A CHARACTERIZATION OF SEMI-SIMPLE RINGS WITH THE DESCENDING CHAIN CONDITION

OSCAR GOLDMAN

H. Weyl[1] has defined a semi-simple algebra (of finite rank) to be an algebra which admits a faithful semi-simple linear representation. Now, algebras are rings with a field of operators; Artin and others[2] have shown that the theory of semi-simple algebras can be generalized to a theory of semi-simple rings (without the field of operators) provided we replace the condition of finite rank by suitable finiteness conditions. (Both the ascending and descending chain conditions were assumed, but it was later shown that the descending chain condition was sufficient.[3]) The notion of semi-simplicity is defined by the condition that the radical reduces to $\{0\}$, there being several equivalent definitions of the radical. We introduce another one below.

The question now arises whether the Weyl definition could not be extended to the case of rings. To do this, we must extend to an arbitrary ring the notion of a linear representation of an algebra. This can be done by replacing the consideration of the algebra of matrices by the more general notion of the ring of endomorphisms of an abelian group: a *representation* of a ring A will be a homomorphism ρ of A into the ring of endomorphisms of an additive group $\mathfrak{M}$. Let such a representation be given; we can define a law of composition, $(a, m) \rightarrow am$, between elements of A and of $\mathfrak{M}$ by writing $am = \{\rho(a)\}(m)$. The composite object formed by $\mathfrak{M}$ and this law of composition is called an *A-module.* A *sub-module* of an *A-module* $\mathfrak{M}$ is a subset $\mathfrak{N}$ of $\mathfrak{M}$ which is a subgroup of the additive group of $\mathfrak{M}$ and is such that $A\mathfrak{N} \subset \mathfrak{N}$. ($A\mathfrak{N}$ is defined to be the set of all finite sums $\sum_i a_i m_i$, $a_i \in A$, $m_i \in \mathfrak{N}$.) A *homomorphism* of an A-module $\mathfrak{M}$ into an A-module $\mathfrak{M}'$ is a homomorphism h of the additive group of $\mathfrak{M}$ into the additive group of $\mathfrak{M}'$ which is such that $h(am) = ah(m)$ for all $a \in A$, $m \in \mathfrak{M}$.

An A-module $\mathfrak{M}$ is said to be *simple* if its only submodules are $\{0\}$ and itself. Concerning simple modules, we have the well known lemma:

SCHUR'S LEMMA. *A homomorphism h of a simple A-module $\mathfrak{M}$ into*

Received by the editors May 3, 1946, and, in revised form, June 26, 1946.

[1] H. Weyl, *The classical groups*, Princeton University Press, 1939, p. 85.

[2] E. Artin, Abh. Math. Sem. Hamburgischen Univ. vol. 5 (1927) p. 251; J. Levitzki, Compositio Math. vol. 5 (1937) p. 392.

[3] R. Brauer, Bull. Amer. Math. Soc. vol. 48 (1942) p. 752; C. Hopkins, Duke Math. J. vol. 4 (1938) p. 664; J. Levitzki, Compositio Math. vol. 7 (1939) p. 214.

an A*-module* $\mathfrak{M}'$ *either maps* $\mathfrak{M}$ *on* $\{0\}$ *or is an isomorphism of* $\mathfrak{M}$ *with some sub-module of* $\mathfrak{M}'$. *If* $\mathfrak{M}'$ *is also simple, and* $h(\mathfrak{M}) \neq \{0\}$, *then* $h(\mathfrak{M}) = \mathfrak{M}'$.

If $\{\mathfrak{N}_\lambda\}_{\lambda \in L}$ is a family of sub-modules of an A-module $\mathfrak{M}$, we call the *sum* of this family of sub-modules, and we denote by $\sum_{\lambda \in L} \mathfrak{N}_\lambda$, the set composed of all sums $\sum_{\lambda \in L} n_\lambda$, where for each λ, $n_\lambda \in \mathfrak{N}_\lambda$, and $n_\lambda = 0$ except for a finite number of the indices λ. It is clear that $\sum_{\lambda \in L} \mathfrak{N}_\lambda$ is again a sub-module of $\mathfrak{M}$. The sum is said to be *direct* if the representation of an element of the sum in the form $\sum_{\lambda \in L} n_\lambda$ uniquely determines the n_λ. (It is clear that $\mathfrak{N}_1 + \mathfrak{N}_2$ is direct if, and only if, $\mathfrak{N}_1 \cap \mathfrak{N}_2 = \{0\}$.)

