A. KUROSCH
1. *Theory of groups* (in Russian), Moscow, 1944.
MARSTON MORSE AND G. A. HEDLUND
1. *Symbolic dynamics*, Amer. J. Math. vol. 60 (1938) p. 815.
G. T. WHYBURN
1. *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, New York, 1942.

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A ZERO-DIMENSIONAL TOPOLOGICAL GROUP WITH A ONE-DIMENSIONAL FACTOR GROUP

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As can be easily shown, if a locally compact topological group is zero-dimensional, all of its factor groups are zero-dimensional. In this note we give an example of a non locally compact zero-dimensional group with a factor group which is topologically isomorphic to the real numbers, hence one-dimensional.[1]

1. **Preliminaries.** Let $\{\lambda\}$ be a set of indices of cardinality c, and for each λ, let R_λ be a topological isomorph of the additive group of rational numbers. We form the weak product R of the R_λ: an element r of R is a collection $r=\{r_\lambda\}$, $r_\lambda \in R_\lambda$, such that for only a finite number of the λ's is $r_\lambda \neq 0_\lambda$. Under the definitions $r+r'=\{r_\lambda+r'_\lambda\}$, $0=\{0_\lambda\}$, R forms a group.

Now for each $r\in R$, we define $\|r\| = \sum_\lambda |r_\lambda|$. Since all but a finite number of the $r_\lambda = 0_\lambda$, this sum exists. Clearly $\|r+r'\| \leqq \|r\| + \|r'\|$, and $\|-r\| = \|r\|$, hence, as can be easily shown, $\|r\|$ defines a metric in R under the definition: the distance from r to r' is $\|r-r'\|$.

LEMMA 1. *Let $\{d_\lambda\}$ be a set of positive real numbers bounded away from zero, that is, there exists $d>0$ such thal $d_\lambda \geqq d$ for all λ. Then*

$$U = \left\{ r \;\middle|\; \sum_\lambda \left| \frac{r_\lambda}{d_\lambda} \right| < 1 \right\}$$

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[1] Cf. Bourbaki, *Topologie generale*, chap. III, p. 21, exercise 12, for an example of a totally disconnected group with a factor group topologically isomorphic to the reals. This example was pointed out to me by I. Kaplansky.

is an open set containing the origin, and

$$\bar{U} \subset \left\{ r \,\middle|\, \sum_{\lambda} \left| \frac{r_\lambda}{d_\lambda} \right| \leqq 1 \right\}.$$

PROOF. We need only prove that the real valued function $f(r) = \sum_\lambda |r_\lambda/d_\lambda|$ is continuous. Given $r = \{r_\lambda\}$ and $\epsilon > 0$, choose $\delta < d\epsilon$. Now, for any $r' = \{r_\lambda'\}$,

$$\left| \sum_\lambda \left| \frac{r_\lambda'}{d_\lambda} \right| - \sum_\lambda \left| \frac{r_\lambda}{d_\lambda} \right| \right| = \left| \sum_\lambda \frac{|r_\lambda'| - |r_\lambda|}{d_\lambda} \right| \leqq \sum_\lambda \left| \frac{|r_\lambda'| - |r_\lambda|}{d_\lambda} \right|$$

$$\leqq \frac{1}{d} \sum_\lambda \big| |r_\lambda'| - |r_\lambda| \big|$$

$$\leqq \frac{1}{d} \sum_\lambda |r_\lambda' - r_\lambda|$$

$$= \frac{1}{d} \|r' - r\|.$$

Thus, if $\|r' - r\| < \delta$, this last expression is less than ϵ, which proves the continuity and hence the lemma.

