however, will be made in this paper to results obtained in previous papers. Moreover, no appeal will be made to known theorems on differential equations. Theorem 3 of the present paper is believed to be a new theorem, even for linear networks.

Let E be a Euclidean space of m dimensions. A vector x will have components $x_1, x_2, \cdots, x_m$ which are real numbers. If x and y are vectors, then the inner product is given by $(x, y)=x_1y_1+x_2y_2+ \cdots +x_my_m$ and the norm of x is $\|x\|=(x, x)^{1/2}$. All constants and functions shall be assumed real. The symbol A will stand for a positive constant whose value, however, may differ in different sections.

If B is a matrix, the manifold of all vectors of the form Bx will be designated as M_B. A semi-definite matrix B satisfies the relation $(Bx, x)\geqq 0$ for all vectors x.

LEMMA 1. *If B is a symmetric matrix which satisfies the inequality $0 \leqq (Bx, x) \leqq A_1\|x\|^2$ for all vectors x, then*:

(a) $\|Bx\|^2 \leqq A_1(Bx, x)$.

(b) $(Bx, x) \leqq A_2\|Bx\|^2$ *where A_2 is a constant.*

(c) $\|z\|^2 \leqq A_2(Bz, z)$ *where $z = Bx$.*

PROOF. The inequalities of this lemma are obvious if B is a diagonal matrix; however, the expressions are invariant under an orthogonal transformation, and it is well known that there is an orthogonal transformation which reduces B to diagonal form.

LEMMA 2. *Let M_B and M' together span E. Then any vector v has a unique representation $v = Bb + v'$ where b is orthogonal to M' and v' is contained in M'. There is a constant A independent of v such that $\|b\|$, $\|v'\| \leqq A\|v\|$.*

PROOF. If the manifold spanned by the vectors of the form $Bb+v'$ were not E, there would be a nonzero vector c such that $(Bb+v', c) = 0$ for all b and v'. Let $b = 0$, then $(v', c) = 0$, so c is orthogonal to M'; therefore, it must be in M_B. Let $b = c$; therefore $(Bc, c) = 0$, and by Lemma 1 (c) it follows that $c = 0$, which is a contradiction.

From $v = Bb + v'$, we have $(v, b) = (Bb, b)$, so $(Bb, b) \leqq \|v\| \, \|b\|$. Then from Lemma 1 (c) we have $\|b\|^2 \leqq A\|v\| \, \|b\|$ or $\|b\| \leqq A\|v\|$. Because $v' = v - Bb$, it is obvious that v' satisfies an inequality of the same form. The uniqueness of the representation follows directly from these inequalities.

In what follows, L, R, and S will denote symmetric semi-definite matrices such that M_L, M_R, and M_S together span E. A simple application of Lemma 1 (a) reveals that the matrix $L+R+S$ is nonsingular.

THEOREM 1. *Any vector v has a unique representation in the form $v = Ll + Rr + Ss$ where $l \in M_L$, $r \in M_R$, and $s \in M_S$ and $Rl = Sl = Sr = 0$. Moreover, $\|l\|$, $\|r\|$, $\|s\| \leqq A\|v\|$ where the constant A is independent of v.*

PROOF. In Lemma 2 let $B = L$ and let M' be the manifold spanned by M_R and M_S. Then $v = Ll + v'$. The manifold M' is also a Euclidean space, so again by Lemma 2, $v' = Rr + v''$ where $v'' \in M_S$; hence $v'' = Ss$. The remainder of the statements of the theorem follow easily from Lemma 2.