A module is said to be *semi-simple* if it can be represented as the direct sum of simple sub-modules. The following facts can easily be shown.

I. If a module is the sum of a family $\mathfrak{F}$ of simple sub-modules, it is also the direct sum of some sub-family of $\mathfrak{F}$.

II. For a module $\mathfrak{M}$ to be semi-simple, it is necessary and sufficient that, given any sub-module $\mathfrak{N}$ of $\mathfrak{M}$, there should exist a sub-module $\mathfrak{N}'$ such that $\mathfrak{M}$ is the direct sum of $\mathfrak{N}$ and $\mathfrak{N}'$.

If $\mathfrak{M}$ is an A-module, define $\mathfrak{M}_T$ to be the set of all $m \in \mathfrak{M}$ for which $Am = \{0\}$. $\mathfrak{M}_T$ is clearly a sub-module of $\mathfrak{M}$; we call $\mathfrak{M}_T$ the *trivial sub-module* of $\mathfrak{M}$. The set $\mathfrak{A}$ of all $a \in A$ such that $a\mathfrak{M} = \{0\}$ is clearly a two-sided ideal in A, called the *annihilator* of $\mathfrak{M}$. The two extreme cases are the one in which $\mathfrak{A} = \{0\}$, in which case we say that $\mathfrak{M}$ is *faithful*, and the case in which $\mathfrak{A} = A$ and therefore $\mathfrak{M}_T = \mathfrak{M}$.

The *radical*[4] is now defined to be the intersection of the annihilators of all simple A-modules. It is clear that the radical is a two-sided ideal in A. The ring A is said to be *semi-simple* if the radical $\mathfrak{R}$ reduces to $\{0\}$. It is readily seen that the factor ring $A/\mathfrak{R}$ is semi-simple.

The following theorem justifies our use of the term "radical":

THEOREM I. *Let* A *be a ring, and* $\mathfrak{R}$ *its radical. Then every nilpotent left ideal is contained in* $\mathfrak{R}$ *and, furthermore, if* A *satisfies the descending chain condition for left ideals,* $\mathfrak{R}$ *is itself nilpotent.*

If $\mathfrak{M}$ is any simple A-module, $\mathfrak{A}$ any left ideal in A, $\mathfrak{A}\mathfrak{M}$ is a sub-module of $\mathfrak{M}$, and is therefore $\{0\}$ or $\mathfrak{M}$. If $\mathfrak{A}\mathfrak{M} = \mathfrak{M}$, then $\mathfrak{A}^n\mathfrak{M} = \mathfrak{M}$ for any positive integer n. Thus if $\mathfrak{A}^n = \{0\}$ for some n, we have $\mathfrak{A}\mathfrak{M} = \{0\}$ and $\mathfrak{A} \subset \mathfrak{R}$.

[4] This definition of the radical is due to C. Chevalley. I am indebted to Professor Chevalley for many interesting discussions on the subject of this paper.

We suppose now that A satisfies the descending chain condition for left ideals and that $\mathfrak{R}$ is not nilpotent. Consider the chain $\mathfrak{R} \supset \mathfrak{R}^2 \supset \cdots$; under our hypotheses, there is some N such that $\mathfrak{R}^N = \mathfrak{R}^{N+1} = \cdots \neq \{0\}$. Call $\mathfrak{R}^N = \mathfrak{S}$; we have $\mathfrak{S}^2 = \mathfrak{S}$ and since $\mathfrak{R}$ is two-sided, $\mathfrak{S}$ is also. Let E be the set of all left ideals $\mathfrak{A} \neq \{0\}$ such that $\mathfrak{S}\mathfrak{A} = \mathfrak{A}$. E is not empty, since $\mathfrak{S}$ is in E. Since the descending chain condition holds, we see that there is a minimal element in E, say $\mathfrak{A}_0$, such that no ideal in E is properly contained in $\mathfrak{A}_0$. Let $\mathfrak{A}_1$ be the set of all $x \in \mathfrak{A}_0$ for which $\mathfrak{S}x = \{0\}$. Since $\mathfrak{S}$ is two-sided, it follows that $\mathfrak{A}_1$ is a left ideal. If $x \in \mathfrak{A}_0$, $x \notin \mathfrak{A}_1$, $\mathfrak{S}x$ is a left ideal, not $\{0\}$, is contained in $\mathfrak{A}_0$, and is such that $\mathfrak{S}(\mathfrak{S}x) = \mathfrak{S}^2 x = \mathfrak{S}x$; whence it follows that $\mathfrak{S}x = \mathfrak{A}_0$. $\mathfrak{A}_0$, $\mathfrak{A}_1$ being left ideals, the factor group $\mathfrak{A}_0/\mathfrak{A}_1$ has a natural structure as an A-module. By the preceding remark, we have for any nonzero $\bar{x} \in \mathfrak{A}_0/\mathfrak{A}_1$, $\mathfrak{S}\bar{x} = \mathfrak{A}_0/\mathfrak{A}_1$. But then for any nonzero $\bar{x} \in \mathfrak{A}_0/\mathfrak{A}_1$, we have $A\bar{x} = \mathfrak{A}_0/\mathfrak{A}_1$, from which it is clear that $\mathfrak{A}_0/\mathfrak{A}_1$ is a simple A-module. Because of this we have $\mathfrak{R}\mathfrak{A}_0/\mathfrak{A}_1 = \{0\}$ and therefore $\mathfrak{S}\mathfrak{A}_0/\mathfrak{A}_1 = \{0\}$ which is contrary to our assumptions. From this it follows that $\mathfrak{R}$ is nilpotent.

