# THE NOVEMBER MEETING IN LOS ANGELES

The four hundred forty-first meeting of the American Mathematical Society was held at the University of California, Los Angeles, on Saturday, November 27, 1948. The attendance was approximately 125, including the following 93 members of the Society:

M. I. Aissen, O. W. Albert, Richard Arens, W. G. Bade, J. L. Barnes, E. F. Beckenbach, H. W. Becker, M. M. Beenken, Clifford. Bell, Gertrude Blanch, H. F. Bohnenblust, J. L. Brenner, Herbert Busemann, W. D. Cairns, L. H. Chin, L. M. Coffin, F. E. Cothran, F. G. Creese, E. L. Crow, D. R. Curtiss, J. H. Curtiss, Tobias Dantzig, P. H. Daus, W. W. Denton, C. R. DePrima, R. P. Dilworth, Roy Dubisch, W. D. Duthie, D. T. Finkbeiner, G. E. Forsythe, Arie Gaalswyk, P. R. Garabedian, H. H. Germond, H. E. Glazier, W. H. Glenn, J. W. Green, G. J. Haltiner, H. J. Hamilton, E. M. Henderson, M. R. Hestenes, P. G. Hoel, Alfred Horn, R. E. Horton, D. G. Humm, D. H. Hyers, Rufus Isaacs, Glenn James, P B. Johnson, Samuel Karlin, M. B. Lehman, D. H. Lehmer, Eugene Lukacs, J. C. C. McKinsey, J. E. McLaughlin, A. V. Martin, W. A. Mersman, A. B. Mewborn, A. D. Michal, W. E. Milne, S. F. Neustadter, A. B. J. Novikoff, L. J. Paige, R. S. Phillips, H. H. Price, E. S. Quade, W. C. Randels, L. T. Ratner, Saul Rosen, Wladimir Seidel, Max Shiffman, R. A. Siegel, Ernst Snapper, I. S. Sokolnikoff, R. H. Sorgenfrey, D. V. Steed, Robert Steinberg, E. G. Straus, L. M. Straus, A. C. Sugar, J. D. Swift, Alfred Tarski, A. E. Taylor, C. J. Thorne, Elmer Tolsted, S. M. Ulam, F. A. Valentine, L. F. Walton, Morgan Ward, P. A. White, A. L. Whiteman, D. V. Widder, V. A. Widder, E. R. Worthington.

The morning session was devoted to research papers and to the invited address, *The geometry of Finsler spaces*, by Professor Herbert Busemann, of the University of Southern California. Professor M. R. Hestenes presided. In the afternoon there were two sections for research papers, at which Professors D. H. Hyers and Morgan Ward, respectively, presided.

Following the meetings, there was a tea at the Institute for Numerical Analysis of the National Bureau of Standards, on the campus.

Abstracts of papers read at the meetings follow. Abstracts whose numbers are followed by the letter "*t*" were presented by title. Mrs. Lehmer was introduced by Professor J. W. Green, Mr. Lackness by Professor G. C. Evans, and Mr. Thompson, Professor Mostowski, and Mrs. Szmielew by Professor Alfred Tarski. Paper number 69 was read by Miss Chin, paper 73 by Mr. McLaughlin, and paper 91 by Professor Phillips.

## Algebra and Theory of Numbers

69. Louise H. Chin and Alfred Tarski: *Distributive and modular laws in relation algebras.*

By the left distributive law for two binary operations $\odot$ and $\oplus$ is meant the formula $a\odot(b\oplus c)=(a\odot b)\oplus(a\odot c)$ where $a$, $b$, $c$ are elements of a set $S$. Similarly, by the left modular law is meant the formula $a\odot[b\oplus(a\odot c)]=(a\odot b)\oplus(a\odot c)$. Right distributive and modular laws are analogously defined. Relation algebras (A. Tarski, Journal of Symbolic Logic vol. 6) are systems where four binary operations—absolute $+$ and $\cdot$, relative $\dot{+}$ and ;—play a fundamental role. Hence there is a variety of such laws to consider. In some cases, that is, for some couples of binary operations, the laws are satisfied by arbitrary $a$, $b$, $c$ of the algebra; for example, distributive laws for ; and $+$, and for $\dot{+}$ and $\cdot$. In other cases, elements $a$ are distinguished which are distributive, or modular, for a given couple of operations, that is, which together with arbitrary $b$, $c$ satisfy a distributive or modular law in question. In most cases the sets of elements thus distinguished prove important even independent of formal reasons. For example, elements $a$ which are left distributive for ; and $\cdot$ (that is, such that $a;(b\cdot c)=(a;b)\cdot(a;c)$ for any $b$, $c$) coincide with the functional elements, that is, with functions in the ordinary interpretation of relation algebras. Similarly, elements $a$ which are left and right modular for $\cdot$ and ; are equivalence elements, that is, symmetric and transitive relations in the ordinary interpretation. (Received October 8, 1948.)

## 70. R. P. Dilworth: *A theorem on the structure of the lattice of continuous functions.*

A prime ideal $P$ of a distributive lattice $L$ is said to be *isolated* if for every prime ideal $Q$ either $Q\supseteq P$, $P\supseteq Q$, or $P\cup Q=L$. The following theorem is proved: let $C(S)$ denote the lattice of continuous, bounded, real functions on a topological space $S$. Then $C(S)$ has the property that each element (as a principal ideal) is an intersection of isolated prime ideals. This property is, in a sense, a weak form of the requirement that the lattice be a direct union of chains, for it can be shown that a finite-distributive lattice has this property if and only if it is a direct union of chains. (Received October 16, 1948.)

