ON THE CONVERGENCE OF DOUBLE SERIES

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1. **Introduction.** We consider three convergence definitions for double series, denoted by (p), (σ), and (reg), which are respectively Pringsheim, Sheffer, and regular convergence. Definitions will be given in §2.

Convergence (σ) has been defined by I. M. Sheffer in a paper[1] which will be referred to as [S]. Convergence (p) and (reg) are well known, the latter having been discussed by G. H. Hardy[2] and others.

[S] established the relation $(\sigma) \subseteq (\text{reg})$; that is, every series which is convergent (σ) is also convergent (reg) and to the same sum. The question arises as to whether the relation between these two types of convergence is actually equivalence. It is part of the purpose of this paper to answer this question in the negative. An infinite set of convergence definitions will be presented, denoted by (σ_n), $n=1, 2, \cdots$, with the property:

$$(\sigma) \subseteq (\sigma_{n+1}) \subseteq (\sigma_n) \subseteq (\sigma_1) \subseteq (\text{reg}), \qquad n = 1, 2, \cdots,$$

and it will then be shown that every inclusion sign but the first may be replaced by equivalence. Of these the most difficult to prove is $(\sigma_2) \equiv (\sigma_1)$. The others just escape being trivial.

The words "to the same sum" will always be understood in the symbols $\subset$ and $\equiv$.

2. **Definitions.** Definitions 3 to 3.2 are adapted from [S].

DEFINITION 1. *A region is a* finite *set of values of the indices.* Examples of regions are the following:

(i) Triangular region: the set of indices (p, q) with $p+q \leqq N$ for a given N.

(ii) Rectangular region: the set of (p, q) with $p \leqq M$, $q \leqq N$, for given M, N.

By varying N in (i) (M, N, in (ii)) we obtain a set of triangular (rectangular) regions. These two examples have in common the following important property:

PROPERTY (1). *Given any square region containing* (0, 0) *there is a region of the set under consideration which includes the square region.*

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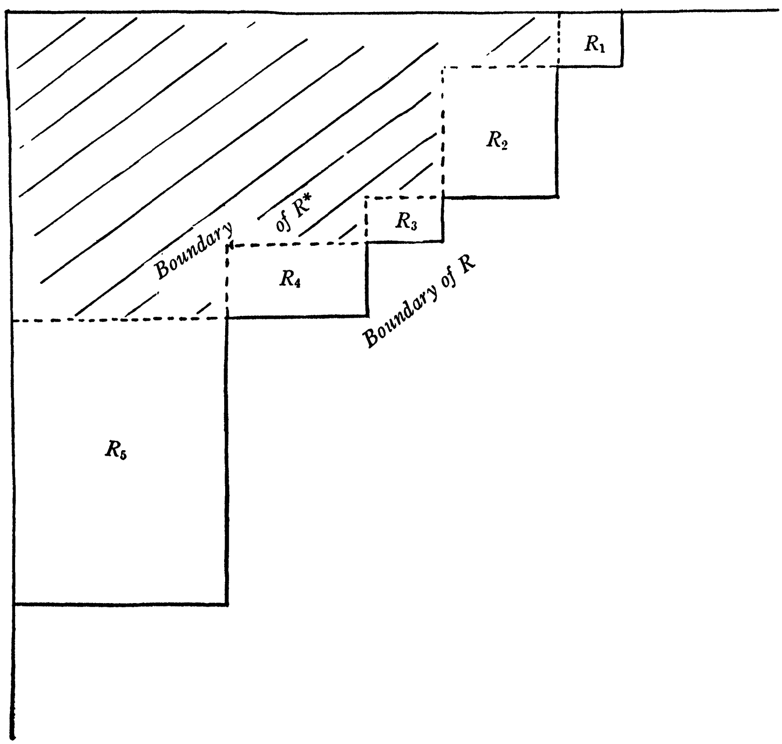
[1] Amer. Math. Monthly vol. 52 (1945) pp. 365–376.

[2] Proc. Cambridge Philos. Soc. vol. 19 (1916–1919) pp. 86–95; Lemmas γ and δ are wrong but Theorem 10 is correct.

The statement "R is a region containing (p, q)*" will be written "R is a region* (p, q)*."*

DEFINITION 2. *If R is a region,* $\sum(R)$ *will denote* $\sum_{(i,j)\in R} a_{ij}$, $|R|$ *will denote* $|\sum(R)|$.



DEFINITION 3. *Let* $\{P\}$ *be a set of regions with Property* (1). *Convergence of* $\sum a_{ij}$ *to the sum A by means of* $\{P\}$ *is defined as: to every* $\epsilon>0$ *corrrespond indices p and q such that* $|A-\sum(R)|<\epsilon$ *for every region* $R\in\{P\}$ *which is a region* (p, q).