The natural generalization of Weyl's definition, that a ring is semi-simple if it has a faithful semi-simple module, is, by Theorem II, equivalent to our definition.

THEOREM II. *A necessary and sufficient condition that the radical $\mathfrak{R}$ of a ring A be $\{0\}$ is that there exists a faithful semi-simple A-module.*

Let $\mathfrak{M}$ be a semi-simple A-module. If we write $\mathfrak{M}$ as the direct sum, $\sum_{\lambda \in L} \mathfrak{N}_\lambda$, of simple sub-modules, it is clear that the annihilator of $\mathfrak{M}$ is the intersection of the annihilators of the modules $\mathfrak{N}_\lambda$. If $\mathfrak{M}$ is faithful, its annihilator is $\{0\}$, giving us a system of simple modules, the intersection of whose annihilators is $\{0\}$. Thus $\mathfrak{R} = \{0\}$.

If $\mathfrak{R} = \{0\}$, let $\{\mathfrak{M}_\lambda\}_{\lambda \in L}$ be a system of simple A-modules chosen in such a manner that $\bigcap_{\lambda \in L} \mathfrak{A}_\lambda = \{0\}$, where $\mathfrak{A}_\lambda$ is the annihilator of $\mathfrak{M}_\lambda$. Let $\mathfrak{M}'$ be the strong direct product of the additive groups of the $\mathfrak{M}_\lambda$. We make $\mathfrak{M}'$ into an A-module by writing $a(\cdots, m_\lambda, \cdots, m_{\lambda'}, \cdots) = (\cdots, am_\lambda, \cdots, am_{\lambda'}, \cdots)$. Let $\mathfrak{M}_\lambda^*$ be the sub-module of $\mathfrak{M}'$ consisting of all elements of $\mathfrak{M}'$ all of whose coordinates other than the λth are zero. $\mathfrak{M}_\lambda^*$ is clearly isomorphic to $\mathfrak{M}_\lambda$ and is therefore simple. Let $\mathfrak{M} = \sum_{\lambda \in L} \mathfrak{M}_\lambda^*$. Since the $\mathfrak{M}_\lambda^*$ are simple, $\mathfrak{M}$ is semi-simple. The annihilator of $\mathfrak{M}$ is the intersection of the annihilators of the $\mathfrak{M}_\lambda^*$; their intersection was chosen to be $\{0\}$, whence it follows that $\mathfrak{M}$ is faithful. This completes the proof of the theorem.

The class of semi-simple rings is much wider than the class of semi-simple rings with the descending chain condition.[5] Our object is to characterize the latter class of rings by properties of their modules. It is known that if A is a semi-simple ring with the descending chain condition, then every A-module is the direct sum of its trivial sub-module and a semi-simple sub-module. We propose to show that this property is characteristic of the class of rings we are studying:

THEOREM III. *Let A be a ring which has the property that every A-module can be represented as the direct sum of its trivial sub-module and a semi-simple sub-module. Then A is a semi-simple ring in which the descending chain condition holds.*

For convenience we introduce two modules, $\mathfrak{A}^L$ and $\mathfrak{A}^L \times Z$, which can be defined for any ring. The additive group of $\mathfrak{A}^L$ is that of A, while, for $a_1 \in A$, $a_2 \in \mathfrak{A}^L$, $a_1 a_2$ is the element of $\mathfrak{A}^L$ formed by taking the product of a_1 and a_2 in A. Sub-modules of $\mathfrak{A}^L$ correspond to left ideals in A, and conversely. $\mathfrak{A}^L \times Z$ consists of all pairs $\{(a, n)\}$ where $a \in A$ and n is an integer. We define

$$(a, n) + (a', n') = (a + a', n + n'), \qquad a(a', n) = (aa' + na, 0).$$

Since $a(0, 1) = (a, 0)$, the element $(0, 1)$ is annihilated by no nonzero a.