We are to consider vector equations of the form

$$Ly'' + Ry' + Sy = g. \tag{1}$$

We first wish to show that it is possible to make a linear transformation of variables so that these equations break up into three independent sets, two of which are of very simple form; the remaining set we shall call the canonical form. First by an orthogonal transformation of y'', y', y, and g, we may employ a rotated coordinate system in which S is a diagonal matrix. Since the matrix elements of S are then non-negative, a second transformation, $D^{-1}y''$, $D^{-1}y'$, $D^{-1}y$, and Dg (where D is a nonsingular diagonal matrix), may be chosen in such wise that the new matrix elements of S are either 0 or 1. The new matrices L, R, and S are expressed in terms of the old by DLD, DRD, and DSD. Moreover, the new matrices are symmetric semi-definite. By Lemma 2, any vector y of E may be uniquely expressed in the form y_1+Ss where y_1 is in the manifold M_L+M_R and s is orthogonal to this manifold and is in the manifold M_S. But now $S^2=S$, so $Ss=s$. Let $y_2=s$, then we have a unique decomposition of the vector y in the form y_1+y_2. Then $Sy=Sy_1+Sy_2=Sy_1+y_2$ where $Sy_1 \in M_L+M_R$. To show the latter statement, let x be any vector orthogonal to M_L+M_R, so it is in M_S. Then $(Sy_1, x)=(y_1, Sx)=(y_1, x)=0$. The vectors y' and y'' are split in the same fashion. Noting that $Ly_2''=Ry_2'=0$, we may write the equations in the form $Ly_1''+Ry_1'+Sy_1+Sy_2=g$. Both sides of this relation can be split into components in the manifold M_L+M_R and its orthogonal complement. Thus $Ly_1''+Ry_1'+Sy_1=g_1$, and $y_2=g_2$. We could transform the first set of equations in an analogous fashion but splitting L instead of S; this gives the desired decomposition into the three sets $y_2=g_2$, $y_4''=g_4$, and $Ly_3''+Ry_3'+Sy_3=g_3$. *The latter set of equations is the canonical form and is such that if E_3 is the manifold of these equations, then M_L and M_R span E_3 and M_R and M_S also span E_3.* In other words, $L+R$ and $R+S$ are both nonsingular.

Let $v=V(x)$ be a vector function with corresponding differential transformation $dv=V'dx$ where V' is the matrix of differential coefficients. If R is a symmetric semi-definite matrix, we shall say that $V(x)$ is a *quasi-linear* replacement of the linear function Rx if:

(I) $V(0)=0$.

(II) *$V'(x)$ exists for all x as a symmetric matrix.*

(III) *There is a positive constant A such that for all vectors x and y*

$$A^{-1}(Ry, y) \leqq (V'y, y) \leqq A(Ry, y).$$

By use of Lemma 1 we may derive other inequalities such as $\|Ry\|^2 \leqq A_3(V'y', y')$, $\|V'y\|^2 \leqq A_4(V'y, y)$, and $\|V'y\| \leqq A_5\|Ry\|$. It is apparent that V' is semi-definite and that it is made up of uniformly bounded matrix elements. By the mean value theorem $V(x)-V(y)=V'(x-y)$ where V' is evaluated at the point $y+\theta(x-y)$; $0\leqq\theta\leqq 1$. Setting $y=0$, we have $V(x)=V'x$.

In performing the transformation of equations (1) to a canonical form, we note that the new matrix R_n is given in terms of the old by the relation $R_n = B_t R B$ where B is a nonsingular matrix and B_t is its transpose. The quasi-linear replacement would become $V_n(x) = B_t V(Bx)$, and hence $V_n' = B_t V' B$. Clearly, then, $V_n(x)$ is a quasi-linear replacement of $R_n x$.

We now wish to show that all vectors $v = V(x)$ are contained in the manifold M_R. Because $V(x) = V'x$, the manifold of vectors $V'y$ contains v. If the manifold of vectors $V'y$ were not contained in the manifold M_R, there would be a vector z perpendicular to M_R; that is, $Rz = 0$ but $V'z \neq 0$. By virtue of the inequalities given for a quasi-linear replacement, this implies that $z = 0$. It follows, therefore, that the same transformation which reduces equations (1) to canonical form will also reduce the following equations:

$$Ly'' + V(y') + Sy = g. \tag{2}$$

We now regard the vectors of a space E to have components which are functions of the time t. The primes in equations (1) and (2) refer to differentiation with respect to time; thus $Ry' = (d/dt)Ry$. We adopt the convention that writing Ry' does not imply that all components of y are differentiable but only those in the manifold M_R. Moreover, in what follows, "differentiable" means absolutely continuous; that is, we differentiate only those functions which are the integrals of Lebesgue integrable functions. It follows, of course, that the derivatives may not exist everywhere, but only almost everywhere. Except where the contrary is indicated we shall require only that the equations (1) and (2) hold almost everywhere.

We define the real Hilbert space H (of electromotive force) to be the set of all vectors of E whose components are periodic functions of the time t of period 2π and which are of integrable square; that is, they belong to $L_2(0, 2\pi)$. The bilinear form of two vectors x and y is given by $((x, y)) = (1/2\pi)\int_0^{2\pi} (x, y)dt$. If x and y are constant vectors, it follows that $((x, y)) = (x, y)$. The norm of x is given by $\|x\|^2 = ((x, x))$. The context will prevent confusion between norms in the space E and H. The subspace of constant vectors will be denoted by C. The subspace H_0 will be defined as the space of all vectors orthogonal to C.

