SET FUNCTIONS AND SOUSLIN'S HYPOTHESIS

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1. Introduction. It is known[1] that Souslin's hypothesis[2] is *implied* by the existence of a nontrivial outer measure on every field of sets satisfying certain conditions. We shall here prove that Souslin's hypothesis is *equivalent* to the existence, on a wide class of fields of sets, of set-functions of a certain type. The axiom of choice is assumed, but not the continuum hypothesis.

Instead of working with fields of sets, it is more convenient to use the equivalent notion of a (finitely additive) Boolean algebra, E.[3] We say that $x, y \in E$ are *disjoint* if $x \wedge y = o$, and that they *intersect* otherwise. A set S of elements of E will be called a *Souslin system* if it satisfies the following three postulates:

(1) $S \not\ni o$, and whenever $s, s' \in S$, then either $s \wedge s' = o$, or $s \geqq s'$, or $s' \geqq s$.

(2) If $A \subset S$ consists of pairwise disjoint elements only, then A is(at most) countable.

(3) If $A \subset S$ is such that every two of its elements intersect, then A is countable.

Souslin's hypothesis is known to be equivalent to the assertion that every Souslin system is countable.[4]

THEOREM. *Souslin's hypothesis is true if and only if there exists, on each non-atomic Boolean algebra E satisfying the countable chain condition, a real-valued function f such that* (i) $x \geqq y \rightarrow f(x) \geqq f(y)$, (ii) $f(x) = 0 \leftrightarrow x = o$, *and* (iii) *given* $x \in E - (o)$ *and* $\epsilon > 0$, *there exists* $y \in E - (o)$ such that $y < x$ and $f(y) < \epsilon$.

2. **"If."** Suppose an uncountable Souslin system exists. Then, as easily follows from [2, §7], there exists a complete Boolean algebra E, satisfying the countable chain condition, and an uncountable Souslin system $S \subset E$ having the following additional properties:

(4) $S = \bigcup S_\alpha$, where α ranges over all countable ordinals, and the elements of each S_α are pairwise disjoint.

(5) If $\alpha < \beta$, then for each $s_\beta \in S_\beta$ there exists an s_α $(\in S_\alpha)$ such that $s_\alpha > s_\beta$.

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[1] See [2]; in the case of a measure, the result is due to K. Gödel. Numbers in brackets refer to the bibliography at the end of the paper.

[2] Souslin, Fund. Math. vol. 1 (1920) p. 223.

[3] See [2] for notations, and so on.

[4] This follows from [3], together with some results in [1].

(6) If $\alpha<\beta$, $s_\alpha=\bigvee\{s_\beta \mid s_\beta<s_\alpha\}$.

(7) For each $x\in E$, there exists an α such that

$$x = \bigvee\{s_\alpha \mid s_\alpha \leqq x\}.$$

(In the notation of [2], we have only to take $E=(D+_2N)/N$, and the elements s_α are the equivalence classes c_α mod N. The notation s_α is intended to imply that $s_\alpha\in S_\alpha$, and so on.)

Clearly E is non-atomic, so by hypothesis there exists on E a real-valued function f having the properties (i)–(iii) of the theorem.

Now, for a given positive integer n, each $s\in S$ for which $f(s)<1/n$ is contained in a *maximal* such element of S, say $m(s, n)$; in fact, if $s=s_\beta$, we have that for each $\alpha\leqq\beta$ there is a (unique) $s_\alpha\geqq s_\beta$, and we take $m(s, n)=s_\alpha$ for the smallest α for which $f(s_\alpha)<1/n$. Let M_n denote the set of all elements $m(s, n)$ (for fixed n); clearly the elements of M_n are pairwise disjoint, so that each M_n is countable. The desired contradiction is now obtained by showing that S is countable after all; and this will follow (in virtue of (3)) once it is established that:

(8) Each $s\in S$ is greater than or equal to some $m(s', n)$.

But, given s, we have from (ii) that $f(s)>1/n$ for some n. By (iii), there is an $x\in E$ such that $o<x<s$ and $f(x)<1/n$. From (7) there exists an $s'\in S$ such that $s'\leqq x$; hence $m(s', n)$ exists and is greater than or equal to s'. Now $m(s', n)$ and s are not disjoint (for both are greater than or equal to s'); and $m(s', n)\not> s$, for $f(s)>1/n>f(m(s', n))$. Hence $s\geqq m(s', n)$, by (1).

3. **"Only if."** Let E be a non-atomic Boolean algebra satisfying the countable chain condition. By Zorn's lemma (or transfinite induction) there exists a maximal subset $S\subset E$ satisfying (1); and the countable chain condition ensures that (2) and (3) hold also. Hence, by Souslin's hypothesis, S can be enumerated as $\{s_n\}$ ($n=1, 2, \cdots$ to ∞; it is easy to see that S is necessarily infinite), where for convenience we may suppose that the unit element e (which necessarily belongs to S) is s_1. We assert:

(9) Given s_k, there exists $s_m<s_k$.