## 71. D. T. Finkbeiner. *A general imbedding problem for lattices.* Preliminary report.

Consider a property P which holds in a lattice $L$ if and only if P holds in every quotient lattice of $L$. Such properties are distributivity, modularity, upper and lower semi-modularity, and so on. P is said to hold weakly in $L$ if and only if P holds in every quotient lattice $a/b$ where $b$ is not the null element of $L$. If $L'$ is a lattice with the property P, and if $S$ is the set sum of principal ideals generated by a set of elements of $L'$, then $L=L'-S+$null element is a lattice in which P holds weakly. In this paper conditions are determined under which a lattice $L$, in which P holds weakly, can be enlarged to $L'$ in which P holds in general (Received October 18, 1948.)

## 72. Emma Lehmer: *On the quintic character of* 2.

A criterion for the cubic character of 2 was developed by Gauss. It states that 2 is a cubic residue of $p$ if and only if $x\equiv 1 \pmod 3$ is even in the quadratic partition of $4p=x_3^2+27y^2$. Recently S. Chowla gave an expression for $x_3=1+\phi_3(4)$, where $\phi_k(a)=\sum_{\nu=1}^{p-1}\chi(\nu)\chi(\nu^k+a)$ is a sum discussed by Jacobsthal in a Berlin dissertation (1906). (Here $\chi(n)$ denotes the quadratic character of $n$ with respect to $p$.) The present paper gives a similar criterion for the quintic character of 2 as follows: Two is a quintic residue of $p$ if and only if the value of $x\equiv 1 \pmod 5$ is *even* in the simultaneous solution of the pair of Diophantine equations $16p=x_5^2+50u^2+50v^2+125w^2$ and $x_5w=v^2-4uv-u^2$. The uniqueness of this value of $x_5$ has been established by L. E. Dickson (Amer. J. Math. vol. 57 (1935) pp. 391–424). It is also shown that $x_5=1+\phi_5(4)$. From another point of view $x_k$ can be defined in general as $x_k=-\sum_{i=1}^{p-1}a_i$, where $a_i$ are the coefficients in the decomposition of $p$ into reciprocal factors in the field of $k$th roots of unity for $k$ an odd prime. Then $x_k=1+\phi_k(4)$. Obviously $x_k$ is even if and only if 2 is a $k$th power residue, but the Diophantine equations involving $x_k^2$ which are known do not have a unique solution for $x_k$ and therefore cannot serve as criteria for $k>5$. (Received October 15, 1948.)

## 73. Jack McLaughlin and R. P. Dilworth: *Projectivities in relatively complemented lattices.*

It is known that if two points of a complemented modular lattice are projective, this projectivity can be accomplished by means of two transpositions. It is shown here that although this property does not extend to arbitrary relatively complemented lattices $L$, every projectivity between points of $L$ can be accomplished by means of at most $2(n-1)$ transpositions, where $n$ is the dimension of $L$. (Received October 16, 1948.)

## 74. Alfred Tarski: *Arithmetical classes and types of mathematical systems.* Preliminary report.

Consider the class $K$ of all systems $\mathfrak{A}=\langle A, R\rangle$, $A$ any nonempty set, $R$ any $n$-termed relation ($n$ fixed). $L\subseteq K$ is an *arithmetical* (or *elementary*) *class*, $L\in \boldsymbol{AC}$, if the condition $\langle A, R\rangle\in L$ is equivalently expressible by a formula containing only elementary logical constants (connectives, quantifiers, identity symbol), bound variables representing elements of $A$, and the free variable $R$. (This definition is replaceable by a purely mathematical one. Compare Tarski and Kuratowski-Tarski, Fund. Math. vol. 17.) $\mathfrak{A}$, $\mathfrak{B}\in K$ are *arithmetically equivalent* (*elementarily indistinguishable*), $\mathfrak{A}\equiv\mathfrak{B}$, if $\mathfrak{A}\in L\leftrightarrow\mathfrak{B}\in L$ whenever $L\in \boldsymbol{AC}$ (Tarski, Fund. Math. vol. 26). $\overline{M}$ denotes the class of all $\mathfrak{A}\in K$ with $\mathfrak{A}\equiv\mathfrak{B}$ for some $\mathfrak{B}\in M$; $L$ is an *arithmetical type*, $L\in \boldsymbol{AT}$, if $L=\overline{M}$ for some one-element $M\subseteq K$. Theorems: I. $\boldsymbol{AC}$ is a Boolean algebra under class addition and complementation, with unit class $K$. II. $(L_{i+1}\subset L_i\in \boldsymbol{AC}$ for $i=0, 1, \cdots)\rightarrow(\prod_{i<\infty}L_i\notin C)$. III. $(L\in \boldsymbol{AT})\leftrightarrow L=\prod_{M\in F}M$ for some dual prime ideal $F$ in the Boolean algebra $\boldsymbol{AC}$). IV. $\boldsymbol{AC}$ is countably infinite; also $\boldsymbol{AT}$ for the fixed $n$ equaling 1; $\boldsymbol{AT}$ has the continuum power otherwise. V. For every $\langle A, R\rangle\in K$, $A$ infinite, there is a $\langle B, S\rangle\equiv\langle A, R\rangle$ with $B$ of any prescribed infinite power. All these notions and results extend to systems with many relations and operations. (Received October 12, 1948.)

