COVERINGS AND BETTI NUMBERS

BENO ECKMANN

1. **Introduction.** Let the finite polyhedron P be a regular covering of the polyhedron $\bar{P}$, and let G be the corresponding (finite) group of covering transformations of P. Then G acts as operator group on the homology groups H_n of P. If we consider homology with real coefficients, H_n is a real vector group of finite rank $p_n = n$th Betti number of P, and G operates in H_n as a group of linear transformations; we denote by $s_n(x)$, $x \in G$, the character of this linear representation of G of degree p_n. Let g be the order of G.

In this note we shall prove:

THEOREM 1. *The nth Betti number* $\bar{p}_n$ *of* $\bar{P}$ *is given by*

$$\bar{p}_n = \frac{1}{g} \sum_{x \in G} s_n(x).$$

In the Princeton Bicentennial Conference W. Hurewicz raised the question whether the homology groups of a polyhedron $\bar{P}$ are determined by those of a regular covering P given as groups with operators. According to Theorem 1 the answer is affirmative in the case of *finite* polyhedra P, $\bar{P}$ and real (or rational) coefficients. We shall show elsewhere—in a general theory of complexes with automorphisms—that the same is true for arbitrary (finite or infinite) polyhedra, provided that the group G is finite. Since the proof in our present case, based upon "harmonic chains," is very simple and yields the explicit formula of Theorem 1, which has interesting applications, we give it here independently of other more general considerations.

2. **Simplicial covering and harmonic chains.** To compute the homology groups of P and $\bar{P}$, let K and $\bar{K}$ be finite simplicial complexes which are subdivisions of P and $\bar{P}$ respectively, such that each oriented simplex of K covers one oriented simplex of $\bar{K}$. Then G acts as an *automorphism group*[1] on K; the automorphisms $x \in G$ are permutations of the oriented simplices σ_n of K in each dimension n and preserve all incidence relations. The set of all simplices σ_n covering a simplex $\bar{\sigma}_n$ of $\bar{K}$ is a transitivity domain of G; that is, it contains with any simplex σ_n' all $x\sigma_n'$, $x \in G$, and only those. Furthermore, since in an automorphism $x \neq e$ no simplex is fixed,

Presented to the Society, April 17, 1948; received by the editors February 25, 1948.

[1] For details see for example [1, §6]. Numbers in brackets refer to the references at the end of the paper.

$x\sigma_n' = x_1\sigma_n'$ implies $x = x_1$. The simplicial map which projects each σ_n of K onto the $\bar{\sigma}_n$ it covers is denoted by U; obviously, $Ux\sigma_n = U\sigma_n$ for all $x \in G$.

In the simplicial complexes K and $\bar{K}$ we shall only use chains with *real coefficients*, and no distinction will be made between chains and cochains. An n-chain a_n in K is a linear form $a_n = \sum_i \alpha_i \sigma_{ni}$ with real coefficients, α_i, where the n-simplices σ_{ni} of K play the role of indeterminates. All n-chains a_n in K form a vector space C_n (a real linear space) of finite rank, in which the scalar product of $a_n = \sum_i \alpha_i \sigma_{ni}$ and $b_n = \sum_i \beta_i \sigma_{ni}$ is defined by $a_n \cdot b_n = \sum_i \alpha_i \beta_i$. The boundary operator ∂ is a linear mapping of C_n into C_{n-1}, $n = 0, 1, \cdots$ ($C_{-1} = 0$); the coboundary operator δ is the linear mapping of C_{n-1} into C_n which is *dual* to ∂, that is, δ is defined by $\delta a_{n-1} \cdot b_n = a_{n-1} \cdot \partial b_n$ for all $(n-1)$ chains a_{n-1} and n-chains b_n. If $\partial a_n = \delta a_n = 0$, a_n is called a *harmonic chain*; as in each homology or cohomology class there is exactly one harmonic chain,[2] the vector group H_n of all harmonic n-chains (a linear subgroup of C_n) is isomorphic both to the nth homology and the nth cohomology group of K (the isomorphism is given by representing a homology or cohomology class by the harmonic chain it contains). The rank of H_n is the nth Betti number p_n of K. All chains, groups, and so on, of $\bar{K}$ will be denoted by the same symbols as in K, with a bar (for example, $\bar{a}_n$, $\bar{C}_n$, $\bar{H}_n$), the boundary and coboundary operator as in K by ∂ and δ.