DEFINITION 3.1. *Convergence* (p): *The set* $\{P\}$ (*of Definition* 3) *is the set of rectangular regions* (0, 0) *of the example in Definition* 1.

DEFINITION 3.2. *A region with the following property will be called a* σ*-region, or Sheffer region: "if it contains* (p, q)*, then it contains every* (i, j), $i\leqq p$, $j\leqq q$." An equivalent definition of a σ-region is that it is the logical sum of a finite number of rectangular regions (0, 0). The lower right boundary of a σ-region may be thought of as a "staircase" rising to the right. It should be noted that the set of σ-regions has Property (1), and that each region is finite by definition.

DEFINITION 3.21. *Convergence* (σ): *The set* $\{P\}$ *is the set of all* σ*-regions.*

DEFINITION 3.3. *A* σ*-region whose lower right boundary rises to the right in exactly* n *steps is called a* σ_n*-region.* The figure illustrates $n=5$. It is convenient to count the corners sticking out into the excluded portions of the array. *These are called outer corners. The other* $n-1$ *corners are called inner corners.* It is obvious that $n \geqq 1$.

DEFINITION 3.31. *Convergence* (σ_n), $n \geqq 2$: *The set* $\{P\}$ *is the set of all* σ_k*-regions,* $k \leqq n$.

LEMMA 1. *For* $m \geqq n \geqq 2$, *we have* $(\sigma_m) \subseteq (\sigma_n)$.

For every σ_k-region, $k \leqq n$, is also a σ_k-region, $k \leqq m$.

DEFINITION 3.32. *Convergence* (σ_1): *This is defined as convergence* (p) *together with summability by rows and columns to the same sum.* In other words $\sum a_{ij}$ converges (p) to the sum

$$A = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_{ij}.$$

The extra condition makes possible a more unified theory.

DEFINITION 3.4. *Convergence* (reg): *A double series is defined to be convergent* (reg) *if it is convergent* (p), *and if every row and column has a finite sum.*

LEMMA 2. $(\sigma_1) \equiv$ (reg).

This well known fact was proved by Pringsheim.[3] There is a proof in Bromwich, *Infinite series*, 1926, chap. 5, p. 81.

2.1. **Consequences.** The following lemmas are given in [S] for convergence (σ). The proofs apply formally unchanged to convergence (σ_n). We suppose throughout that $n \geqq 1$ is a fixed integer.

LEMMA 3. *A convergent* (σ_n) *series has a unique sum.*

LEMMA 4. *If* $\sum a_{ij}$ *converges* (σ_n) *to the sum* A, *then* $\sum m \cdot a_{ij}$ *converges* (σ_n) *to* mA.

LEMMA 5 (The general principle of convergence). *For* $n \geqq 2$, *a necessary and sufficient condition that a series converge* (σ_n) *is that to every* $\epsilon > 0$ *correspond indices* p *and* q *such that*

$$\left| \sum (R_1) - \sum (R_2) \right| < \epsilon$$

[3] *Elementare theorie der unendliche Doppelreihen*, Sitzungsberichte der Akademie der Wissenschaften der München pp. 101–152, especially Satz II, p. 117.

for all R_1, R_2, *which are* σ_k*-regions* (p, q), $k \leqq n$.

LEMMA 6. *If* $\sum a_{ij}$ *converges* (σ_n), *then to every* $\epsilon > 0$ *correspond indices* p *and* q *such that* $|a_{ij}| < \epsilon$ *if* $i > p$ *or* $j > q$.

LEMMA 7. *A series which converges* (σ_n) *is summable by rows and columns to the same sum.*

For $n = 1$ this is a restatement of part of the definition. For $n \geqq 2$, Sheffer's proof of Theorem 7 in [S] holds.

Since a series which is convergent (σ_n) is obviously convergent (p) we have:

COROLLARY. $(\sigma_n) \subseteq (\sigma_1)$, $n \geqq 1$.

At this stage the following relations are clear:

$$(1) \qquad (\sigma) \subseteq (\sigma_{n+1}) \subseteq (\sigma_n) \subseteq (\sigma_1) \equiv (\text{reg}) \subset (\text{p}), \qquad n = 1, 2, \cdots.$$

That the last relation is strict inclusion is shown by the simple example: $a_{0j} = 1$, $a_{1j} = -1$, $a_{ij} = 0$ if $i > 1$.

3. **Equivalence of all** (σ_n). If R is a σ_{n+1}-region, let R^* denote that σ_n-region whose outer corners are the inner corners of R. (In the figure R^* is indicated by shading.) Let $R - R^* = R_1 + R_2 + \cdots + R_{n+1}$ as shown in the figure. R and R^* will be thought of as including their boundaries; the R_j must be construed accordingly.

THEOREM I. $(\sigma_m) \equiv (\sigma_n)$ *for all* m, $n \geqq 1$.