For, since E is non-atomic, there exists $x\in E$ such that $o<x<s_k$. If $x\in S$, there is nothing to prove. If not, since S is maximal, there must be an s_m such that $s_m\wedge x\neq o$, and s_n and x are incomparable. But then s_m intersects s_k, so either $s_k\leqq s_m$—which implies $x\leqq s_m$ and so is excluded—or $s_k>s_m$, q.e.d.

Let ϵ_i denote either 1 or -1, and write $\epsilon_i s_i$ to denote s_i if $\epsilon_i=1$, and the complement $-s_i$ if $\epsilon_i=-1$. For each finite sequence, $\epsilon_1\epsilon_2\cdots\epsilon_n$ of ± 1's, we write $\bigwedge_1^n(\epsilon_i s_i)=t(\epsilon_1\epsilon_2\cdots\epsilon_n)$.

Now let $\{t^i\}$ be any infinite sequence of elements $t^i = t(\epsilon_1^i\epsilon_2^i \cdots \epsilon_{n(i)}^i)$ such that $1 \leqq n(1) < n(2) < \cdots$. We shall show that:

(10) If $a \in E$ is such that a is less than or equal to each t^i, then $a = o$.

For suppose $a \neq o$. Then clearly ϵ_j^i is independent of i (provided only that $n(i) \geqq j$), so that we may write $\epsilon_j^i = \epsilon_j$, and have that, for every j,

$$\epsilon_j s_j \geqq a. \tag{11}$$

Hence $a \in S$ (else a could be adjoined to S without violating (1), and S is maximal); say $a = s_k$. From (9), $s_k > s_m$ for some m. But (11) gives $\epsilon_m s_m \geqq s_k$—a contradiction.

Now, given any $x \in E - (o)$, (10) shows that there will be a greatest n, say $n(x)$, for which there exists an element $t(\epsilon_1\epsilon_2 \cdots \epsilon_n) \geqq x$. (Note that $t(1) = e$, so that $n(x)$ is always defined.) We put $f(x) = 1/n(x)$, and complete the definition by setting $f(o) = 0$. Properties (i) and (ii) are immediate. To verify (iii), suppose that $x \in E - (o)$ and $\epsilon > 0$ are given. Choose $n > \max(1/\epsilon, 1/f(x))$, and consider the 2^n (not necessarily distinct) elements $t(\epsilon_1\epsilon_2 \cdots \epsilon_n)$ for all possible choices of $\epsilon_i = \pm 1$. The $\bigvee$ of these elements is e, so at least one of them, say t, intersects x. Write $y = t \wedge x$; thus $o < y \leqq x$, and $n(y) \geqq n$, so that $f(y) < \min(\epsilon, f(x))$, and (iii) is established.

4. **Further remarks.** (a) Let E be a non-atomic Boolean algebra satisfying the countable chain condition. It does *not* follow that the existence of a function f on E *alone*, satisfying conditions (i)–(iii) of the theorem, implies that every Souslin system *in* E is countable. In fact, this is *false*—unless Souslin's hypothesis is true. For if Souslin's hypothesis is false, there will be a Boolean algebra E_1 satisfying our conditions and containing an uncountable Souslin system S. (Cf. §2.) Let E_2 be (say) the algebra of measurable sets modulo null sets on the unit interval. We can regard E_1 and E_2 as the algebras of open-closed subsets of their respective representation spaces R_1 and R_2. The "product" algebra $E = E_1 \times E_2$ can now be defined to consist of all finite unions of open-closed "rectangles" in the topological product $R_1 \times R_2$. For each $x \in E$, say $x = \bigcup_1^n (x_1^i \times x_2^i)$ $(x_2^i \neq o)$ we define $f(x)$ = measure of $\bigcup x_1^i$. It is easy to see that E and f fulfil all the requirements. Yet E contains an uncountable Souslin system S^*, formed by the cylinder sets on S.

It can however be shown that the countability of every Souslin system in E (where E is, as hitherto, non-atomic and satisfies the countable chain condition) is equivalent to the existence, *for every non-atomic subalgebra* F *of* E, of a function f (depending on F, in general), defined on F and satisfying postulates (i)–(iii) for F. In one direction this is an immediate consequence of §3; the proof of the other implication, while using the same ideas as §2, is more complicated.

(b) The arguments of §§2 and 3 also readily give purely algebraic properties equivalent to Souslin's hypothesis. We have, for example:

Souslin's hypothesis is true if and only if each non-atomic Boolean σ-algebra satisfying the countable chain condition contains a double sequence of elements t_{ni} such that $(\alpha)\bigvee_i t_{ni}=e$ *and* (β) *for every function* $i(n)$ *of* n, $\bigwedge_n t_{ni(n)}=o$.

Bibliography

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