84. Edward Kasner and John DeCicco: *Rational harmonic curves.*

A rational harmonic curve is defined by setting the real part of a rational function of $x+iy$, of degree r, equal to zero. Any such curve is given by $P(x, y)=0$, where P is a polynomial of (x, y) of degree $2r-k$, where $0 \leqq k \leqq r$. The curve has k real asymptotes, all of which pass through a fixed point and which make equal angles with each other. The angle between consecutive asymptotes is π/k. The remaining asymptotes are minimal and $2(r-k)$ in number. The theorem generalizes to rational harmonic curves a theorem of Briot and Bouquet concerning the asymptotes of polynomial harmonic curves, which are defined by setting the real part of any rational integral function of a complex variable equal to zero. Other properties are given by means of the foci of systems of confocal curves. Additional results are found about satellites. (Received October 4, 1946.)

85. C. E. Springer: *Union torsion of a curve on a surface.*

The geodesic torsion at a point of a curve on a surface is the torsion of the geodesic which is tangent to the curve at the point. In this paper the union torsion at a point of a curve C on a surface is defined as the torsion of the union curve in the direction of the curve C, the union curve being defined relative to a given rectilinear congruence. The union torsion is given by a formula which reduces to the expression for geodesic torsion in case the congruence is normal to the surface. It is shown that a union curve is a plane curve if, and only if, it is tangent to a curve of intersection of a developable of the congruence with the surface. (Received October 3, 1946.)

86. A. E. Taylor: *A geometric theorem and its application to biorthogonal systems.*

Let S be a bounded and closed point set in E_n (Euclidean space of n dimensions). Let O be a point such that O and S together are not contained in any subspace of $n-1$ dimensions (such a subspace is hereafter called a plane). Then there exist n linearly independent vectors $x_1, \cdots, x_n$ emanating from O, with terminal points $P_1, \cdots, P_n$ in S, and n planes $p_1, \cdots, p_n$ satisfying the following conditions: (a) p_i contains P_i; (b) p_i is parallel to the plane determined by $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$; (c) S and O both lie in the same one of the two closed half-spaces into which p_i divides E_n. This theorem is proved, and then applied to demonstrate the following: If Y_n is an n-dimensional subspace of a normed linear space X, there exist n elements $x_1, \cdots, x_n$ of unit norm in Y_n and n linear functionals $f_1, \cdots, f_n$ of unit norm defined on X, such that $f_i(x_j)=\delta_{ij}$. (Received October 25, 1946.)

Statistics and Probability

87. G. E. Forsythe: *On Nörlund summability of random variables to zero.*

The present paper is an incomplete extension to regular Nörlund summability methods of some previous results (G. E. Forsythe, Duke Math. J. vol. 10 (1943) pp. 397–428, §5) on Cesàro summability in probability of random variables to zero. Corresponding to a sequence p_0 $(=1)$, p_1, $p_2, \cdots$ of non-negative constants, the Nörlund method N_p is defined by the triangular Toeplitz matrix $\|a_{nk}\|$, where $a_{nk} = p_{n-k}(\sum_{k=0}^{n} p_k)^{-1}$. It is conjectured that if $N_p \subset N_q$ with respect to the summability of sequences of real numbers, then $N_p \subset N_q$ (meaning that $N_p \subset N_q$ with respect to the summability in probability to zero of normal families $\{x_k\}$ of independent, real-valued, symmetric random variables). In the present paper the conjecture is proved for only two special cases: (i) when N_p is the Cesàro method C_1; (ii) when N_q is C_1 and when the $\{p_k\}$ defining N_p are nonincreasing. If in case (i) the $\{q_k\}$ defining N_q are nondecreasing it is proved that $C_1 \equiv N_q$ (meaning that $C_1 \subset N_q$ and $N_q \subset C_1$). Sufficient conditions are given for $N_p \subset N_q$ and for $N_p \equiv N_q$. A regular Nörlund method N_p is exhibited for which $N_p \subset C_1$ (and $C_1 \not\subset N_p$) with respect to sequences of real numbers, but for which $N_p \equiv C_1$. Hence the converse of the above conjecture is false. (Received November 23, 1946.)

88. E. J. Gumbel: *The asymptotic distribution of the range.*

Consider a large sample taken from an initial symmetrical unlimited distribution $\phi(x)$ for which all moments exist. Then the joint distribution of the largest and of the smallest values x_n and x_1 splits into the product of the asymptotic distributions of the largest value $f_n(x_n)$, and of the smallest value $f_1(x_1)$. Let u_n be the most probable largest value, and put $\alpha_n = n\phi(u_n)$. Then the asymptotic distribution $\Psi'(R)$ of the reduced range $R = \alpha_n(x_n - x_1 - 2u_n)$ obtained from the convolution of the asymptotic distributions of the two extremes is $\Psi'(R) = e^{-R}\int_{-\infty}^{+\infty} \exp\,[-e^y - e^{-y-R}]dy$. The numerical values of the cumulative probability $\Psi(R) = \int_{-\infty}^{+\infty} \exp\,[y - e^y - e^{-y-R}]dy$ and of the distribution $\Psi'(R)$ may also be calculated from the differential equation $\Psi'' + \Psi' - e^{-R}\Psi = 0$. The distribution $g(w)$ of the range $w = x_n - x_1$ itself is obtained from $\Psi'(R)$ by the usual linear transformation. The asymptotic probabilities and the asymptotic distributions of the mth range and of the range for assymetrical distributions are obtained by the same method and lead to similar integrals which may also be evaluated by numerical methods. (Received November 8, 1946.)