## 75*t*. Alfred Tarski: *Metamathematical aspects of arithmetical classes and types.* Preliminary report.

For notions below see Tarski, Fund. Math. vols. 25, 26. Using symbolism of Bull. Amer. Math Soc. Abstract 54-11-74, consider any $L\subseteq K$. Let Th($L$) be the *arithmetical* (*elementary*) *theory of* $L$, that is, the part of the theory of systems $\langle A, R\rangle\in L$ formalizable within the lower predicate calculus. Th($L$) and Th($\overline{L}$) clearly coincide. (Th($L$) is consistent)$\leftrightarrow$($L\neq 0$) [Godel's result related to II of preceding abstract]; (Th($L$) has a finite system of non-logical axioms)$\leftrightarrow$($\overline{L}\in \boldsymbol{AC}$); (Th($L$) is consistent and complete)$\leftrightarrow$($\overline{L}\in \boldsymbol{AT}$). V of abstract 54-11-74 in metamathematical translation presents an extension of Löwenheim-Skolem's theorem. Call $M$ an *arithmetical L-class*,

$M\in AC(L)$, or *L-type*, $M\in AT(L)$, if $M=L\cdot N$ for some $N\in AC$, or $M=L\cdot N\neq 0$ for some $N\in AT$. If $L\in AC$, then $(M\in AC(L))\leftrightarrow(L\supseteq M\in AC)$ and $(M\in AT(L))\leftrightarrow(L\supseteq M\in AT)$. $AC(L)$ is a Boolean algebra under class addition and complementation, with unit class $L$; its isomorphism type determines uniquely the structural type of $\mathrm{Th}(L)$ and conversely. If one succeeds in establishing a decision procedure for $\mathrm{Th}(L)$ by means of the so-called method of eliminating quantifiers, one also succeeds in exhaustively describing $AC(L)$ and $AT(L)$; if $L\in AC$, the description of $AT(L)$ amounts to that of all consistent and complete extensions of $\mathrm{Th}(L)$, that is, consistent and complete theories obtained by enriching $\mathrm{Th}(L)$ with new non-logical axioms. (Received October 12, 1948.)

### 76*t*. Alfred Tarski: *Arithmetical classes and types of Boolean algebras*. Preliminary report.

For notations see Bull. Amer. Math. Soc. Abstracts 54-11-74 and 54-11-75. $L$ denotes the class of all Boolean algebras. Clearly $L=\overline{L}\in AC$. The decision procedure for the theory $\mathrm{Th}(L)$ (found by the author) leads to the following description of $AC(L)$ and $AT(L)$. Given $\mathfrak{B}\in L$, let $\mathfrak{B}_0=\mathfrak{B}$; let $\mathfrak{B}_{k+1}$ be the quotient algebra $\mathfrak{B}_k|I$ where $I$ is the ideal of the elements $x$ in $\mathfrak{B}_k$ such that the join of all atoms included in $x$ exists. $M$, $N(n)$, $P(n)$, $Q(n, p)$, where $n, p=0, 1, \cdots$, are respectively classes of all $\mathfrak{B}\in L$ such that (i) none of the algebras $\mathfrak{B}_0, \mathfrak{B}_1, \cdots, \mathfrak{B}_k, \cdots$ is atomistic; (ii) $\mathfrak{B}_n$ is atomistic; (iii) $\mathfrak{B}_n$ has infinitely many atoms while $\mathfrak{B}_{n+1}$ has only one element; (iv) $\mathfrak{B}_n$ has exactly $p$ atoms. Theorems: $AC(L)$ is the smallest family containing $L$, all $N(n)$'s and $Q(n, p)$'s, and closed under class addition and subtraction. $AC(L)$ as a Boolean algebra under class operations has a well ordered basis of the type $\omega^2$ (cf. Mostowski-Tarski, Fund. Math. vol. 32). $AT(L)\cdot AC(L)$ consists of all the classes $Q(0, 0)\cdot N(0)$, $Q(n, p+1)\cdot N(n)$ and $Q(n, p)-N(n)$; $AT(L)-AC(L)$ consists of all the classes $M$, $P(n)\cdot N(n)$, and $P(n)-N(n)$. The families $AC(L)$, $AT(L)\cdot AC(L)$, $AT(L)-AC(L)$ are countably infinite. The results above were obtained in 1940. (Received October 12, 1948.)

### 77*t*. Alfred Tarski: *Arithmetical classes and types of algebraically closed and real-closed fields*. Preliminary report.

For notations see Bull. Amer. Math. Soc. abstracts 54-11-74, 54-11-75. $L$ being the class of all algebraically closed fields $\langle A, +, \cdot\rangle$, we have $L=\overline{L}\notin AC$. A decision procedure for $\mathrm{Th}(L)$ has been found. The resulting description of $AC(L)$ and $AT(L)$ follows: $M(p)$ being the class of all $\mathfrak{A}\in L$ with characteristic $p$, $AC(L)$ is the smallest family containing all $M(p)$'s, $p$ a prime, and closed under class addition and complementation with respect to $L$; $AT(L)\cdot AC(L)$ consists of all $M(p)$'s, $p$ a prime; $AT(L)-AC(L)$ consists only of $M(0)$. Hence $AC(L)$, $AT(L)$, and $AT(L)\cdot AC(L)$ are countably infinite. As a Boolean algebra under class operations, $AC(L)$ has a well ordered basis of the type $\omega$. $\mathfrak{A}\equiv\mathfrak{B}$ (for $\mathfrak{A}, \mathfrak{B}\in L$) if, and only if, $\mathfrak{A}$ and $\mathfrak{B}$ have the same characteristic. Let now $L'$ be the class of all real-closed fields. Sturm's theorem for $L'$ is extended from one equation in one unknown to arbitrary systems of algebraic equations and inequalities in many unknowns. By means of this, a decision procedure for $\mathrm{Th}(L')$ is established; $\mathrm{Th}(L')$ proves to be consistent and complete. Hence $L'=\overline{L'}\in AT-AC$; $AC(L')$ consists only of $L'$ and 0, and $AT(L')$ of $L'$; finally $\mathfrak{A}\equiv\mathfrak{B}$ for arbitrary $\mathfrak{A}, \mathfrak{B}\in L'$ (for example, the field $\mathfrak{A}$ of real numbers and the field $\mathfrak{B}$ of real algebraic numbers). (Received October 12, 1948.)

78t. Andrzej Mostowski and Alfred Tarski: *Arithmetical classes and types of well ordered systems.* Preliminary report.