The normed linear space Q will be defined as the space of vectors of the form $y = y_0 + lt^2/2 + rt + s$ where l, r, and s satisfy Theorem 1 and where $y_0 \in H_0$. The norm of y is given by the relation $\|y\|_Q^2 = \|Ly''\|^2 + \|Ry'\|^2 + \|Sy\|^2$. We note that in equation (1) if $g \in C$, we can obtain a solution $y \in Q$ with $y_0 = 0$. This is by virtue of Theorem 1. If $y \in Q$, it follows that Ly'', Ry', and $Sy \in H$, and by the convention adopted above, Ly' and Ry are absolutely continuous.

In what follows it will be tacitly assumed that we are considering a canonical form, because a considerable simplification of the proofs is thereby achieved. It will be apparent, a posteriori, that the main theorems are independent of this assumption. The first simplification is that $l=0$, because by Theorem 1, $(R+S)l=0$, but $R+S$ is nonsingular in the canonical form. We now have that Ly' is periodic as well as being absolutely continuous, so $Ly' \in H$. Thus $(L+R)y' \in H$, and since $L+R$ is nonsingular, y' exists and belongs to H.

LEMMA 3. *If u, v, u', and v' all belong to H, then* $((u, v')) = -((u', v))$.

PROOF. Since $(d/dt)(u, v) = (u', v) + (u, v')$, and since (u, v) is absolutely continuous and of period 2π, it follows that integration of this equation on the left yields zero.

LEMMA 4. *If $y \in Q$, then* $((Sy, y')) = 0$ *and* $((Ly'', y')) = 0$.

PROOF. We have seen above that y' exists, so y is absolutely continuous, and so, also, is Sy. We now have (Sy, y) an absolutely continuous periodic function, and $(d/dt)(Sy, y) = 2(Sy, y')$, so the first part of the lemma is true. The second part follows from the identity $(d/dt)(Ly', y') = 2(Ly'', y')$.

A sequence f_n, $n = 1, 2, \cdots$, of H converges strongly to f if $\|f - f_n\|$ approaches zero as n approaches infinity, and a sequence converges weakly if $((x, f - f_n))$ approaches zero for every x of H. Strong convergence, of course, implies weak convergence. Pointwise convergence is ordinary convergence, and if such convergence is bounded (uniformly) it implies strong convergence by Lebesgue's theorem.

If a sequence f_n is uniformly bounded, it is easy to show, by using the "diagonal process" on an expansion of f_n in terms of a complete orthonormal set, that there is a sub-sequence which is weakly convergent.

LEMMA 5. *If f_n is a sequence of H_0 and f_n' is uniformly bounded and weakly convergent, then there is an $f \in H_0$ such that f_n converges pointwise and boundedly to f and f_n' converges weakly to f'. Conversely, if f_n converges pointwise to f, then f_n' converges weakly to f'.*

PROOF. Suppose f_n' converges weakly to f'. We may take f' to be the derivative of a function f belonging to H_0. Considering the sequence $f_n' - f'$, we see that there is no loss of generality in supposing in the beginning that f_n' converges weakly to zero. Thus $((x, f_n')) \to 0$, and by taking x to be a vector of one component, which component is the characteristic function of the interval a, b, we have, if u_n is the corresponding component of f_n, $\int_a^b u_n' dt \to 0$. But $|u_n(b)-u_n(a)|^2 = |\int_a^b u_n' dt|^2 \leqq |b-a| \int_a^b |u_n'|^2 dt \leqq (2\pi)^2 \|f_n'\|^2$. Thus $u_n(b)-u_n(a)$ is boundedly convergent, so $\int_0^{2\pi} (u_n(b)-u_n(a))da \to 9$. But since $f_n \in H_0$, the integral of the second term vanishes, so $2\pi u_n(b) \to 0$. Thus all the components of f_n approach zero pointwise and boundedly. The converse is obvious.

LEMMA 6. *If $y \in Q$, then $\|y'\| \leqq A\|y\|_Q$ where A is independent of y.*

PROOF. Otherwise there is a sequence y_n such that $\|y_n'\| = 1$ and $\|y_n\|_Q \to 0$. Clearly r_n and s_n approach zero by Theorem 1, so without loss of generality we assume $y \in H_0$. Since Ly_n'' converges strongly to zero, it follows from Lemma 5 that Ly_n' converges to zero. Thus $(L+R)y_n'$ converges to zero but $L+R$ is nonsingular, so y_n' converges to zero, which is incompatible with $\|y_n'\| = 1$.