The linear mappings of the chain groups and the homology groups induced by the simplicial maps U and $x \in G$ will also be denoted by U and x respectively. x is an isomorphism of C_n onto itself, U a homomorphism of C_n onto $\bar{C}_n$, for all n.

The simplicial map U of K onto $\bar{K}$ is locally one-one; that is, if the simplices σ_n and σ_n' of K are both incident with a simplex σ_k, then $U\sigma_n = U\sigma_n'$ implies $\sigma_n = \sigma_n'$. For each simplex σ_n of K one has therefore not only $U\partial\sigma_n = \partial U\sigma_n$, but also

$$U\delta\sigma_n = \delta U\sigma_n; \tag{1}$$

in other words, the linear mapping U of C_n onto $\bar{C}_n$ commutes with ∂ and δ. Let U^* be the dual mapping of $\bar{C}_n$ into C_n, defined by $U^*\bar{a}_n \cdot b_n = \bar{a}_n \cdot Ub_n$ for all n-chains $\bar{a}_n \in \bar{C}_n$ and $b_n \in C_n$; it also commutes with ∂ and δ. Hence

(2) *U and U^* both map harmonic chains onto harmonic chains.*

Since U is a mapping of C_n onto $\bar{C}_n$, the dual mapping U^* is an isomorphism of $\bar{C}_n$ into C_n. For, $U^*\bar{a}_n = 0$ implies $U^*\bar{a}_n \cdot b_n = \bar{a}_n \cdot Ub_n = 0$

[2] See for example [2, pp. 245–246]. Other references are given in [2].

for all $b_n \in C_n$, hence $\bar{a}_n \cdot \bar{b}_n = 0$ for all $\bar{b}_n \in \bar{C}_n$, hence $\bar{a}_n = 0$. This together with (2) yields

THEOREM 2. *U^* induces an isomorphism of $\bar{H}_n$ into H_n, $n = 0, 1, \cdots$.*

COROLLARY 1. *The nth Betti number $\bar{p}_n$ of $\bar{K}$ is not greater than the nth Betti number p_n of K.*

We remark further that U maps $U^*\bar{C}_n \subset C_n$ isomorphically onto $\bar{C}_n$. For, $UU^*\bar{a}_n = 0$ implies $UU^*\bar{a}_n \cdot \bar{a}_n = U^*\bar{a}_n \cdot U^*\bar{a}_n = 0$, hence $U^*\bar{a}_n = 0$. In particular U maps $U^*\bar{H}_n \subset H_n$ isomorphically into $\bar{H}_n$, and since $U^*\bar{H}_n$ is isomorphic to $\bar{H}_n$, this must be a map onto $\bar{H}_n$. Hence

(3) *U maps H_n onto $\bar{H}_n$.*

REMARK. UU^* is not the identity mapping of $\bar{C}_n$ or $\bar{H}_n$ but, as is easily seen, multiplies each chain $\bar{a}_n$ by the order g of G.

3. **Invariant chains.** In the linear mapping of C_n onto itself induced by an automorphism $x \in G$ the scalar product of two n-chains a_n, b_n remains unchanged:

$$xa_n \cdot xb_n = a_n \cdot b_n. \tag{4}$$

This follows simply from the fact that x is a permutation of the oriented simplices (that is, of the basis vectors of C_n). In other words, x is an orthogonal transformation of C_n.