For notations see Bull. Amer. Math. Soc. Abstract 54-11-74, 54-11-75. The set $A$ being ordered by the relation $\leqq$, $\tau(A)$ denotes the order type of the system $\mathfrak{A}=\langle A, \leqq\rangle$. Let $L$ be the class of all $\langle A, \leqq\rangle$ with $A\neq 0$ well ordered by $\leqq$. A decision procedure for $\mathrm{Th}(L)$ is established. Given any ordinal $\gamma$, let $M(\gamma)$, $N(\gamma)$, $P(\gamma)$, $Q(\gamma)$ be respectively classes of all $\mathfrak{A}\in L$ with: $\tau(\mathfrak{A})=\gamma+\eta$ for some ordinal $\eta\neq 0$; $\tau(\mathfrak{A})=\xi+\gamma+\eta$ for some $\xi$ and $\eta$, $\eta<\gamma\cdot\omega$; $\tau(\mathfrak{A})=\gamma$; $\tau(\mathfrak{A})=\omega^\omega\cdot\xi+\gamma$ for some $\xi\neq 0$. Theorems : $\boldsymbol{AC}(L)$ is the smallest family containing all the classes $L$, $M(\omega^n)$, $N(\omega^n\cdot p)$, where $n$, $p=0, 1, \cdots$, and closed under class addition and subtraction. $\boldsymbol{AC}(L)$ as a Boolean algebra under class operations has a well ordered **basis of the type** $\omega^\omega$ $\cdot$ $\boldsymbol{AT}(L)\cdot\boldsymbol{AC}(L)$ consists of all $P(\gamma)$'s, and $\boldsymbol{AT}(L)-\boldsymbol{AC}(L)$ of all $Q(\gamma)$'s, with $\gamma<\omega^\omega$. The families $\boldsymbol{AC}(L)$, $\boldsymbol{AT}(L)$, $\boldsymbol{AT}(L)\cdot\boldsymbol{AC}(L)$, $\boldsymbol{AT}(L)-\boldsymbol{AC}(L)$ are countably infinite. For $\mathfrak{A}$, $\mathfrak{B}\in L$, we have $\mathfrak{A}\equiv\mathfrak{B}$ if, and only if, either $\tau(\mathfrak{A})=\tau(\mathfrak{B})<\omega^\omega$ or $\tau(\mathfrak{A})=\omega^\omega\cdot\alpha+\gamma$, $\tau(\mathfrak{B})=\omega^\omega\cdot\beta+\gamma$ for some $\alpha\neq 0$, $\beta\neq 0$, $\gamma<\omega^\omega$. If $\mathfrak{A}\in L$, $\tau(\mathfrak{A})\geqq\omega$ then $\mathfrak{A}\equiv\mathfrak{B}$ for some $\mathfrak{B}\notin L$; take, for example, $\tau(\mathfrak{B})=\omega+\omega^*.+\tau(\mathfrak{A})$. There is a $\mathfrak{B}\notin\overline{L}$ such that $\mathfrak{B}\in M$ whenever $L\subseteq M\in\boldsymbol{AC}$; take, for example, $\tau(\mathfrak{B})=\omega+\omega^*$. $L\neq\overline{L}$ and $L, \overline{L}\notin\boldsymbol{AC}$ (Tarski, Fund. Math. vol. 26, p. 301). The results above were obtained in 1941. (Received October 12, 1948.)

79t. Wanda Szmielew: *Arithmetical classes and types of Abelian groups.* Preliminary report.

Let $L$ be the class of all Abelian groups. Obviously, $L=\overline{L}\in\boldsymbol{AC}$ (see Bull. Amer. Math. Soc. Abstracts 54-11-74, 54-11-75). A decision procedure for the theory $\mathrm{Th}(L)$ has been established. The resulting description of $\boldsymbol{AC}(L)$ follows: Given $\langle A, +\rangle\in L$ and $n$, $p=1, 2, \cdots$, the elements $a_1, \cdots, a_n\in A$ are called independent modulo $p$ if the formula $p_1\cdot a_1+\cdots+p_n\cdot a_n=p\cdot a$ with any $a\in A$ and $0\leqq p_1<p, \cdots, 0\leqq p_n<p$ implies $p_1=\cdots=p_n=0$; they are called $p$-independent if they are of order $p$ and the formula $p_1\cdot a_1+\cdots+p_n\cdot a_n=0$ with $0\leqq p_1<p, \cdots, 0\leqq p_n<p$ implies $p_1=\cdots=p_n=0$. Let $M(n, p)$, $N(n, p)$, and $P(n, p)$ be classes of all $\langle A, +\rangle\in L$ with at least $n$ different elements which are respectively (i) $p$-independent, (ii) independent modulo $p$, and (iii) of order $p$ and independent modulo $p$; let $Q(n)$ be the class of all $\langle A, +\rangle\in L$ with $n\cdot a=0$ for every $a\in A$. Theorems: $\boldsymbol{AC}(L)$ is the smallest family containing all the classes $L$, $M(n, p^k)$, $N(n, p^k)$, $P(n, p^k)$, $Q(n)$ [$k$, $n=1, 3, \cdots$, $p$ any prime], and closed under class addition and subtraction. (Hence a description of $\boldsymbol{AT}(L)$ can also be derived.) $\boldsymbol{AC}(L)$ as a Boolean algebra under class operations has no well ordered basis. The families $\boldsymbol{AC}(L)$ and $\boldsymbol{AT}(L)\cdot\boldsymbol{AC}(L)$ are countably infinite; $\boldsymbol{AT}(L)$ and $\boldsymbol{AT}(L)-\boldsymbol{AC}(L)$ have the power of the continuum. Analogous (though simpler) results apply to simply ordered Abelian groups $\langle A, +, \leqq\rangle$. (Received October 12, 1948.)