THEOREM 2. *The space Q of electric charge is a Hilbert space.*

PROOF. Let the bilinear form be $((x, y))_Q = ((Lx'', Ly'')) + ((Rx', Ry')) + ((Sx, Sy))$. If $\|y\|_Q = 0$, it follows from Lemma 6 that $y' = 0$. But by Theorem 1, $r = s = 0$, so $y = 0$. It is apparent, therefore, that the bilinear form has the necessary qualifications.

It remains to show that Q is closed, so consider a sequence y_n such that $\|y_n - y_m\|_Q^2 \to 0$. Without loss of generality we may suppose that $y_n \in H_0$. By Lemma 6, y_n' is convergent, and by Lemma 5, y_n converges pointwise to a function y. By the converse statement of Lemma 5, Ly_n'', Ry_n', and Sy_n converge to Ly'', Ry', and Sy respectively. This completes the proof.

We need the following further restriction on a quasi-linear replacement:

(IV) *If V is a function of the time t, then $V(x, t) = V(x, t+2\pi)$ and $(V_1, V_1) \leqq A(V, V)$ where $V_1 = \partial V(x, t)/\partial t$.*

It is evident that the inequalities given for a quasi-linear replacement remain unchanged when the bilinear form (,) is replaced by the bilinear form ((,)).

LEMMA 7. *If Rx_n converges strongly to Rx, then $V(x_n)$ converges strongly to $V(x)$. The same is true for pointwise convergence.*

PROOF. $V(x_n) - V(x) = V'(x_n - x)$, so

$$\|V(x_n) - V(x)\|^2 = \|V'(x_n - x)\|^2 \leqq A_5^2 \|Rx_n - Rx\|^2 \to 0.$$

The proofs below would be a lot simpler if we could say the same thing for weak convergence. However, it is easy to see that the lemma is almost always false under such conditions. Consider sin nt, which converges weakly to zero, but $f(C+\sin nt)$ would not converge weakly to $f(C)$ for all C unless $f(C)$ is a linear function.

The fundamental hypothesis which will be imposed on the matrices L, R, and S in what follows is the relation:

$$Ly'' + Ry' + Sy = 0 \text{ implies } y = 0 \text{ if } y \in Q. \tag{3}$$

An apparently less restrictive condition would be that these equations have no periodic solution in Q, but by virtue of Theorem 1 it is easy to see that these conditions are equivalent. It may be shown that a necessary and sufficient condition for this hypothesis is that the determinant $(-n^2L+(-1)^{1/2}nR+S)\neq 0$ for $n=\pm 1, \pm 2, \cdots$. No use shall be made of this fact, however.

THEOREM 3a. *Let $Ly''+V'(x)y'+Sy=f$ where $y\in Q$ and $x\in H$. Then $\|y\|_Q\leqq A\|f\|$ where A is independent of x and y.*

PROOF. It is to be noted that, in particular, $V'(x)y'$ could be $V(y')$. Because $\|y\|_Q$ is finite, it follows from the inequalities derived for a quasi-linear replacement that $\|f\|$ is finite. If the theorem is not true, it is clear that a sequence y_n exists with $\|y_n\|_Q=1$ and the corresponding sequence f_n converging strongly to zero. By Lemma 6, $\|y'\|\leqq A$, so it may be assumed that the sequence y_n' converges weakly. It may also be assumed that s_n converges; hence, by Lemma 5, y_n converges pointwise to some y.

Employing Lemma 4, we have $((V'y_n', y_n'))=((Ly_n'+V'y_n'+Sy_n, y_n'))=((f_n, y_n'))\to 0$. By the inequalities for a quasi-linear replacement, Ry_n' and $V'y_n'$ converge strongly to zero. Write $Ly_n''=f_n-V'y_n'-Sy_n$. The terms on the right converge strongly; hence, Ly_n'' converges strongly. Thus y_n converges in Q to, say, y, so $Ly''+Sy=0$ and $Ry'=0$. But $Ly''+Ry'+Sy=0$ implies $y=0$ by hypothesis (3). This is incompatible with $1=\|y_n\|_Q=\|y\|_Q$, so the contradiction proves the theorem.

Let $L_n=L+R/n$ where n is a positive integer. The equation $L_ny''+Ry'+Sy=0$ for $y\in Q$ implies $y=0$. This is seen by forming the inner product with y', yielding $((Ry', y'))=0$. Thus $Ry'=0$, and consequently $Ry''=0$. Thus $Ly''+Ry'+Sy=0$ and $y=0$.