A chain $a_n \in C_n$ will be called *invariant* if $xa_n = a_n$ for all $x \in G$. This is the case if and only if

$$a_n \cdot b_n = a_n \cdot xb_n \tag{5}$$

for all $x \in G$ and $b_n \in C_n$. For, if a_n is invariant, $a_n \cdot b_n = xa_n \cdot xb_n = a_n \cdot xb_n$; and conversely, if (5) holds for all $x \in G$, then $a_n \cdot b_n = a_n \cdot x^{-1}b_n = xa_n \cdot b_n$ for all $b_n \in C_n$, hence $a_n = xa_n$. The invariant chains constitute a linear subgroup C_n^i of C_n.

(6) *The isomorphism U^* maps $\bar{C}_n$ onto C_n^i.*

PROOF. For any $\bar{a}_n \in \bar{C}_n$, $U^*\bar{a}_n$ is invariant; for

$$U^*\bar{a}_n \cdot xb_n = \bar{a}_n \cdot Uxb_n = \bar{a}_n \cdot Ub_n = U^*\bar{a}_n \cdot b_n$$

for all $b_n \in C_n$. Conversely, if a_n is invariant, the relation

$$\bar{a}_n \cdot U\sigma_n = a_n \cdot \sigma_n \qquad \text{for all } \sigma_n \text{ of } K$$

defines an n-chain $\bar{a}_n$ of $\bar{K}$ without ambiguity (for replacing σ_n by $\sigma_n' = x\sigma_n$ does not change either side of this equation). Then $U^*\bar{a}_n \cdot \sigma_n = \bar{a}_n \cdot U\sigma_n = a_n \cdot \sigma_n$ for all σ_n, hence $U^*\bar{a}_n = a_n$.

The invariant harmonic n-chains of K form a linear subgroup H_n^i of H_n. For any $a_n \in H_n^i$ there is, by (6), an $\bar{a}_n$ such that $a_n = U^*\bar{a}_n$;

since U^* commutes with ∂ and δ and is an isomorphism, it follows that $\bar{a}_n$ is harmonic. Combining this with Theorem 2 and (6) we obtain the following result.

THEOREM 3. *U^* induces an isomorphism of $\overline{H}_n$ onto H_n^t.*

4. **Proof of Theorem 1.** Since the automorphisms $x \in G$ of K permute the n-simplices, for each n, and preserve the incidence relations, the linear mappings x of C_n onto itself, $n=0, 1, \cdots$, commute with both ∂ and δ. Hence they induce linear mappings of H_n onto itself; these mappings describe *the operation of G on the n-dimensional homology and cohomology groups of K.* By (4) these linear mappings of H_n onto itself are orthogonal. Hence for each n one has an orthogonal representation $\mathfrak{R}_n$ of the group G in the vector group H_n. The subspace H_n^t of H_n consists of those elements of H_n which are fixed under this representation; by Theorem 3 it may be considered as the homology or cohomology group $\overline{H}_n$ of $\overline{K}$. This proves that the latter *is determined by H_n and by the operation of G in H_n* (cf. §1).

The rank of H_n^t, which is the nth Betti number $\bar{p}_n$ of $\overline{K}$, is equal to the number of trivial irreducible representations of G contained in the representation $\mathfrak{R}_n$ of G in H_n, hence by a well known character relation[3] is equal to the average over the group G of the character $s_n(x)$ of the representation $\mathfrak{R}_n$. This proves Theorem 1. Replacing $s_n(e)$ by the degree p_n of the representation, where e is the unit element of G, we may write the formula in the form

$$g \cdot \bar{p}_n = p_n + \sum_{x \in G, x \neq e} s_n(x). \tag{7}$$

REMARK. $p_n = \bar{p}_n$ holds if and only if $H_n = H_n^t$, hence if all harmonic n-chains of K are invariant under the operations of G; in other words, if each n-dimensional homology class is mapped into itself by all $x \in G$. This result may be stated in the following way:

THEOREM 4. *The nth Betti numbers p_n and $\bar{p}_n$ of K and $\overline{K}$ are equal if and only if in the dimension n all automorphisms $x \in G$ are of the homology type of the identity (for real coefficients).*[4]

5. **Products,[5] manifolds.** The Alexander $\cup$-product associates with two cohomology classes of K of dimensions n and k a third one of di-

[3] Cf. [3, p. 201, formula (16)].

