THE CONVERSE OF A THEOREM OF TCHAPLYGIN ON DIFFERENTIAL INEQUALITIES

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1. **Introduction.** Suppose that $y(x)$ is a solution of the linear differential equation

$$y'' - p_1 y' - p_2 y - q = 0, \qquad x \geqq x_0, \tag{1.1}$$

where $p_1(x)$, $p_2(x)$ and $q(x)$ are continuous when $x \geqq x_0$, and that

$$y(x_0) = y_0, \qquad y'(x_0) = y_0'. \tag{1.2}$$

Then if $v(x)$ satisfies the differential inequality

$$v'' - p_1 v' - p_2 v - q > 0, \qquad x \geqq x_0, \tag{1.3}$$

and the same boundary conditions as $y(x)$ at x_0, it is clear that the inequality

$$v(x) > y(x) \tag{1.4}$$

holds in some right-hand neighborhood of x_0. Tchaplygin[1] has proved that the inequality (1.4) holds when $x_0 < x \leqq x_1$ provided that there exists a solution $\lambda(x)$ of the Riccati equation

$$\lambda' + \lambda^2 + p_1\lambda + (p_1' - p_2) = 0 \tag{1.5}$$

which is continuous when $x_0 < x < x_1$. Let $X(x_0)$ be the least upper bound of values x_1 for which the Riccati equation admits a continuous solution when $x_0 < x < x_1$. Then the inequality (1.4) holds when $x_0 < x \leqq X(x_0)$, and Petrov [2] has shown that if p_1 and p_2 are constants no better result is true. That is, if p_1 and p_2 are constants and $X(x_0)$ is finite, then there exists a function $v(x)$ satisfying (1.3) and (1.2) for which $v(x) = y(x)$ at a point arbitrarily close to but greater than $X(x_0)$. It is the purpose of this paper to show that this last result is true without the restriction that p_1 and p_2 are constants. We prefer to state our results in terms of and make our proofs depend

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[1] The author knows this result only by virtue of a reference to it contained in the paper of Petrov [2], and there it is not made clear whether or not Tchaplygin took the interval from x_0 to x_1 to be open, as we have written it, or closed or half-open. It follows, however, from the results obtained in §2 that this statement is true for the open interval and hence is a fortiori true for the closed and half-open intervals. Numbers in brackets refer to the bibliography at the end of the paper.

upon properties of solutions of the homogeneous linear differential equation

$$u'' - p_1 u' - p_2 u = 0, \tag{1.6}$$

rather than solutions of the Riccati equation (1.5). Apart from the trivial advantage that we do not need to assume that $p_1(x)$ is differentiable, it is possible to give simpler and more natural proofs using (1.6) than (1.5).

2. **The main theorem.** We are going to prove the following theorem.

THEOREM. *If $y(x)$ satisfies* (1.1) *and* (1.2), *and if $v(x)$ satisfies* (1.3) *and* (1.2), *then the inequality* (1.4) *holds when $x_0 < x \leqq x_1$ provided that there exists a solution $u(x)$ of* (1.6) *which does not vanish when $x_0 < x < x_1$. Let $u_0(x)$ be a solution of* (1.6) *such that $u_0(x_0) = 0$, $u_0'(x_0) = 1$, and let $X(x_0)$ be the first zero of $u_0(x)$ to the right of x_0 if any such zero exists; otherwise, let $X(x_0) = +\infty$. Then the interval where $x_0 < x \leqq X(x_0)$ is the largest interval in which the inequality* (1.4) *can be asserted to hold. In other words, either $X(x_0) = +\infty$ or else there exists a function $v(x, k)$ for each sufficiently small positive k which satisfies* (1.3) *and* (1.2) *and for which $v[X(x_0)+k, k] < y[X(x_0)+k]$.*

