NICHOLSON'S INTEGRAL FOR $J_n^2(z)+Y_n^2(z)$

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The integral in question is

$$J_n^2(z) + Y_n^2(z) = (8/\pi^2)\int_0^\infty K_0(2z \sinh t) \cosh 2nt dt, \tag{1}$$

and its validity for arbitrary complex n when the real part of z is positive is proved in [1, pp. 441–444][1] with the help of Hardy's theory of generalized integrals and integrations over contours in the complex plane. It is the purpose of this paper to give a much more elementary proof of (1).

We begin by observing [1, p. 146] that if $D=z(d/dz)$, then three linearly independent solutions of the equation

$$[D(D^2-4n^2)+4z^2(D+1)]y = 0 \tag{2}$$

are $J_n^2(z)$, $Y_n^2(z)$ and $J_n(z)Y_n(z)$. Equation (2) may be written as

$$z^2y''' + 3zy'' + (1-4n^2+4z^2)y' + 4zy = 0. \tag{3}$$

We shall now show that $y(z)=\int_0^\infty K_0(2z \sinh t) \cosh 2nt dt$ is a solution of (3). When the real part of z is positive it is clear that $K_0(2z \sinh t)$ is sufficiently small at ∞ to permit us to differentiate under the integral sign as many times as we please. Therefore,

$$y'(z) = 2\cdot\int_0^\infty \sinh t K_0'(2z \sinh t) \cosh 2nt dt. \tag{4}$$

If we make use of the differential equation

$$xK_0''(x) + K_0'(x) - xK_0(x) = 0 \tag{5}$$

satisfied by $K_0(x)$, then we find that

$$y'' = \int_0^\infty \{4 \sinh^2 t K_0(2z \sinh t) - 2z^{-1} \sinh t K_0'(2z \sinh t)\} \cosh 2nt dt,$$

$$y''' = \int_0^\infty \{(8 \sinh^3 t + 4z^{-2} \sinh t)K_0'(2z \sinh t) - 4z^{-1} \sinh^2 t K_0(2z \sinh t)\} \cosh 2nt dt.$$

It follows that

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[1] Numbers in brackets refer to the reference cited at the end of the paper.

$$
\begin{aligned}
&z^2y''' + 3zy'' + (1 + 4z^2)y' + 4zy \\
(6)\qquad &= \int_0^\infty \{4z^2 \sinh 2t \cosh tK_0'(2z \sinh t) \\
&\qquad + 4z \cosh 2tK_0(2z \sinh t)\} \cosh 2ntdt.
\end{aligned}
$$

If now (4) is integrated by parts and use is made of (5) we find that

$$4n^2y' = -4nz \int_0^\infty \sinh 2tK_0(2z \sinh t) \sinh 2ntdt,$$

whence another integration by parts shows that $4n^2y'$ is equal to the right-hand side of (6). Therefore $y(z)$ is a solution of (3). Consequently, there exist constants A, B, C such that

$$y(z) = AJ_n^2(z) + BY_n^2(z) + CJ_n(z)Y_n(z). \tag{7}$$

We shall now show that

$$\lim_{z=\infty} zy(z) = \lim_{z=\infty} \int_0^\infty zK_0(2z \sinh t) \cosh tdt = \frac{\pi}{4}, \tag{8}$$

the last equality being a consequence of the result [1, p. 388]

$$\int_0^\infty K_0(u)du = \frac{\pi}{2}\cdot$$

In (8), z is restricted to real values. In fact, the difference of the integrands in the limitands in (8) is

$$F(z, t) = zK_0(2z \sinh t)(\cosh 2nt - \cosh t).$$

Now $x^{1/2}e^xK_0(x)$ is bounded on $(0, \infty)$, so that

$$|F(z, t)| \leqq A_0(z \operatorname{csch} t)^{1/2}e^{-2z\sinh t}|\cosh 2nt - \cosh t|.$$

Moreover, $\operatorname{csch} t \leqq 1/t$ and the mean value theorem shows that

$$|\cosh 2nt - \cosh t| \leqq (2|n| + 1)t(\sinh 2|n|t + \sinh t),$$

whence we see that

$$|F(z, t)| \leqq A_1(zt)^{1/2}e^{-2z\sinh t}(\sinh 2|n|t + \sinh t).$$

We can suppose that $z \geqq 1$. Since $\sinh t \geqq t$ and $(zt)^{1/2}e^{-zt}$ is bounded, we find that

$$
\begin{aligned}
|F(z, t)| &\leqq A_2(\sinh 2|n|t + \sinh t)e^{-z\sinh t} \\
&\leqq A_2(\sinh 2|n|t + \sinh t)e^{-\sinh t}.
\end{aligned}
$$

Therefore, $F(z, t)$ converges dominatedly to zero as z approaches ∞, and this suffices to prove (8).

It is known [1, p. 199] that

$$J_n(z) = (2/\pi z)^{1/2} \cos\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O(z^{-3/2}),$$

$$Y_n(z) = (2/\pi z)^{1/2} \sin\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O(z^{-3/2}).$$

From (7) we conclude that

$$\frac{\pi z y(z)}{2} = A + (B - A) \sin^2\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) + \frac{C}{2} \sin\left(2z - n\pi - \frac{\pi}{2}\right) + O(z^{-1}).$$

This result is incompatible with (8) unless $A = \pi^2/8$, $B = A$, $C = 0$, and in this case $y(z) = (\pi^2/8)\{J_n^2(z) + Y_n^2(z)\}$. This completes the proof of (1).

Reference

1. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, 1945.

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