[4] The sufficiency of the condition has previously been proved by G. Hirsch, with an application to closed Lie groups; see [4, p. 226].

[5] For definitions and properties see [5]; for products in a covering complex the remarks of [1, §6] have to be used.

mension $n+k$, hence with two harmonic chains a_n and b_k a unique harmonic $(n+k)$-chain $a_n \cup b_k$. The Čech-Whitney $\cap$-product associates with a cohomology class of dimension k and a homology class of dimension $k+p$ $(p \geqq 0)$ a homology class of dimension p, hence with two harmonic chains b_k and c_{k+p} a unique harmonic p-chain $b_k \cap c_{k+p}$. From the fact that all incidence relations are preserved by the automorphisms $x \in G$ it may easily be deduced that

$$xa_n \cup xb_k = x(a_n \cup b_k), \tag{8}$$

$$xb_k \cap xc_{k+p} = x(b_k \cap c_{k+p}) \tag{9}$$

for all harmonic chains a_n, b_k, c_{k+p} and all $x \in G$.

It follows that the $\cup$- or $\cap$-product of invariant harmonic chains is again an invariant harmonic chain. The direct sum of all H_n^i, $n=0, 1, \cdots$, together with the $\cup$-product, constitutes a subring R^i of the Alexander ring (the cohomology ring) R of K. Since U^* preserves the $\cup$-product, it follows from Theorem 3:

(10) *U^* induces a ring-isomorphism of the cohomology ring $\overline{R}$ of $\overline{K}$ onto R^i.*

We now assume K to be a (closed) orientable N-dimensional manifold. Let m_N be the fundamental harmonic N-chain corresponding to an orientation of K; that is, m_N is a chain with coefficients ± 1 generating H_N. The *duality operator* D in K may be defined by

$$Da_n = a_n \cap m_N \tag{11}$$

for harmonic chains a_n; if a_n is considered as a representative of an n-dimensional cohomology class, Da_n is the harmonic representative of the dual $(N-n)$-dimensional homology class. As is well known, D is an isomorphism of H_n onto H_{N-n}.

(12) *For any $x \in G$ one has $xDa_n = \gamma \cdot Dxa_n$, where $\gamma = \pm 1$ is the (topological) degree of x.*

Proof. $xDa_n = x(a_n \cap m_N) = xa_n \cap xm_N = xa_n \cap \gamma m_N = \gamma \cdot (xa_n \cap m_N) = \gamma \cdot Dxa_n$.

As a consequence we have:

Theorem 5. *If K is an orientable N-dimensional manifold, then the characters $s_n(x)$ and $s_{N-n}(x)$ of the representations of G in H_n and H_{N-n} are related by*

$$s_n(x) = \gamma \cdot s_{N-n}(x),$$

where γ is the degree of $x \in G$.

6. An application. Let $\overline{K}$ be a non-orientable N-dimensional manifold, and K an orientable two-sheeted covering of K. The covering transformation group consists of the unit element e and an element y of degree -1. Various relations between the Betti numbers of $\bar{K}$ and K may be deduced from (7) and (12). (7) becomes in this case (we write s_n for $s_n(y)$)

$$2\bar{p}_n = p_n + s_n, \qquad n = 0, 1, \cdots, N. \tag{13}$$

Since y has degree -1, one has, by (12), $s_n = -s_{N-n}$, hence $2\bar{p}_{N-n} = p_{N-n} + s_{N-n} = p_n - s_n = 2\bar{p}_n - 2s_n$, hence

$$\bar{p}_n - \bar{p}_{N-n} = s_n, \tag{14}$$

$$\bar{p}_n + \bar{p}_{N-n} = p_n. \tag{15}$$

This is a duality theorem for non-orientable N-dimensional manifolds. We shall use (13)–(15) to prove the following theorem.