We return to the proof of the theorem. Let us write $\mathfrak{A}^L$ as the direct sum of $\mathfrak{A}^L_T$ and a semi-simple sub-module $\mathfrak{B}$, where $\mathfrak{A}^L_T$ is the trivial sub-module of $\mathfrak{A}^L$, consisting of those $a \in A$ for which $Aa = \{0\}$. We shall show that $\mathfrak{A}^L_T = \{0\}$. If $\mathfrak{M}$ is any A-module, we write $\mathfrak{M} = \mathfrak{N} + \mathfrak{M}_T$, the sum being direct so that $\mathfrak{M}_T \cap \mathfrak{N} = \{0\}$. Since $\mathfrak{N}$ is a sub-module of $\mathfrak{M}$, and $\mathfrak{A}^L_T$ a subset of A, we have $\mathfrak{A}^L_T \mathfrak{N} \subset \mathfrak{N}$. However, $A(\mathfrak{A}^L_T \mathfrak{N}) = (A\mathfrak{A}^L_T)\mathfrak{N} = \{0\}$, since $A\mathfrak{A}^L_T = \{0\}$, so that $\mathfrak{A}^L_T \mathfrak{N} \subset \mathfrak{M}_T$. We have then $\mathfrak{A}^L_T \mathfrak{N} = \{0\}$. Since $\mathfrak{A}^L_T \mathfrak{M} = \mathfrak{A}^L_T \mathfrak{N} + \mathfrak{A}^L_T \mathfrak{M}_T$, $A\mathfrak{M}_T = \{0\}$ and therefore $\mathfrak{A}^L_T \mathfrak{M}_T = \{0\}$, so that $\mathfrak{A}^L_T \mathfrak{M} = \{0\}$ for every $\mathfrak{M}$. But we have already seen that $(0, 1) \in \mathfrak{A}^L \times Z$ is annihilated only by the zero element of A, whence it follows that $\mathfrak{A}^L_T = \{0\}$, which proves our assertion. This result under the condition of the theorem shows that $\mathfrak{A}^L$ is a semi-simple A-module.

We shall need the following two lemmas.

LEMMA I. *If A is a ring which satisfies the condition of the theorem, and $\mathfrak{A}$ is a two-sided ideal in A, then the factor ring $A/\mathfrak{A}$ also satisfies the condition of the theorem.*

Let π be the natural homomorphism of A onto $A/\mathfrak{A}$. If $\mathfrak{M}$ is

[5] The ring of integers, which does not satisfy the descending chain condition, is seen to be semi-simple by consideration of the cyclic groups of prime order as simple modules for this ring.

any $A/\mathfrak{A}$-module, we can consider $\mathfrak{M}$ as an A-module by writing $am=\pi(a)m$ for each $a\in A$, $m\in\mathfrak{M}$, and retaining the additive group structure of $\mathfrak{M}$. It is readily verified that the trivial A-sub-module of $\mathfrak{M}$ is the trivial $A/\mathfrak{A}$-sub-module; and that every A-sub-module of $\mathfrak{M}$ is an $A/\mathfrak{A}$-sub-module, and conversely. Writing $\mathfrak{M}=\mathfrak{M}_T+\mathfrak{N}$, where $\mathfrak{N}$ is a semi-simple A-sub-module, it is clear that $\mathfrak{N}$ is semi-simple as an $A/\mathfrak{A}$-sub-module, which proves the lemma.

LEMMA II. *Let A be a ring which satisfies the condition of the theorem. Furthermore, suppose that there exists an integer $q\neq 0$ such that $qA=\{0\}$. Then the ring A has a unit element.* (qA is the set of all elements of the form qa, $a\in A$.)

