BOOK REVIEWS

Lectures on the calculus of variations. By G. A. Bliss. University of Chicago Press, 1946. 9+296 pp. $5.00.

Calculus of variations is in the main the study of properties of a real-valued function on a class $\mathfrak{M}$ with particular attention to maximizing and minimizing properties of certain elements in $\mathfrak{M}$. Normally the elements of $\mathfrak{M}$ are curves or surfaces embedded in an euclidean space. There are two main problems in the calculus of variations, problems in the large and problems in the small. The first of these is concerned with existence theorems of minimizing elements of various types, the classification and the counting of these elements and the connections between these minimizing elements and the topology of $\mathfrak{M}$. The calculus of variations in the small is concerned primarily with the properties of a particular minimizing element and the determination of those properties that will insure a minimum. Besides these two aspects, there is the fascinating study of the relations between the calculus of variations and differential equations, geometry, physical applications, and the like.

The book under review is concerned with the problem in the small. It contains the best introduction to the calculus of variations known to the reviewer, no matter what phase of the subject one wishes to pursue. The book undoubtedly will become the standard text for the beginner and will be used by the specialist as a source of material and ideas.

Professor Bliss restricts himself to the case in which the class $\mathfrak{M}$ referred to above is a class of continuous arcs C: $y_i(x)$ $(x_1 \leqq x \leqq x_2)$ $(i=1, \cdots, n)$ in euclidean $(x, y_1, \cdots, y_n)$-space. These arcs are assumed to have a piecewise continuously turning tangent with elements (x, y, y') in a prescribed region R of (x, y, y')-space. Professor Bliss begins first with the case when the end values $[x_1, y(x_1), x_2, y(x_2)]$ are fixed and develops the theory in a simple and direct manner. Later he permits these end values to lie on a prescribed end manifold. Finally he requires the arcs to satisfy in addition a given set of differential equations $\phi_\beta(x, y, y')=0$ $(\beta=1, \cdots, m<n)$. The function to be minimized is of the form

$$I(C) = g[x_1, y(x_1), x_2, y(x_2)] + \int_{x_1}^{x_2} f(x, y, y')dx,$$

with $g\equiv 0$ in the fixed end point case. By this procedure the author begins with the simplest problem in the calculus of variations and skillfully leads the reader step by step to the development of the theory of the most comprehensive problem, known as the problem of Bolza, for which there exists a satisfactory theory of maxima and minima. In fact the book under review contains the first published account of this theory that is relatively complete. The problem of Bolza contains as special cases all but a very few of the hitherto formulated problems involving simple integrals. The development of its theory is largely due to Professor Bliss and his students, inspired by him.

The book is divided into two parts. Part I deals with the simpler problems and Part II with the problem of Bolza. Part I is divided into six chapters. The author has chosen to begin with the fixed end point problem in three dimensions. This is the simplest problem that is general in the sense that the extension to higher-dimensional spaces represents little more than formal notational changes.

Chapter I is concerned with necessary conditions for a minimum together with properties of the extremal family. Sufficient conditions for a minimum are established in Chapter II. In Chapter III a detailed discussion of the Hamiltonian theory is given, together with a discussion of Hamilton's principle in mechanics. These results are extended in Chapter IV to the plane and to spaces of higher dimensions. A similar extension is made in Chapter V to parametric problems. The case when the end points of the arc are allowed to vary is treated in Chapter VI.

The last three chapters, which constitute Part II, are concerned with the general problem of Bolza. In this part the author develops a theory of minima that is analogous to the theory developed for the simpler problem. It is the only published account of this theory that is relatively complete. The presence of differential equations introduces a concept of normality that is absent in the simpler case. The discussion of this concept, given in this book, is the clearest published exposition of this important topic. The author restricts himself for the most part to the normal case. This is the most interesting case since it includes most of the important problems of the Bolza type.

At the end of the book is found a bibliography for the problem of Bolza. In the papers here listed one can find additional material on the problem of Bolza. There is also found in the appendix a simple and elegant treatment of implicit function theorems and existence theorems for differential equations. These theorems are formulated so as to be conveniently applicable in the calculus of variations.

The book is exceptionally well written and easy to read. The author has succeeded in setting forth the theory of the calculus of variations in a simple and appealing manner. The reviewer's only regret is that Professor Bliss did not have occasion to include various other topics in the calculus of variations in which he has been interested and to which he has made numerous contributions. The book is a valuable addition to a mathematician's library.

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The theory of potential and spherical harmonics. By W. J. Sternberg and T. L. Smith. (Mathematical Expositions, No. 3.) The University of Toronto Press, 1944. 312 pp. $3.35.

This is a book on a classical field of analysis treated in the classical way and restricted to classical theorems. Potential theory in the 19th century sense is no longer a familiar subject in a mathematical curriculum, and so the value of such a publication is to be considered with respect to a rather special audience. Its needs might equally well be served by a compact presentation in English covering the fundamental notions similar, for instance, to that in Courant-Hilbert, volume 2 (which actually presents a more complete picture as well) or the third volume of Goursat's *Cours d'analyse* with its illuminating problems. The preface suggests the research worker may find some ideas here but this is not borne out by the contents.

The chapter headings indicate the topics taken up: The Newtonian law of gravity, concept of the potential, the integral theorems of potential theory, analytic character of the potential, spherical harmonics, behavior of the potential at points of the mass, relation of potential to theory of functions, the boundary value problems of potential theory, the Poisson integral in the plane, the Poisson integral in space, the Fredholm theory of integral equations, general solution of the boundary value problem. The last chapters deserve commendation, especially for the excellent presentation of the Fredholm theory. The Riemann integral is used exclusively, and only an elementary acquaintance with real variable theory and the rudiments of functions of a complex variable is needed as a prerequisite. The book can be read by the first year graduate student in an American university.

The exposition is notably lucid, and in fact almost conversational in its naturalness, though there is little emphasis. Perhaps the principal results could have been singled out and more said about the power and generality of these theorems. The preface remarks on a "consistent" use of vector analysis. Without implying any criticism, it should be noted, however, that the treatment is not vectorial in spirit. The vectors are usually brought in as shorthand notations for the more complicated Cartesian expressions actually manipulated.