For definiteness we suppose $m > n$. In view of (1) we have only to prove $(\sigma_n) \subseteq (\sigma_m)$. Suppose that $\sum a_{ij}$ is convergent (σ_n), $n \geqq 1$; we now show that it is convergent (σ_{n+1}) to the same sum. Let us suppose that the sum in question is zero; this can be brought about by the addition of a suitable constant to a_{00}.

Given $\epsilon > 0$, we may by hypothesis choose p, q depending on ϵ, such that $|S| < \epsilon/3$ for any region S that is a σ_k-region (p, q), $k \leqq n$. Now let R be an arbitrary σ_{n+1}-region (p, q). There are two possibilities:

Case I: $(p, q) \in R^*$.

Consider the identity $R = (R - R_1) + [(R - R_2) - (R - R_1 - R_2)]$. It is seen that $|R| \leqq |R - R_1| + |R - R_2| + |R - R_1 - R_2| < \epsilon$, the last inequality holding since the three regions are σ_k-regions (p, q), $k \leqq n$.

Case II: $(p, q) \in R_r$, $1 \leqq r \leqq n+1$.

Suppose at first that $n \geqq 2$. In this case it is possible to chose s, t distinct from r and each other such that $1 \leqq s$, $t \leqq n+1$. Then we can write $R = (R - R_s) + [(R - R_t) - (R - R_s - R_t)]$, and noting that the regions on the right-hand side are again σ_k-regions (p, q), $k \leqq n$, we have $|R| < \epsilon$.

Case II *when* $n=1$. We require two lemmas.

LEMMA 8. *If* $\sum a_{ij}$ *is summable by rows and columns to the sum* A, *then a necessary and sufficient condition that it be convergent* (p) *to the same sum is the following: to every* $\epsilon > 0$ *correspond indices* p, q *such that* $|\sum_{i=r+1}^{\infty}\sum_{j=0}^{s} a_{ij}| < \epsilon$ *if* $r > p$ *and* $s > q$.

This follows immediately from consideration of the identity

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} - \sum_{i=0}^{r}\sum_{j=0}^{s} a_{ij} = \sum_{i=r+1}^{\infty}\sum_{j=0}^{s} a_{ij} + \sum_{i=0}^{\infty}\sum_{j=s+1}^{\infty} a_{ij}.$$

The inversion is justified by the hypothesized summability by rows and columns to the same sum. The series is convergent (p) if and only if the left-hand side can be made smaller than ϵ in absolute value for sufficiently large r, s. The last term on the right-hand side is the remainder of a convergent simple series whose general term is $a_j = \sum_{i=0}^{\infty} a_{ij}$.

Note that in the statement of Lemma 8, the symbol (p) can be replaced by (σ_1) without any alteration in the meaning of the lemma.

LEMMA 9. *If* $\sum a_{ij}$ *is convergent* (σ_1) *the following condition is satisfied: to every* $\epsilon > 0$ *corresponds an index* p *such that if* $r > p$, *then* $|\sum_{i=r+1}^{\infty}\sum_{j=0}^{s} a_{ij}| < \epsilon$, *for all* s.

First choose p' and q' as in Lemma 8, then choose p'' depending on q' and ϵ such that for $r > p''$ we have $|\sum_{i=r+1}^{\infty} a_{ij}| < \epsilon/(q'+1)$ for $j = 0, 1, 2, \cdots, q'$. Finally let $p = \max(p', p'')$. Then if $r > p$, we have

$$\left|\sum_{i=r+1}^{\infty}\sum_{j=0}^{s} a_{ij}\right| \leqq \sum_{j=0}^{q'}\left|\sum_{i=r+1}^{\infty} a_{ij}\right| < \epsilon, \qquad s \leqq q';$$

$$\left|\sum_{i=r+1}^{\infty}\sum_{j=0}^{s} a_{ij}\right| < \epsilon, \qquad s > q'.$$

This proves Lemma 9

To complete the discussion of Case II with $n=1$, we assume $(p, q) \in R_1$. In view of the symmetry in all formulae, i and j can always be interchanged; thus the following argument will cover the case $(p, q) \in R_2$.

From $R = (R - R_2) + R_2$ follows $|R| < \epsilon/3 + |R_2|$ since the first region is a σ_1-region (p, q). Next

$$| R_2 | = \left| \sum_{i=r}^{\infty} \sum_{j=0}^{s} a_{ij} - \sum_{i=t}^{\infty} \sum_{j=0}^{s} a_{ij} \right|$$

$$\leqq \left| \sum_{i=r}^{\infty} \sum_{j=0}^{s} a_{ij} \right| + \left| \sum_{i=t}^{\infty} \sum_{j=0}^{s} a_{ij} \right|,$$

where r, t, s are determined by R_2 and $t>r>p$. Lemma 9 shows the possibility of making p so large (depending only on ϵ) that each of the quantities on the right is less than $\epsilon/3$. Thus $|R|<\epsilon$. This completes Case II. The relation $(\sigma_n)\subseteq(\sigma_{n+1})$ having been proved it is clear that Theorem I is established.

