THE MULTIPLICATIVE COMPLETION OF SETS OF FUNCTIONS

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1. Introduction. A set $\{f_n(x)\}_1^\infty$ of functions of $L^2(a, b)$, where (a, b) is finite or infinite, is called complete if $g(x) \in L^2$ and $\int_a^b f_n(x)g(x)dx = 0$, $n = 1, 2, \cdots$, imply that $g(x) = 0$ almost everywhere on (a, b); a well known equivalent property ("closure") is that every element of L^2 can be approximated in the L^2 metric by finite linear combinations of the $f_n(x)$.

Suppose that $\{f_n(x)\}$ is not complete. It will sometimes be possible to find a function $m(x)$ such that the set $\{m(x)f_n(x)\}$ is complete. This can also be considered as completeness after a change of weight function or a change of measure; but we shall not attempt to consider the most general change of measure here. We give some results on when a set can or cannot be completed by multiplication; the problem of finding necessary and sufficient conditions is left open.

We first state our results.

THEOREM 1. *If $\{f_n(x)\}_1^\infty$ is an orthonormal set which is not complete, but can be completed by the addition of a finite number of functions to the set, then there is a bounded measurable function $m(x)$ such that $\{m(x)f_n(x)\}_1^\infty$ is complete.*

The condition of Theorem 1, while necessary, is not sufficient, as Theorem 2 shows.

THEOREM 2. *The orthogonal set $\{e^{-x/2}L_{2n}(x)\}_0^\infty$, where $L_{2n}(x)$ is the 2nth Laguerre polynomial, cannot be completed on $(0, \infty)$ by the addition of a finite number of functions, but is completed on multiplication by $m(x) = e^{-x/2}$.*

Our next three theorems give examples of sets which cannot be completed by multiplication.

THEOREM 3. *A set of even functions cannot be completed by multiplication by an integrable function in any interval containing* 0.

THEOREM 4. *The set $\{e^{2inx}\}_{-\infty}^{\infty}$ cannot be completed in $(-\pi, \pi)$ by multiplication by an integrable function.*

THEOREM 5. *The set $\{x^{\lambda_n}\}$, where $\lambda_n > 0$, $\sum 1/\lambda_n < \infty$, cannot be completed in any interval by multiplication by a continuous function.*

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2. **Proof of Theorem 1.** For simplicity we suppose that the functions are real. We first consider the case where the set has deficiency 1, that is, can be completed by the addition of one function f_0. We may suppose that f_0 is orthogonal to f_n, $n=1, 2, \cdots$; for otherwise we could replace f_0 by $f_0 - \sum_1^\infty f_n(x)\int_a^b f_0(t)f_n(t)dt$, where $\sum$ implies convergence in L^2, that is, convergence in mean square. Suppose that $m(x)$ is measurable, bounded, never 0, but such that $f_0(x)/m(x)$ does not belong to L^2 on (a, b); then if

$$\int_a^b m(x)g(x)f_n(x)dx = 0, \qquad n = 1, 2, \cdots, \tag{1}$$

with $g(x) \in L^2$, we have, since $\{f_n(x)\}_0^\infty$ is orthogonal and complete, $m(x)g(x) = cf_0(x)$ almost everywhere for some constant c. Then c must be 0, since otherwise $g(x) = cf_0(x)/m(x)$ would not belong to L^2 on (a, b); that is, since $m(x)$ is never 0, $g(x)=0$ almost everywhere. In other words, the set $\{m(x)f_n(x)\}_1^\infty$ is complete.

It remains to construct $m(x)$. Let E be a bounded set of positive measure on which $|f_0(x)| \geqq \epsilon > 0$; choose $m(x)$ on E so that $m(x)$ is bounded and measurable and $1/m(x)$ is never ∞ but does not belong to L^2 on E; let $m(x)=1$ elsewhere. This function has the desired properties.

We now consider the general case. Here there are k functions $f_n(x)$, $n=0, -1, -2, \cdots, -k+1$, such that $\{f_n(x)\}_{-k+1}^\infty$ is complete. We may again suppose that $\{f_n(x)\}_{-k+1}^\infty$ is an orthogonal set. It is enough to construct a bounded measurable $m(x)$, never 0, such that $\{m(x)\}^{-1}\sum_{-k+1}^0 a_jf_j(x)$ belongs to L^2 only if all the a_j are zero. For, if (1) is true,

$$m(x)g(x) = \sum_{j=-k+1}^{0} a_jf_j(x), \tag{2}$$

and unless all the a_j are zero, (2) contradicts the fact that $g(x)$ belongs to L^2.