THEOREM 6. *The following relations hold between the Betti numbers $\bar{p}_n$ of a closed non-orientable N-dimensional manifold $\bar{K}$ and the Betti numbers p_n of a two-sheeted orientable covering K of $\bar{K}$:*

(16) *for* $N = 3, \quad p_1 = 2\bar{p}_1 - 1$;[6]

(17) *for* $N = 2L, \quad p_L = 2\bar{p}_L$;

(18) *for* $N = 2L + 1, \ \sum_{n=0}^{L} (-1)^n p_n = 2\sum_{n=0}^{L} (-1)^n \bar{p}_n$ *and*

$$\sum_{n \text{ even}} p_n = 2 \sum_{n \text{ even}} \bar{p}_n.$$

PROOF. (a) If $N=3$, the Euler characteristic $\bar{p}_0 - \bar{p}_1 + \bar{p}_2 - \bar{p}_3 = 0$, where $\bar{p}_0 = 1$, $\bar{p}_3 = 0$, hence $\bar{p}_2 = \bar{p}_1 - 1$; by (15), $\bar{p}_1 + \bar{p}_2 = 2\bar{p}_1 - 1 = p_1$. (b) (17) follows immediately from (15). (c) If $N = 2L+1$, the characteristics of K and $\bar{K}$ are 0 and we deduce from (13)

$$\sum_{n=0}^{N} (-1)^n s_n = 0, \tag{19}$$

hence $\sum_{n=0}^{L}(-1)^n s_n = -\sum_{n=L+1}^{N} (-1)^n s_n$; since $s_n = -s_{N-n}$, this is equal to $-\sum_{n=L+1}^{N}(-1)^{N-n} s_{N-n} = -\sum_{n=0}^{L}(-1)^n s_n$, hence $\sum_{n=0}^{L} (-1)^n s_n = 0$. Using $s_n = -s_{N-n}$ again, we may also write this as $\sum_{n \text{ even}} s_n = 0$. These two results together with (13) yield (18). We remark that since the characteristics of K and $\bar{K}$ are 0 $(N=2L+1)$ the first of the formulas (18) is equivalent to $\sum_{n=L+1}^{N}(-1)p_n = 2\sum_{n=L+1}^{N}(-1)\bar{p}_n$, the second one to $\sum_{n \text{ odd}} p_n = 2\sum_{n \text{ odd}} \bar{p}_n$.

[6] For 3-dimensional manifolds $\bar{K}$ satisfying certain restrictive conditions (16) is proved in a note by T. H. Kiang [6].

REMARK. The formula (19), which here was proved directly, may also be deduced from a well known result concerning the number of fixed simplices in a simplicial self-mapping.[7] In the general case of an arbitrary automorphism group G and arbitrary finite complexes K and $\overline{K}$ it follows from this result that the "Lefschetz number" $\lambda(x) = \sum_{n=0}^{N} (-1)^n s_n(x)$ is 0 for any automorphism $x \neq e$.

REFERENCES

1. B. Eckmann, *Der Cohomologiering einer beliebigen Gruppe*, Comm. Math. Helv. vol. 18 (1945–1946) pp. 232–282.

2. ———, *Harmonische Funktionen und Randwertaufgaben in einem Komplex*, Comm. Math. Helv. vol. 17 (1945) pp. 240–255.

3. B. L. van der Waerden, *Moderne Algebra*, vol. 2, Berlin, 1931.

4. G. Hirsch, *Sur les groupes d'homologie de certains complexes de recouvrement*, Portugaliae Math. vol. 4 (1945) pp. 225–237.

5. H. Whitney, *On products in a complex*, Ann. of Math. vol. 39 (1938) pp. 397–432.

6. T. H. Kiang, *An application of the addition formulas of Mayer-Vietoris*, Academia Sinica Science Record vol. 7 (1945) pp. 275–276.

7. P. Alexandroff and H. Hopf, *Topologie*, Berlin, 1935.

INSTITUTE FOR ADVANCED STUDY AND
UNIVERSITY OF LAUSANNE

[7] Cf. [7, p. 530].