80t. Andrzej Mostowski: *A general theorem concerning the decision problem.* Preliminary report.

Let $\mathfrak{A}$ be a mathematical system formed by a set $A$ and arbitrary relations and operations $R_1, \cdots, R_n$. Let $\mathfrak{B}$ be the direct (cardinal) product of arbitrarily many factors each identical with $\mathfrak{A}$. The term "direct product" can be understood in the strong or the weak sense; in the latter case any element $z\in A$ definable in terms of $R_1, \cdots, R_n$ can be taken as 0. (See, for example, A. Tarski, *Cardinal algebras*, New

York, 1948.) Let $\mathrm{Th}(\mathfrak{A})$ and $\mathrm{Th}(\mathfrak{B})$ be the arithmetical theories of $\mathfrak{A}$ and $\mathfrak{B}$ respectively (cf. Bull. Amer. Math. Soc. Abstract 54-11-75). Theorem: if the theory $\mathrm{Th}(\mathfrak{A})$ is decidable (that is, if the set of all theorems in $\mathrm{Th}(\mathfrak{A})$ is generally recursive), the same applies to $\mathrm{Th}(\mathfrak{B})$. Example: Let $\mathfrak{A}=\langle A, +\rangle$ and $\mathfrak{B}=\langle B, \cdot\rangle$, where $A$ is the set of all non-negative integers, $B$ the set of all positive integers, and $+$ and $\cdot$ have their usual arithmetical meanings. A decision procedure for $\mathrm{Th}(\mathfrak{A})$ has been established (by Presburger); by the theorem stated above it implies the existence of a decision procedure for $\mathrm{Th}(\mathfrak{B})$ (a result obtained in a different way by Skolem). (Received October 12, 1948.)

### 81. F. B. Thompson: *A note on the unique factorization of abstract algebras.*

In their monograph, *Direct decompositions of finite algebraic systems*, Notre Dame Mathematical Lectures, No. 5, 1947, Jónsson and Tarski establish the unique factorization theorem and several related results for finite algebras consisting of a set $A$ of elements closed under a binary operation $+$, such that $A$ contains an element $z$ which is a zero element for $+$. They raised the problem whether these results can be extended to algebras differing from those mentioned above in that the element $z$ need only be idempotent under $+$ (that is, $z+z=z$). The solution of this problem proves to be negative. In fact, let $\bigcirc$ and $\square$ be operations on integers such that (i) $m\bigcirc n=0$ for all $m$ and $n$, (ii) $m\square n=2m'+n'$ where $0\leqq m', n'<2, m\equiv m' \pmod 2), n\equiv n' \pmod 2)$. Let $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_1$, $\mathfrak{C}_2$ be respectively the algebras constituted by: (1) the set $\{0, 1\}$ and the operation $\bigcirc$; (ii) the set $\{0, 1, 2, 3, 4, 5\}$ and the operation $\square$; (iii) the set $\{0, 1, 2\}$ and the operation $\bigcirc$; (iv) the set $\{0, 1, 2, 3\}$ and the operation $\square$. It is seen that each of these algebras has an idempotent element and is indecomposable, and that $\mathfrak{B}_1\times\mathfrak{B}_2\cong\mathfrak{C}_1\times\mathfrak{C}_2$ This contradicts the unique factorization theorem. (Received September 22, 1948.)

## ANALYSIS

### 82t. Paul Civin: *Approximation in* Lip $(\alpha, p)$.

If $f(x)\in\mathrm{Lip}\ (\alpha, p)$ and $\{P_n(x)\}$ is a sequence of trigonometric polynomials of order $n$ such that $(\int_{-\pi}^{\pi}|f(x)-P_n(x)|^p dx)^{1/p}\leqq Kn^{-\alpha}$, then it is shown that $D=(\int_{-\pi}^{\pi}|P_n'(x)|^p dx)^{1/p}\leqq A(1-\alpha)^{-1}n^{1-\alpha}$ for $0<\alpha<1$, $D\leqq A\log n$ for $\alpha=1$, and $D\leqq A(\alpha-1)^{-1}$ for $\alpha>1$, where in each case $A$ depends only on $\alpha$ and the sequence $P_n(x)$ but not on $n$. The corresponding result for functions in Lip $\alpha$ was established by M. Zamansky (C. R. Acad. Sci. Paris vol. 224 (1947) pp. 704–706). (Received August 10, 1948.)

### 83. G. E. Forsythe: *Certain inequalities concerning the Legendre polynomials.* Preliminary report.

For integers $n\geqq 0$, $k\geqq h\geqq 1$ and a real variable $x$, let $\Delta=\Delta(n, h, k; x)$ $=P_n(x)P_{n+h+k}(x)-P_{n+h}(x)P_{n+k}(x)$, where $P_r(x)$ is the Legendre polynomial of $r$th degree. When $h+k$ is an even [odd] number, $\Delta$ is an even [odd] function of $x$. For given $n, h, k$, the form $\Delta$ is said to have property T when $0<x<1$ implies that $\Delta<0$. The general purpose of this investigation is to see which of the forms $\Delta$ have property T. Turán showed that $\Delta(n, 1, 1; x)$ has property T for each $n\geqq 0$ (see G. Szegö, *On an inequality of P. Turán concerning Legendre polynomials*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 401–405). The present author proves that $\Delta(n, 1, 2; x)$ and $\Delta(2n+1, 2, 2; x)$ both have property T for each $n\geqq 0$. The proofs involve applying Szegö's first two methods of proof (loc. cit.) to $\Delta$, $d\Delta/dx$, or $d^2\Delta/dx^2$ in various subintervals of $(0, 1)$. On the other hand, the present author shows that $\Delta$ has one or more zeros $x_0$ with $0<x_0<1$ for the following values of the parameters: (i) for each $n\geqq 0$, when $h+k=4M+1$ $(M\geqq 1)$; (ii) for each even $n\geqq 0$, when $h+k=2M$ $(M\geqq 2)$, except when $h=k=$odd number$\geqq 3$; (iii) for each odd $n\geqq 1$, when $h+k=2M$ $(M\geqq 2)$, except when $h=k\geqq 2$; (iv) for $n=0$, $h=1$, when $k\geqq 3$; (v) for $n=0$, $h=2$ or 3, when $k\geqq h$. By computation when necessary it is further shown that for $n+h+k\leqq 10$ the only $\Delta$ with property T are those of types $\Delta(n, 1, 1; x)$, $\Delta(n, 1, 2; x)$, and $\Delta(2n+1, 2, 2; x)$. (Received October 16, 1948.)