Let us define $z(x) = v(x) - y(x)$. Then

$$z'' - p_1 z' - p_2 z = \phi(x) > 0, \qquad z(x_0) = z'(x_0) = 0. \tag{2.1}$$

If $u(x)$ is a solution of (1.6) and one defines the Wronskian of $u(x)$ and $z(x)$ to be

$$W(x) = u(x)z'(x) - u'(x)z(x),$$

then it is easy to see that

$$W'(x) - p_1(x)W(x) = u(x)\phi(x), \qquad W(x_0) = 0, \tag{2.2}$$

whence we have that

$$W(x) = \int_{x_0}^{x} u(s)\phi(s)P(x, s)ds, \tag{2.3}$$

where $P(x, s)$ is defined as

$$P(x, s) = \exp \int_{s}^{x} p_1(t)dt. \tag{2.4}$$

Suppose now that $z(x)$ vanishes at some point to the right of x_0, and that x^* is the first such point. Then when $x = x^*$, equation (2.3) reduces to

$$u(x^*)z'(x^*) = \int_{x_0}^{x^*} u(s)\phi(s)P(x^*, s)ds. \tag{2.5}$$

Since $z(x)>0$ to the left of x^* we have that $z'(x^*)\leqq 0$. It now follows from equation (2.5) that $u(s)$ must change sign when $x_0<s<x^*$, for otherwise the two sides of (2.5) could not have the same sign. This remark is equivalent to the first sentence of the theorem.

If $X(x_0)=+\infty$, then $z(x)$ can never vanish to the right of x_0. In this case the inequality (1.4) holds whenever $x>x_0$. Let us now suppose that $X(x_0)$ is finite. Let x_2 be any point to the right of $X(x_0)$ such that $u_0(x)$ has no zeros between $X(x_0)$ and x_2. Let $u(x)$ be the solution of (1.6) such that $u(x_2)=0$, $u'(x_2)=-1$. It follows from the separation theorem for the zeros of solutions of second order linear homogeneous differential equations [1, p. 224] that there is a unique point x_1 between x_0 and $X(x_0)$ such that $u(x_1)=0$. For this function $u(x)$ we get from equation (2.3) that

$$z(x_2) = \int_{x_0}^{x_2} u(s)\phi(s)P(x_2, s)ds.$$

To complete the proof of the theorem it is sufficient to show that for each x_2 subject to the restrictions already imposed on it a positive function $\phi(s, x_2)$ can be found such that

$$\int_{x_0}^{x_2} u(s)\phi(s, x_2)P(x_2, s)ds < 0. \tag{2.6}$$

If $b>0$, the function $\phi(s, x_2)$ defined as

$$\begin{aligned} \phi(s, x_2) &= [1 - bu(s)]/P(x_2, s), & x_0 < s < x_1, \\ \phi(s, x_2) &= 1/P(x_2, s), & x_1 < s < x_2, \end{aligned}$$

is surely positive. For this function the integral (2.6) has the value

$$\int_{x_0}^{x_2} u(s)ds - b\int_{x_0}^{x_1} u^2(s)ds$$

and will certainly be negative if b is sufficiently large.

To round out the results of this paper we shall now show that the number $X(x_0)$ defined in the theorem coincides with the number $X(x_0)$ defined in the introduction by means of the Riccati equation (1.5). We prefer to replace (1.5) by

$$\mu' + \mu^2 - p_1\mu - p_2 = 0, \tag{2.7}$$

where $\mu=\lambda+p_1$, and define $X_1(x_0)$ as the least upper bound of numbers x_1 such that the equation (2.7) admits a solution continuous when $x_0<x<x_1$. Since p_1 is continuous this number is the same as that defined for the equation (1.5) in case p_1 is differentiable, but $X_1(x_0)$ is defined even though p_1 is not differentiable. A function $\mu(x)$ satisfies (2.7) if and only if $\mu(x)=u'(x)/u(x)$, where $u(x)$ is a nontrivial solution of (1.6). Hence solutions $\mu(x)$ of (2.7) which are continuous when $x_0<x<x_1$ correspond in a one-to-one fashion with families of functions $cu(x)$, where c is an arbitrary nonzero constant, which satisfy (1.6) and which do not vanish when $x_0<x<x_1$. It follows that $X_1(x_0)$ is the least upper bound of values x_1 for which the equation (1.6) admits a solution not vanishing when $x_0<x<x_1$. Since every solution of (1.6) must vanish when $x_0<x\leqq X(x_0)$ we have that $X_1(x_0)\leqq X(x_0)$, and since $u_0(x)$ does not vanish when $x_0<x<X(x_0)$ we have that $X(x_0)\leqq X_1(x_0)$. Therefore, $X(x_0)=X_1(x_0)$.

Bibliography

1. E. L. Ince, *Ordinary differential equations*, Dover, New York, 1944.

2. V. N. Petrov, *The limits of applicability of S. Tchaplygin's theorem on differential inequalities to linear equations with usual derivatives of the second order*, C. R. (Doklady) Acad. Sci. URSS. vol. 51 (1946) pp. 255–258.

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