4. **On convergence** (σ). Let us introduce the following notations:

Let S_k be a σ_k-region defined as follows: Its inner corners are at points (p, q) with $p+q=k$; its outer corners are at points (p, q) with $p+q=k+1$, with the exception of the two points $(k+1, 0)$ and $(0, k+1)$. S_k is bounded by a staircase which rises to the right in k equal steps from $(k, 0)$ to $(0, k)$.

Let T_k be the "diagonal" region of points (p, q) with $p+q=k$. Thus

$$(2) \qquad \sum (S_k - S_{k-1}) = \sum (T_{k+1}) - a_{k+1,0} - a_{0,k+1} + a_{k,0} + a_{0,k}.$$

LEMMA 10. *A necessary condition for convergence* (σ) *is that the diagonal sums* $|T_k|$ *approach zero as* $k\rightarrow\infty$.

This follows from (2) and Lemma 6, taking account of (1).

THEOREM II. $(\sigma)\subset(\sigma_n)$, *strictly, for every* $n\geqq 1$.

The relation of inclusion is a trivial consequence of the definitions since every σ_k-region, $k\leqq n$, is a σ-region. We now exhibit a class of double series each of which converges (σ_1) and thus (σ_n) for every n. This class will be shown to contain a member which does not converge (σ).

Let $a_{ij}=v_{i+j}-v_{i+j+1}$ where $\{v_n\}$ is a sequence such that $\sum v_n$ converges to the sum v. Such double series are considered by Pringsheim.[4] Verification of (σ_1) convergence is trivial, the sum being v. We have

$$\sum (T_k) = \sum_{r=0}^{k} a_{r,k-r} = \sum_{r=0}^{k} (v_k - v_{k+1}) = (k+1)(v_k - v_{k+1})$$

and if we choose $v_n=(-1)^n/(n+1)$ we have $\lim |T_k| = 2$ and, by Lemma 10, $\sum a_{ij}$ is not convergent (σ).

[4] Loc. cit. pp. 127–128.

4.1 **Remarks.** Suppose that the hypothesis of a theorem involves convergence (σ). It may happen (and does indeed in the case of all theorems in [S]) that convergence (σ_1) may be substituted in the hypothesis as follows: each σ-region which occurs in the proof is a σ_n-region for some n; if there is a largest such n call it N. Then convergence (σ_N) may be substituted in the hypothesis and may, by Theorem I, be replaced by convergence (σ_1). Lemma 10 is an example of an exception to this statement; there is no largest n and convergence (σ) is essential.

If a series is convergent (σ_1) then, for a given n, the series is convergent (σ_n). Hence, given $\epsilon>0$ there are indices (p, q) such that $|A-\sum(R)|<\epsilon$ for every σ_n-region R which is a region (p, q). In general (p, q) will depend on n; if the choice does not depend on n then the series is convergent (σ).

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THE DIFFERENTIABILITY AND UNIQUENESS OF CONTINUOUS SOLUTIONS OF ADDITION FORMULAS

NELSON DUNFORD AND EINAR HILLE

The problem of representing a one-parameter group of operators (that is, a family T_ξ, $-\infty<\xi<\infty$, of bounded linear operators on a Banach space which satisfies $T_{\xi+\zeta}=T_\xi T_\zeta$) reduces according to several well known methods of attack to establishing differentiability of the function T_ξ at $\xi=0$. The derivative $Ax=\lim_{\xi\to 0}\xi^{-1}(T_\xi-I)x$ exists as a closed operator with domain $D(A)$ dense, providing T_ξ is continuous in the *strong* operator topology (that is, $\lim_{\xi\to\xi_0}T_\xi x=T_{\xi_0}x$, $x\in\mathfrak{X}$). It is then possible to assign a meaning to exp (ξA) in a natural way and so that $T_\xi=\exp(\xi A)$, $-\infty<\xi<\infty$. The operator A is bounded if and only if T_ξ is continuous in ξ in the *uniform* operator topology (that is, $\lim_{\xi\to\xi_0}|T_\xi-T_{\xi_0}|=0$) in which case $A=\lim_{\xi\to 0}\xi^{-1}(T_\xi-I)$ exists in the uniform topology. This implies that T_ξ is an entire function of ξ; conversely, if T_ξ is analytic anywhere, then A is bounded. These considerations extend to the semi-group case in which $T_{\xi+\zeta}=T_\xi T_\zeta$ is known to hold only for positive values of the parameters, although

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