## 84. P. R. Garabedian (National Research Fellow): *The sharp form of the principle of hyperbolic measure.*

Let $D$ be a Riemann surface of genus $g$ bounded by $n$ continua, let $G$ be a Riemann surface with a universal covering surface of hyperbolic type, and let $z_0$ be a point of $D$, $w_0$ a point of $G$. Denote by $E$ the class of functions $f(z)$, analytic in $D$, which assume values in $G$ and satisfy $f(z_0)=w_0$, and denote by $F(z)$ a function in $E$ with $|F'(z_0)|=\max|f'(z_0)|$ for $f$ in $E$. It is shown that $F(z)$ maps $D$ upon a covering surface $S$ of $G$ which has no boundary components over $G$, that is, upon which any point can move indefinitely, provided its projection lies interior to $G$. While $S$ may have infinitely many sheets, it has at most $4g+2n-2$ branch points, with multiple branch points counted according to their multiplicities. These branch points lie at the zeros of a quadratic differential $R(t)dt^2$ which is regular on $S$ except for a possible double pole at $t=w_0$ and which is positive on the boundary of $S$. These results can be extended to problems of Pick-Nevanlinna type. The proofs are based on variational methods. The variations are defined by Green's formula and the construction of simply- and doubly-connected covering surfaces of $G$ which correspond to the boundary components of $D$. (Received October 1, 1948.)

## 85*t*. M. R. Hestenes: *A note on the Lagrange multiplier rule in the calculus of variations.*

Consider a Hilbert space $H$. Let $Q(x, y)$ be a real symmetric bilinear form on $HH$ and denote by $H_0$ the set of all $y$ such that $Q(y, x)=0$ for all $x$ in $H$. Suppose that there is a positive number $m$ such that $Q(x, x)\geqq m|x|^2$ on the subset of $H$ that is orthogonal to $H_0$. Then every linear form $L(x)$ that vanishes on $H_0$ is expressible in the form $L(x)=Q(y, x)$. In the present paper it is shown that the Lagrange multiplier rule for the problem of Bolza is a simple consequence of this result. (Received October 15, 1948.)

## 86*t*. R. M. Lakness: *Green's theorems in multiple leaved domains.*

A multiple leaved space $\mathfrak{M}$ in 3 dimensions is the analog of a Riemann surface in the plane. Let $\Omega$ be a bounded domain on $\mathfrak{M}$ with boundary consisting of a finite number of branch curves $\{s_i\}$ and a bounded exterior frontier $\Omega^*$ such that $\Omega$ is topologically equivalent to an $m$-leaved sphere where the $\{s_i\}$ correspond to branch circles within the sphere, no two of which loop or have a point in common. The $\{s_i\}$ are to be of zero capacity as closed sets in space. Since the boundary $\Omega^*$ is not involved by the branch curves we may take the parts of the boundary to be regular surfaces. Let $u$, $v$ be functions which are linear combinations of bounded functions of class $C_1$ whose Laplacians exist almost everywhere and are non-negative. Then $\nabla^2 u$, $(\nabla u)^2$, $(\nabla u\cdot\nabla v)$ are summable over $\Omega$ and the usual Green's theorems are valid in which the boundary integral involves only $\Omega^*$ and not the branch curves. The main tool used in the proof is Kellogg's uniqueness theorem for such domains, which has been proved by Professor G. C. Evans (Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947) p. 272). (Received October 11, 1948.)

87. S. F. Neustadter: *Multiple-valued harmonic functions with circle as branch-curve.*

The three-dimensional Euclidean space is referred to toroidal coordinates. A multiple-leaved Riemann space with the fundamental circle of the coordinate system as branch curve is obtained by joining a finite or infinite number of ordinary spaces along the disk spanned by the fundamental circle. One is concerned with the multiple-leaved domain $T$ of which the branch-circle is the sole boundary. There is obtained in the form of an integral the solution of Laplace's equation which has an ordinary pole at some point of $T$, continuous derivatives everywhere else in $T$, and vanishes at infinity in all the leaves of $T$. It is shown that if a point on the branch-circle is approached, the function which results for the finitely-leaved case approaches one $n$th of the reciprocal of the distance from the pole to the point on the branch-circle. In the infinitely-leaved case this limit value is zero. Analogous results hold for the limits of the derivative of the function with respect to the toroidal coordinates, while the directional derivatives are shown to be unbounded near the branch-circle. (Received October 12, 1948.)