LEMMA 2. *Let* $\{\alpha_\lambda\}$ *be a bounded set of positive irrational numbers linearly independent with respect to the rationals, that is*:

$$(1) \qquad a_1\alpha_{\lambda(1)} + \cdots + a_h\alpha_{\lambda(h)} = a \qquad (a_1, \cdots, a_h, a \text{ rational})$$

implies

$$a_1 = \cdots = a_h = a = 0.$$

Then if, in Lemma 1, *we take* $d_\lambda = 1/\alpha_\lambda$, *the resulting* U *has a vacuous boundary.*

PROOF. From the second part of Lemma 1, we need only prove that there is no r such that $\sum_\lambda |r_\lambda/d_\lambda| = 1$. Assume there is. Then, since replacing an r_λ by its negative does not change absolute values, we can assume all the r_λ are non-negative. Then we have $\sum_\lambda r_\lambda/d_\lambda = 1$. But each $d_\lambda = 1/\alpha_\lambda$, hence $\sum_\lambda r_\lambda\alpha_\lambda = 1$, which contradicts the hypothesis on the α_λ's.

Using Lemma 2, we now define a special sequence of neighborhoods $\{U_n\}$ $(n = 0, 1, \cdots)$ which form a basis around the origin. We first take the set of real numbers $1/2 \leqq c < 1$ and set them in one-one correspondence with the λ's: $\{c_\lambda\}$. (Our purpose in bringing these in will become clear in §2.) We then choose a set of irrational numbers $\{\alpha_\lambda\}$ with the property (1) (the existence of such a set follows from the existence of a Hamel basis for the reals), and such that for each λ, $c_\lambda < \alpha_\lambda < 1$. This last can always be accomplished by multiplying α_λ by a suitable rational. We then have

$$(2) \qquad 1/2 \leqq c_\lambda < \alpha_\lambda < 1 \qquad \text{for all } \lambda.$$

Now for each n, we define U_n by taking

$$(3) \qquad d_\lambda^{(n)} = \frac{1}{2^n} \cdot \frac{1}{\alpha_\lambda},$$

and letting

$$(4) \qquad U_n = \left\{ r \,\middle|\, \sum_\lambda \left| \frac{r_\lambda}{d_\lambda^{(n)}} \right| < 1 \right\}.$$

Since, for $r \in U_n$,

$$\|r\| = \sum_\lambda |r_\lambda| = \sum_\lambda d_\lambda^{(n)} \left| \frac{r_\lambda}{d_\lambda^{(n)}} \right| \leqq \frac{2}{2^n} \sum_\lambda \left| \frac{r_\lambda}{d_\lambda^{(n)}} \right| < \frac{1}{2^{n-1}},$$

the diameter of U_n is less than $1/2^{n-2}$, hence approaches zero as n goes to infinity. Thus $\{U_n\}$ constitutes a basis around the origin. Since, from Lemma 2, the boundary of each U_n is vacuous, it follows that R is zero-dimensional.

2. **The example.** Let R_* be the additive group of real numbers $\{r_*\}$ with distance defined by $\|r_*\| = 1$ for all r_* different from zero. This makes it discrete. Let $G = R_* \times R$ with distance defined as follows: If $g = (r_*, r)$ then $\|g\| = \|r_*\| + \|r\|$. Since R_* is discrete, the U_n's, now considered as subsets of G, form a basis around the origin of G, hence G is zero-dimensional.

We define the subgroup H of G as the set of all $g = (r_*, r)$ such that $r_* + \sum_\lambda c_\lambda r_\lambda = 0$ (cf. (2)).

LEMMA 3. *H is a closed subgroup of G.*

PROOF. H is a subgroup, for if $g, g' \in H$ then $r_* + \sum_\lambda c_\lambda r_\lambda = 0$ and $r'_* + \sum_\lambda c_\lambda r'_\lambda = 0$, hence $(r_* - r'_*) + \sum_\lambda c_\lambda (r_\lambda - r'_\lambda) = 0$. To prove H is closed, it is sufficient to show that the real-valued function $f(g) = r_* + \sum_\lambda c_\lambda r_\lambda$ is continuous. Given $g = (r_*, r)$ and $\epsilon > 0$, choose $\delta < \min(\epsilon, 1)$. Consider any $g' = (r'_*, r')$ such that $\|g' - g\| < \delta$. We note first that $r'_* = r_*$, for otherwise $\|r'_* - r_*\| = 1$ and hence $\|g' - g\| \geqq 1 \geqq \delta$. Then