THEOREM 3b. *Theorem* 3a *is valid if L is replaced by L_n for an A independent of n if $f\in H$.*

PROOF. For n fixed we may introduce a space Q_n. Then by the above remarks we see that Theorem 3 is applicable. Thus $\|L_ny''\|^2$

$+\|Ry'\|^2+\|Sy\|^2\leqq A^2\|f\|^2$. Since L_n is nonsingular, $\|L_ny''\|\geqq A_1\|y''\|$ $\geqq A_2\|Ly''\|$. We may assume $0<A_2\leqq 1$, so if $A_n=A/A_2$, then $\|y\|_Q\leqq A_n\|f\|$.

If it is not possible to obtain a uniform bound, there is a sequence of integers, say $\{m\}$, with a corresponding sequence y_m and f_m with the properties assumed in the proof of Theorem 3a. Then as before we have that L_my_m'' is strongly convergent. But $L_my_m'=Ly_m'+Ry_m'/m$, and this converges to Ly'. Thus by Lemma 5, L_my_m'' converges strongly to Ly''. As before, we obtain the contradiction $Ly''+Ry'+Sy=0$.

THEOREM 4. *Let T be a transformation of a linear vector space Q on a Hilbert space H such that*: (a) *if y_n is a sequence of Q such that $\|Ty_n\|\leqq A$, then there is a sub-sequence y_m such that $T(y_m)$ converges weakly to $T(y)$ for some y of Q*; (b) *for each x and y of Q, $\lim_{h\to 0}(T(y+hx)-T(y))/h=T'(y, x)$ exists where h is real*; (c) *for each y the relation $((T'(y, x), f))=0$ for all x implies $f=0$.*

Then for any g of H the equation $g=T(y)$ is solvable.

PROOF. Let b be the greatest lower bound of $\|g-T(y)\|$ for all y of Q. Let y_n be a sequence such that $\|g-T(y_n)\|\to b$. Clearly $\|T(y_n)\|$ is uniformly bounded, so there is a weakly convergent subsequence, y_m, and by condition (a) there is a y such that $T(y_m)\to T(y)$ weakly. Thus $\|g-T(y)+T(y)-T(y_m)\|^2=\|g-T(y)\|^2+2((g-T(y),\ T(y)-T(y_m)))$ $+\|T(y)-T(y_m)\|^2$. The limit of the left side is b^2, and the limit of the right is not less than $\|g-T(y)\|^2$, so $b^2\geqq\|g-T(y)\|^2$; hence $b=\|g-T(y)\|$. Let $f=T(y)-g$, then $\|g-T(y\pm hx)\|^2=\|f+T(y\pm hx)$ $-T(y)\|^2=\|f\|^2+2((f,\ T(y\pm hx)-T(y)))+\|T(y\pm hx)-T(y)\|^2$. Thus $0\leqq 2((f, T(y\pm hx)-T(y)))+\|T(y\pm hx)-T(y)\|^2$. Divide through by h taken to be positive, and let h approach zero. Then $0\leqq 2((f, T'(y, x)))$, and $0\leqq -2((f,\ T'(y, x)))$, so by (c) it follows that $f=0$. Q. E. D.

THEOREM 5. *For each g of H there is a unique y of Q such that $Ly''+V(y')+Sy=g$.*

PROOF. First assume that L is nonsingular. Let $T(y)=Ly''$ $+V(y')+Sy$, and consider the conditions of Theorem 4. Let $T(y_n)$ be a uniformly bounded and weakly convergent sequence. By Theorem 3a, Ly_n'' is uniformly bounded. There is a sub-sequence, therefore, over which Ly_n'' is weakly convergent, and since L is nonsingular, y_n'' is also weakly convergent. Also by Theorem 3a, Ry_n' and Sy_n are uniformly bounded, so we may assume that a sub-sequence has been selected in such wise that r_n and s_n converge. Thus by Lemma 5 there is a y of Q such that y_n converges pointwise boundedly to y, y_n' converges pointwise boundedly to y', and y_n'' converges weakly to