88. A. E. Taylor: *Spectral theory for closed distributive operators.*

Let $T$ be a closed distributive operator with domain and range in a complex Banach space $X$. Let the resolvent set $\rho(T)$ be nonempty. Let $G(T)$ be the class of complex functions $f(\lambda)$, each analytic in some domain containing the spectrum $\sigma(T)$ and a neighborhood of $\lambda=\infty$; it is assumed that $f$ is regular at $\lambda=\infty$. Define $f(T) = f(\infty)I+(1/2\pi i)\int f(\lambda)(\lambda I-T)^{-1}d\lambda$, the integration being over the boundary of a certain region containing $\sigma(T)$ and a neighborhood of $\lambda=\infty$. The correspondence $f(\lambda)\rightarrow f(T)$ defines a homomorphism of the ring $G(T)$ into the ring $[X]$ of bounded linear operators on $X$ into itself. Polynomials in $T$ are defined in the natural way. If $P(\lambda)$ is a polynomial of degree $n$ and if $f\in G(T)$ has a zero of order at least $n$ at $\lambda=\infty$, the operator corresponding to $P(\lambda)f(\lambda)$ is $P(T)f(T)$. The operational calculus and spectral theory founded in this way include the work of Dunford on bounded operators (Trans. Amer. Math. Soc. vol. 54 (1943) pp. 185–217) and afford a uniform development of the theory in which the role of boundedness at particular places is made clear. Various spectral criteria for $T$ to be in $[X]$ are obtained. If $T\notin[X]$, it is proved that there exists no polynomial $P(\lambda)\neq 0$ for which $P(T)$ is the 0 operator on its domain. (Received October 16, 1948.)

89. A. L. Whiteman: *Theorems analogous to Jacobstahl's theorem.*

For any prime $p$ of the form $8n+1$ the diophantine equation $p=x^2+2y^2$ has a unique solution in $x$ and $y$ except for signs. It is proved in this paper that $x=4^{-1}\sum_{m=1}^{p-1}(m/p)((m^4+1)/p)$, where $(m/p)$ denotes the Legendre symbol. From this result it follows that the number of solutions of the congruence $y^2\equiv x^5+x \pmod{p}$ ($p\equiv 1 \pmod{8}$) is $p+O(p^{1/2})$. The method is based upon the theory of the division of the circle and is used to obtain similar formulas for primes of other forms. In particular, the analogous theorems of E. Jacobstahl (J. Reine Angew. Math. vol. 132 (1907) pp. 238–245) and S. Chowla (Proceedings of the Lahore Philosophical Society vol. 7

(1945) 2 pp.) for primes of the forms $4n+1$ and $3n+1$, respectively, are derived. (Received September 28, 1948.)

## Applied Mathematics

### 90. H. W. Becker: *The symmetric reciprocity theorem.*

A source and receiver of any internal impedances $s$ and $z$ may be bodily interchanged in a network $\theta$ without change of current through $z$, if they are in the same summand of the branch symmetry partition, and both impedances are the same in each pair of branches permuted by the transformation $\theta_{-+} \to \theta_{+-}$. The latter are the here topologically identical networks to which $\theta$ reduces, when $s$ is shorted and $z$ opened, and vice versa. Moreover the same equalities (graphically deduced, with no time lost in any calculations whatever) yield the same interchangeability in the dual network ${}^{-}\theta$, since ${}^{-}(\theta_{\mp\pm}) = ({}^{-}\theta)_{\pm\mp}$. If $s$ and $z$ are not symmetrical in $\theta$, $\theta_{-+} \neq \theta_{+-}$ topologically; and the conditions for their algebraic equality, by adjustment of any one branch between $s$ and $z$, are neither simple, dual, nor necessarily realizable. (Received October 16, 1948.)

### 91. R. S. Phillips and Henry Malin. *A helical wave guide.* Preliminary report.

This paper is concerned with an idealized helical wave guide. This guide consists of a circular cylindrical surface which has perfect conductivity in a given helical direction and zero conductivity in the direction normal to this on the surface. For each $n$ ($n=0, \pm 1, \pm 2, \cdots$) there is a general solution of the wave equation in cylindrical coordinates involving $n$th order Bessel functions. For each such solution there are precisely as many non-attenuated modes as there are real and imaginary solutions in $v$ of the following characteristic equation: $\delta/\alpha = -(n/v^2)[1+(v/\alpha)^2]^{1/2} \pm \{-[I_n'(v)K_n'(v)]/[v^2 I_n(v)K_n(v)]\}^{1/2}$. It is found that there are no imaginary solutions. The paper is thus devoted to a study of the real solutions of this equation for all possible $\alpha$ and $n$ and their relations to one another. The results are obtained from a study of certain ordinary differential equations of the Riccati type. In the course of the study numerous approximations for and inequalities between Bessel functions are found. (Received October 20, 1948.)

### 92*t*. Edmund Pinney: *Elastic waves from a point source within a sem-infinite elastic solid.*

A solution is given to the problem of harmonic elastic waves originating at a point source in the interior of a semi-infinite elastic solid. (Received October 18, 1948.)

### 93*t*. C. J. Thorne: *Symmetrically loaded rectangular plates fixed at points.*

The work begun in the previous paper *Square plates fixed at points* (Journal of Applied Mechanics (1948)) is extended to include additional design data for the square plate and similar results for plates with side ratios 2:1 and 3:1. Deflection functions for thin plates with symmetric loads are given as a sum of biharmonic polynomials with coefficients determined by the slopes and deflections at equally spaced points on each edge of the plate for what are called full fixed and not full fixed edge conditions. Results and design data are then given numerically and graphically for a center point load and a uniform load. (Received October 18, 1948.)

## Geometry

### 94. Alfred Horn: *Some generalizations of Helly's theorem on convex sets.*

Let $S_n$ be the surface of the unit sphere $x_1^2 + \cdots + x_n^2 = 1$ in $n$-dimensional space $R_n$. A subset $A$ of $S_n$ is convex if whenever $a$ and $b$ are linearly independent points of $A$, then either the minor or the major arc of the great circle through $a$ and $b$ lies in $A$. Let $S_p$ denote the intersection of $S_n$ with a $p$-dimensional plane through the origin. Theorem: Let $F$ be a family of closed convex subsets of $S_n$ which is such that every $p$ members of $F$ have a point in common. Then through every $S_{n-p}$ there passes an $S_{n-p+1}$ which intersects every member of $F$, $1 \leqq p \leqq n$. From this can be deduced the following theorem concerning convex sets in the ordinary sense: Let $F$ be a family of compact convex subsets of $R_n$ which is such that every $p$ members of $F$ have a point in common. Then through every $(n-p)$-dimensional plane, there passes an $(n-p+1)$-dimensional plane which intersects every member of $F$, $1 \leqq p \leqq n$. The case $p = n+1$ is Helly's theorem, if a $-1$-dimensional plane is defined as the empty set. (Received May 17, 1948.)