$$\begin{aligned}\left|r_*' + \sum_\lambda c_\lambda r_\lambda' - r_* - \sum_\lambda c_\lambda r_\lambda\right| &= \left|\sum_\lambda c_\lambda r_\lambda' - \sum_\lambda c_\lambda r_\lambda\right| \\ &\leqq \sum_\lambda c_\lambda |r_\lambda' - r_\lambda| \\ &\leqq \sum |r_\lambda' - r_\lambda| \qquad \text{(from (2))} \\ &= \|g' - g\| < \delta < \epsilon.\end{aligned}$$

This proves the lemma.

LEMMA 4. *G/H is algebraically isomorphic to the real numbers.*

PROOF. Since $R_* \times 0 \subset G$ is algebraically isomorphic to the real numbers, it is sufficient to prove that G/H is algebraically isomorphic to this subgroup.

(i) Every coset of H contains an element of $R_* \times 0$.

For, let $g = (r_*, r)$ be any element of G, where $r = \{r_\lambda\}$. Then if $g' = (r_* + \sum_\lambda c_\lambda r_\lambda, 0)$, $g' - g = (\sum_\lambda c_\lambda r_\lambda, -r) \in H$. Since $g' \in R_* \times 0$, this proves (i).

(ii) Different elements of $R_* \times 0$ lie in different cosets.

For, let $g = (r_*, 0)$, $g' = (r_*', 0)$, with $r_* \neq r_*'$. Then $g' - g = (r_*' - r_*, 0)$, and since $(r_*' - r_*) + 0 \neq 0$, $g' - g \notin H$. This proves (ii), and with it, Lemma 4.

We can thus denote each element of G/H by a unique real number. From the proof of (i) above, we see that the real number is given by the mapping

$$\pi(r_*, r) = r_* + \sum_\lambda c_\lambda r_\lambda. \tag{5}$$

LEMMA 5. $\pi(U_n) = [-1/2^n < x < 1/2^n]$ *for all* $n = 0, 1, \cdots$.

PROOF. Since the argument is the same for all n, it is sufficient to prove this for U_0. From (4) and (3),

$$U_0 = \left\{ g = (0, r) \,\middle|\, \sum_\lambda |\alpha_\lambda r_\lambda| < 1 \right\}.$$

(i) $\pi(U_0) \subset [-1 < x < 1]$.

For, if $g = (0, r) \in U_0$, then from (5),

$$\begin{aligned}|\pi(g)| = \left|\sum_\lambda c_\lambda r_\lambda\right| &\leqq \sum_\lambda |c_\lambda r_\lambda| \\ &< \sum_\lambda |\alpha_\lambda r_\lambda| \qquad \text{(from (2))} \\ &< 1.\end{aligned}$$

(ii) $[-1<x<1]\subset\pi(U_0)$.

For consider any real number x such that $-1<x<1$. Since the c_λ's run through all the real numbers from 1/2 to 1, there is a $c_{\lambda'}$ such that

$$x = \epsilon c_{\lambda'},$$

where ϵ is one of the values ± 1, $\pm 1/2$. Hence, if g is the element of G whose λ'-coordinate is ϵ and whose remaining coordinates are 0, we have from (5) that

$$\pi(g) = c_{\lambda'}\epsilon = x.$$

Thus x has an inverse in U_0 under π. Since x was any element of $[-1<x<1]$, (ii) is proved. This establishes the lemma.

Since the set $\{U_n\}$ is a basis around the origin of G, the set $\{\pi U_n\}$ is by definition a basis around zero in G/H. Hence, from Lemma 5, G/H has the topology of the real numbers.

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