y''. By Lemma 7, $V(y_n')$ converges pointwise to $V(y')$. It follows, therefore, that $T(y_n)$ converges weakly to $T(y)$. This shows that (a) is satisfied. To show that (b) is satisfied, we note that $(V(y'+hx')-V(y'))/h$ has the limit $V'(y')x'$. Thus $T'(y, x)=Lx''+V'x'+Sx$, and T' is a linear transformation of x. To show that (c) is satisfied, consider the relation $((f, Lx''+V'x'+Sx))=0$. First let x be a constant vector; then $0=((f, Sx))$, so the component of f in C is orthogonal to the range of S. The same will be true of f_i where f_i is the first i terms of the Fourier series for f. An x may be chosen, therefore, so that $x'=f_i$; then $((f, Lx''))=((x', Lx''))=0$, and $((f, Sx))=((x', Sx))=0$. Thus we obtain $((f, V'x'))=0$, or $((f, V'f_i))=0$. Allowing i to approach infinity in this expression, we obtain $((f, V'f))=0$. By the properties of a quasi-linear replacement, this implies that $Rf=0$; thus, f belongs to H_0. Let $f=u''$, and let u and x belong to H_0; then $0=((u'', Lx''-Rx'+Sx))$. By Lemma 3, $((x'', Lu''+Ru'+Su))=0$, but x'' is an arbitrary element of H_0, so $Lu''+Ru'+Su=0$; thus $u=0$ and hence, $f=0$. This proves the existence of a solution under the assumption that L is nonsingular.

The matrix L_n is nonsingular, so let g be a function of H with a bounded first derivative. By what we have just proved, there is a y of Q such that $L_ny''+Vy'+Sy=g$. Thus $y''=L_n^{-1}(g-V(y')-Sy)$ almost everywhere. Since y_n' is absolutely continuous, $y'=\int_0^t L_n^{-1}(g-V(y')-Sy)dt+C$. But the integrand is continuous, so actually the expression for y'' holds in the pointwise sense. It is clear, therefore, that y'' has a first derivative which is uniformly bounded in the pointwise sense; hence, by a well known theorem of Lebesgue integration, y'' is absolutely continuous, and therefore $y'\in Q$. We may write $L_ny'''+V'y''+Sy'=g_1$ where $g_1=g'-V_1(y')$. By hypothesis IV, $\|V_1(y')\|\leqq A\|V(y')\|$ and from Theorem 3b applied to the equation for g, we have $\|V(y')\|\leqq A\|g\|$. Thus $\|g_1\|$ is uniformly bounded, so applying Theorem 3b to the equation for g_1, we have that $\|Ly'''\|$, $\|Ry''\|$, $\|Sy'\|$ are uniformly bounded. Allow n to approach infinity and consider the sequence y_n. We may suppose that the sequence is so chosen that Ly_n''', Ry_n'', and Sy_n' are weakly convergent. Thus Ly_n'', Ry_n', and Sy_n converge pointwise. Clearly, $V(y_n')$ converges pointwise. The equation for g_1 may be solved for Ry_n'''/n in terms of uniformly bounded expressions. It may be assumed, therefore, that it is uniformly bounded in H and that Ry_n''/n is pointwise convergent. But $\|Ry_n''\|$ is uniformly bounded, so Ry_n''/n must approach zero. If y is the limit of y_n, then $Ly''+V(y')+Sy=g$, *and this equation holds in the pointwise sense, and all terms appearing are continuous.*

If g does not have a bounded derivative, let g_i equal the first i terms in its Fourier series, and we can satisfy the equation for g_i. If y_i and y_j correspond to g_i and g_j, let $z=y_i-y_j$; then $Lz''+V'z'+Sz=g_i-g_j$. By Theorem 3a, $\|z\|_Q$ approaches zero as i and j approach infinity, so y_i converges in Q to some y. This y satisfies the equation for g, and the proof that a solution exists is complete.

To prove the uniqueness of the solution, we note that if z is the difference of two solutions, then $Lz''+V'z'+Sz=0$. Theorem 3a shows directly that $z=0$.

THEOREM 6. *Suppose that all solutions of equations* (3) *are such that* $y'\to 0$ *as* $t\to\infty$. *Let* y_1 *be the periodic solution of equations* (2) *when* g *is a function with a bounded first derivative. If* y_2 *is any other solution of these equations with a continuous first derivative, then* $\int_0^\infty \|y_1'-y_2'\|^2 dt < \infty$.

PROOF. This is a special case of a theorem which the writer proved in a previous note [1].[1] To apply this theorem here it is necessary to know that y_1' is continuous, but this is apparent from the proof of Theorem 5. In the previous note it was not explicitly stated that V could be a function of the time. The proof, however, remains unchanged for this generalization.

A later note will consider similar theorems for networks of linear resistors and iron-core inductors.

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[1] Numbers in brackets refer to the references cited at the end of the paper.