## Logic and Foundations

### 95*t*. Julia Robinson: *General recursive functions.*

A function will be called general recursive if it can be obtained from the constant, identity and successor functions by repeated use of substitution, primitive recursion and the $\mu$-rule. Functions obtained without using the $\mu$-rule are called primitive recursive. The $\mu$-rule has the form $F(x_1, \cdots, x_n) = \mu y\{A(x_1, \cdots, x_n, y) = 0\}$, where the expression on the right denotes the smallest $y$ such that the condition in braces holds, it being assumed that for every $x_1, \cdots, x_n$ there exists a $y$ satisfying this condition. The purpose of this paper is to give a mathematical treatment of general recursive functions; part of the results obtained were previously established metamathematically by Kleene. In particular, the question whether all general recursive functions are obtained by restricted operations is discussed. One result is that the recursions may be omitted, if suitable functions are adjoined to the initial functions. On the other hand, using primitive recursions, a single application of the $\mu$-rule is sufficient. It is also shown that all general recursive functions of one variable may be obtained without using functions of more than one variable, by starting with just two primitive recursive functions and using only substitution and the $\mu$-rule in the form $F(x) = \mu y\{A(y) = x\}$. (Received October 11, 1948.)

### 96. A. C. Sugar: *A rational reconstruction of dimensional analysis.* Preliminary report.

A dimensional number is defined as a class $(N, P)$ where $N$ is a real number and the dimensional part $P$ is the class $((D_1, \alpha_1), \cdots, (D_n, \alpha_n))$, $D_i \neq D_j$, when $i \neq j$; the $D_i$ are dimensions, and the $\alpha_i$ rational numbers. In the usual interpretation $D_i$ is a standard physical or geometric object such as the standard meter bar, the standard gram, or a radian. As a consequence of this definition, the so-called dimensionless quantity can be regarded as a special kind of dimensional number, that is, $(N, (D_1, 0), \cdots, (D_n, 0))$. The set $(D_1, \cdots, D_n)$, $D_i \neq D_j$, when $i \neq j$, is called a dimensional system of order $n$. The order of a system is fixed in a given analysis. Using these concepts and the definitions of multiplication and division of dimensional numbers it is possible to analyze the fundamental functions of the physical sciences. Three typical examples of fundamental physical functions are length, $L$, mass, $M$, and acceleration, $A$. A fundamental physical function will have a dimensional number for its value and may have several arguments one of which may be a physical object. Consider, for example, the mass function, $M$. Here, the value of the function, $M(Q)$, is a dimensional number and the argument, $Q$, is a physical object. Included is a logical critique of the methods of obtaining physical laws by dimensional analysis. (Received October 6, 1948.)

## Numerical and Graphical Methods

### 97t. C. J. Thorne: *A table of biharmonic polynomials.*

The polynomials $z^n/n!$ and $\bar{z}(z^n/n!)$ satisfy $\nabla^4 W=0$. The values of the first twenty-five such polynomials both real and imaginary are given at intervals of .05 for $y=0$, $0\leqq x\leqq 1$; $x=1$, $0\leqq y\leqq 1$; $x=y$, $0\leqq y\leqq 1$; $x=2y$, $0\leqq y\leqq 1/2$; $x=3y$, $0\leqq y\leqq 1/3$; $y=1/2$, $0\leqq x\leqq 1$; $y=1/3$, $0\leqq x\leqq 1$. The partial derivatives to the third order are given for the real parts of $\bar{z}(z^n/n!)$. The construction of the tables and their use is discussed. (Received October 18, 1948.)

## Statistics and Probability

### 98t. A. L. Whiteman. *Distribution theory of unrestricted runs.*

A run is a sequence of like elements preceded and succeeded by different elements. An unrestricted run is a sequence of like elements without regard to the elements which precede or follow the sequence. The number of elements in a run is its length. Consider a random arrangement of $n_1$ $a$'s and $n_2$ $b$'s with $n_1+n_2=n$. Let $r_{1j}$ denote the number of runs of $a$'s of length $j$ and $r_{2j}$ denote the number of runs of $b$'s of length $j$. Let $r_1$ and $r_2$ denote the total number of runs of $a$'s and $b$'s, respectively. In a similar manner, define $u_{1j}$, $u_{2j}$, $u_1$ and $u_2$ for unrestricted runs. Typical theorems are the following. The expected value and variance of $u_{13}$ are given by the following expressions: $E(u_{13})=n_1(n_1-1)(n_1-2)/n(n-1)$ and $\sigma^2(u_{13})=n_1(n_1-1)(n_1-2)n_2(n_2+1)\cdot(5n^2-4n_2n-19n+6n_2+12)/n^2(n-1)^2(n-2)(n-3)$. When $r_1$ is given, the expected value and variance of $u_{13}$ are given by the following expressions: $E(u_{13})=(n_1-r_1)(n_1-r_1-1)/(n_1-1)$ and $\sigma^2(u_{13})=(n_1-r_1)(n_1-r_1-1)r_1(r_1-1)/(n_1-1)^2(n_1-2)$. New derivations of some theorems of A. M. Mood (*The distribution theory of runs*, Ann. Math. Statist. vol. 11 (1940) pp. 367–392) are obtained by expressing the $r$'s as linear functions of the $u$'s. (Received May 4, 1948.)

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