We now construct $m(x)$. Let E_0 be a bounded set of positive measure on which $f_0(x) \neq 0$. Construct a bounded measurable $m_0(x)$, never 0, such that $f_0(x)/m_0(x)$ does not belong to L^2 on E_0 and $m_0(x)=1$ outside. We now proceed by induction. Suppose that $0 \leqq n < k-1$ and that we have determined a bounded measurable $m_n(x)$ and a bounded set E_n of positive measure such that $\{m_n(x)\}^{-1}\sum_{-n}^0 a_jf_j(x)$ belongs to $L^2(a, b)$ only if all a_j are zero. Consider all linear combinations $F = a_0f_0 + a_1f_{-1} + \cdots + a_{-n}f_{-n} + f_{-n-1}$. At most one function $F(x)/m_n(x)$ can belong to $L^2(a, b)$, since, if F_1/m_n and F_2/m_n both did, their difference would not involve f_{-n-1} and so could not belong to L^2 by the induction hypothesis. If no $F(x)/m_n(x)$ belongs to L^2, take $E_{n+1}=E_n$, $m_{n+1}(x)=m_n(x)$. If some $F(x)/m_n(x)$ belongs to L^2, $|F(x)|>0$ on some bounded measurable set E^* of positive measure; by making E^* smaller if necessary, we may suppose that either (a) E^* is disjoint from E_n or (b) E^* is a subset of E_n. In case (a), let $E_{n+1}=E_n+E^*$; in case (b) let $E_{n+1}=E^*$. Define a bounded measurable $m_{n+1}(x)$ so that $m_{n+1}(x)\leqq m_n(x)$ and so that $F(x)/m_{n+1}(x)$ does not belong to L^2 on E_{n+1}. Then $m_{n+1}(x)$ has the properties of $m_n(x)$, since $m_n(x)\leqq m_{n+1}(x)$, and, in addition, $\{m_{n+1}(x)\}^{-1}\sum_{-n-1}^{0} a_jf_j(x)$ does not belong to L^2 unless all the a_j are zero. This completes the induction; take $m(x)=m_{k-1}(x)$.

3. **Proof of Theorem 2.** (a) The functions $e^{-t/2}L_{2n+1}(t)$ are orthogonal to all $e^{-t/2}L_{2n}(t)$; hence no finite set of additional functions will complete the latter set.

(b) Suppose that $g(t)\in L^2(0,\ \infty)$ and $\int_0^\infty e^{-t}g(t)L_{2n}(t)dt=0$, $n=0,1,2,\cdots$. That is,

$$\sum_{k=0}^{2n} C_{2n,k}\frac{(-1)^k}{k!}\int_0^\infty e^{-t}g(t)t^k dt = 0,$$

or $\Delta^{2n}\mu_0=0$, where

$$\mu_k = (1/k!)\int_0^\infty t^k e^{-t}g(t)dt.$$

Also,

$$|\mu_k| \leqq \frac{\{(2k)!\}^{1/2}}{2^{k+1/2}k!}\left\{\int_0^\infty g^2(t)dt\right\}^{1/2} \leqq \text{constant};$$

that is, $\{\mu_k\}$ is bounded. By a theorem of Agnew [**1**][1] and Fuchs [**2**], [**3**], $\mu_k=0$, $k=0,1,2,\cdots$. Hence [**4**, p. 20] $g(t)=0$ almost everywhere.

4. **Proof of Theorem 3.** Let the functions $f_n(x)$ be even and belong to $L^2(-a,a)$. We shall show that for every integrable $m(x)$ there is a bounded $g(x)$, not almost everywhere zero, such that

$$(3)\qquad \int_{-a}^{a} g(x)m(x)f_n(x)dx = 0, \qquad n=1,2,\cdots.$$

[1] Numbers in brackets refer to the references at the end of the paper.

This is trivial if $m(x)=0$ on some set of positive measure. Otherwise, since $m(x)$ is integrable, there is a set E of positive measure in $(0, a)$ on which $m(x)$ is bounded and not zero. Let E_1 be the symmetric set in $(-a, 0)$. E_1 contains a subset E_2 of positive measure on which $m(x)$ is bounded and not zero. Let E_3 be the symmetric set in $(0, a)$. Let $g(x)=m(-x)$ in E_2, $g(x)=-m(-x)$ in E_3, $g(x)=0$ elsewhere. Then $m(x)g(x)$ is odd and $f_n(x)$ is even, so (3) follows.

5. **Proof of Theorem 4.** As in §4, if $m(x)$ is integrable we have to find $g(x)$, not almost everywhere zero, such that

$$\int_{-\pi}^{\pi} g(x)m(x)e^{2inx}dx = 0, \qquad n = 0, \pm 1, \pm 2, \cdots .$$

This is trivial if $m(x)=0$ on some set of positive measure. Otherwise, we can find E, of positive measure, in $(-\pi, 0)$, with $m(x)$ bounded and bounded from zero. The set E_1 obtained by adding π to every point of E is a subset of $(0, \pi)$; it contains a subset E_2 of positive measure on which $m(x)$ is bounded and bounded from zero. Let E_3 be the set obtained by subtracting π from every point of E_2 and let $E_4=E_2+E_3$. Let $f(x)=1/m(x)$ in E_4, $f(x)=0$ elsewhere. Then $f(x)m(x)$ has period π and so is orthogonal to $e^{(2n+1)ix}$, $n=0, \pm 1, \pm 2, \cdots$, and is different from 0 on a set of positive measure.

The same argument shows that $\{e^{2inx}\}$ cannot be completed by multiplication on any interval of length exceeding π.

6. **Proof of Theorem 5.** It is known that the L^2 span of $\{x^{\lambda_n}\}$, $\sum 1/\lambda_n < \infty$, on any interval not containing 0, contains only functions analytic in that interval [5]. Suppose that $\{m(x)x^{\lambda_n}\}$ were complete, hence closed, with $m(x)$ continuous. Let I be an interval in which $|m(x)| > \epsilon > 0$. Then to every $f(x)$ of $L^2(I)$ and every positive δ there would exist constants a_k such that

$$\int_I \left| m(x) \sum a_k x^{\lambda_k} - f(x) \right|^2 dx < \delta\epsilon^2,$$

and hence

$$\int_I \left| \sum a_k x^{\lambda_k} - f(x)/m(x) \right|^2 dx < \delta.$$

Thus $f(x)/m(x)$ would be in the span of $\{x^{\lambda_n}\}$, but we can certainly find $f(x)$ of L^2 with $f(x)/m(x)$ not analytic in I.

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