89. A. M. Mood: *On Hotelling's weighing problem.*

The paper contains some solutions of the weighing problem proposed by Hotelling (Ann. Math. Statist. vol. 15 (1944) pp. 297–306). The experimental designs found are essentially two-level multifactorial designs for determining main effects. They are particularly applicable to the problem of measuring several objects. The chemical balance problem (in which objects may be placed in either of the two pans of the balance) is almost completely solved by means of designs constructed from Hadamard matrices. Designs are provided both for a balance which has a bias and for one which has no bias. The spring balance problem (in which objects may be placed in only one pan) is completely solved when the balance is biased. For an unbiased spring balance, designs are given for small numbers of objects and weighing operations. Also the most efficient designs are found for the unbiased spring balance, but it is shown that in some cases these cannot be used unless the number of weighings is as large as the binomial coefficient $C_{p,p/2}$ or $C_{p,(p+1)/2}$ where p is the number of objects. It is found that when p objects are weighed in $N \geqq p$ weighings, the variances of the estimates of the weights are of the order of σ^2/N in the chemical balance case (σ^2 is the variance of a single weighing), and of the order of $4\sigma^2/N$ in the spring balance case. (Received October 23, 1946.)

90. Isaac Opatowski: *Simple Markoff chains with reverse transitions: the time moments.*

Consider a chain of transitions $(i \rightarrow i+1)$, $(i+1 \rightarrow i)$ $(i = 0, 1, \cdots, n-1)$ between the states $0, 1, \cdots, n$. Let the usual conditional probabilities of these transitions within any time Δt be respectively $k_{i+1}\Delta t+o(\Delta t)$ and $g_i\Delta t+o(\Delta t)$. Let the probability of any other transition during Δt be $o(\Delta t)$. Let k_i's and g_i's be constant. Let $P_i(t)$ be the probability of the existence of the state i at the time t if the state 0 existed at the time $t=0$. The paper gives $\int_0^\infty t^m P_i(t)dt$ or $\int_0^\infty t^m P_i'(t)dt$ as algebraic expressions of the k_i's and g_i's. The result is obtained by using the complete homogeneous symmetric functions of the poles of the Laplace transform of $P_i(t)$ or $P_i'(t)$. (Received November 20, 1946.)

91. Gerhard Tintner: *The statistical estimation of the dimensionality of a given set of observations.*

There is a set of N observations of p variables $X_{it}=M_{it}+y_{it}$, where M_{it} is the systematic part and y_{it} the random element ($i=1, 2, \cdots, p$; $t=1, 2, \cdots, N$). The y_{it} are normally and independently distributed with means zero. There are in the population R linear independent relationships of the form: $k_{s0}+\sum_i k_{si}M_{it}=0$ ($s=1, 2 \cdots R$). A method based upon results of R. A. Fisher (Annals of Eugenics, 1938) and P. L. Hsu (ibid. 1941) assumes that we have a large sample estimate of the covariance matrix of the y_{it}: $\|V_{ij}\|$. The determinantal equation is $|a_{ij}-\lambda V_{ij}|=0$, where the a_{ij} are the sample covariances of the X_{it}. λ_1 is the smallest root of the equation, λ_2 the next smallest, and so on. $\Lambda_r=(N-1)\sum_1^r\lambda_i$. These sums of squares are distributed like χ^2 with $r(N-p-1+r)$ degrees of freedom and may be used to estimate the number R of linear relations between the M_{it}. By inserting the R smallest roots into the determinantal equation matrices are found for the computation of the coefficients k_{si} (G. Tintner, Ann. of Math. Statist., 1945). (Received October 15, 1946.)

92. Jacob Wolfowitz: *On the efficiency of unbiased sequential estimates.*

Let $f(x, \theta)$ be the distribution function (density function or probability function) of a chance variable X, which depends upon the parameter $\theta=\theta_1, \cdots, \theta_l$. Let n successive independent observations be made on X, where n is itself a chance variable, and the decision to terminate the drawing of observations depends upon the observations already obtained. Let $\theta_1^*(x_1, \cdots, x_n), \cdots, \theta_l^*(x_1, \cdots, x_n)$ be joint unbiased estimates of $\theta_1, \cdots, \theta_l$. Let $\|\lambda_{ij}\|$ be the nonsingular matrix of their covariances, and $\|\lambda^{ij}\|$ its inverse. Under certain regularity conditions it is proved that the concentration ellipsoid $\sum\lambda^{ij}(k_i-\theta_i)(k_j-\theta_j)=l+2$ always contains within itself the ellipsoid $\sum\mu_{ij}(k_i-\theta_i)(k_j-\theta_j)=l+2$, where $\mu_{ij}=EnE((\partial \log f/\partial\theta_i)(\partial \log f/\partial\theta_j))$. (Received October 21, 1946.)

Topology

93. R. F. Arens: *Pseudo-normed algebras.*

A pseudo-norm defined on a linear algebra A over, for example, the reals, is a real-valued function having the formal properties of a norm in a normed ring except that the pseudo-norm of some nonzero elements may vanish. Consequently, a pseudo-normed algebra is required by definition to have a complete system of pseudo-norms. The continuity of inversion is considered; and the singular elements related to the closed divisorless proper ideals. The space of the latter, and conditions for its compactness, are considered. As a by-product a characterization of rings of all continuous complex-valued functions on a topological space is obtained. Special attention is given to the question of completeness of quotient rings. (Received November 19, 1946.)