Form the module $\mathfrak{A}^L\times Z$. As an A-module, $\mathfrak{A}^L\times Z$ can be expressed as the direct sum $\mathfrak{A}_1+\mathfrak{A}_2$, where $\mathfrak{A}_1$ is the trivial sub-module of $\mathfrak{A}^L\times Z$. Suppose that the element $(0, 1)\in\mathfrak{A}^L\times Z$ decomposes into $(-a_0, 1-n)+(a_0, n)$, with $(-a_0, 1-n)\in\mathfrak{A}_1$, $(a_0, n)\in\mathfrak{A}_2$. We have $(0, qn)=(qa_0, qn)=q(a_0, n)$, which is in $\mathfrak{A}_2$ since (a_0, n) is. However, for each $a\in A$, $a(0, qn)=(qna, 0)=(0, 0)$, so that $(0, qn)\in\mathfrak{A}_1$. But then $(0, qn)=(0, 0)$ or $n=0$. The element $(-a_0, 1-n)=(-a_0, 1)$ is in $\mathfrak{A}_1$, so that $(-aa_0+a, 0)=a(-a_0, 1)=(0, 0)$ or $aa_0=a$ for all $a\in A$. Since $a(b-a_0b)=0$ for every a, $b\in A$, and since $\mathfrak{A}^L_T=\{0\}$, we have $a_0b=b$. Thus a_0 is a unit element for the ring A, concluding the proof of the lemma.

We shall now show that the ring A has a unit element in every case. We start by showing that the trivial sub-module $\mathfrak{A}_1$ of $\mathfrak{A}^L\times Z$ has at least two elements. If $\mathfrak{A}_1=\{0\}$, then $\mathfrak{A}^L\times Z$ is semi-simple. The set $(A, 0)$ is a proper sub-module of $\mathfrak{A}^L\times Z$; we can then write, as a direct sum, $\mathfrak{A}^L\times Z=(A, 0)+\mathfrak{A}_3$, with $\mathfrak{A}_3\neq\{0\}$. Since $\mathfrak{A}_3$ is a sub-module of $\mathfrak{A}^L\times Z$, we have $A\mathfrak{A}_3\subset\mathfrak{A}_3$. However every element of $A\mathfrak{A}_3$ has a zero in the second coordinate, so that $A\mathfrak{A}_3\subset(A, 0)$. But then $A\mathfrak{A}_3=\{0\}$, and therefore $\mathfrak{A}_3\subset\mathfrak{A}_1$, contradicting the supposition that $\mathfrak{A}_1=\{0\}$, and proving the assertion.

Now let (a', n') be any nonzero element of $\mathfrak{A}_1$, that is, $a(a', n')=(0, 0)$ or $aa'+n'a=0$ for all $a\in A$. Clearly n' is not zero, for, if it were, a' would be zero, contrary to the condition that (a', n') is not zero. If $n'a=0$ for all a, Lemma II shows that A has a unit element.

Let $\mathfrak{A}_0$ be the set of all $a\in A$ for which $n'a=0$, and let $\mathfrak{B}=n'A$. Clearly both are two-sided ideals in A. We assert that A is the direct sum of $\mathfrak{A}_0$ and $\mathfrak{B}$. We have already seen that $\mathfrak{A}^L$ is semi-simple; write $\mathfrak{A}^L$ as the direct sum, $\sum_{\lambda\in L}\mathfrak{A}_\lambda$, of simple left ideals. The mapping, $\mathfrak{A}_\lambda\rightarrow\mathfrak{A}_\lambda$, which sends $a\in\mathfrak{A}_\lambda$ into $n'a$, is an endomorphism of the simple module $\mathfrak{A}_\lambda$, which, by Schur's Lemma, is either zero or an automorphism. Let $L_0 \subset L$ be the set of all λ for which this endomorphism is zero. $\mathfrak{A}_0$ is then $\sum_{\lambda \in L_0} \mathfrak{A}_\lambda$. Furthermore,

$$\sum_{\lambda \in L-L_0} \mathfrak{A}_\lambda = n' \sum_{\lambda \in L-L_0} \mathfrak{A}_\lambda = n' \sum_{\lambda \in L} \mathfrak{A}_\lambda = n'A = \mathfrak{B}.$$

Thus, $A = \mathfrak{A}_0 + \mathfrak{B}$, the sum being direct.

Since both $\mathfrak{A}_0$ and $\mathfrak{B}$ are two-sided ideals, $\mathfrak{A}_0$ is isomorphic with the factor ring $A/\mathfrak{B}$. By Lemma I, $\mathfrak{A}_0$ then satisfies the conditions of Lemma II with $q = n'$, and hence $\mathfrak{A}_0$ has a unit element which we shall denote by e_0. Because of the two-sidedness of $\mathfrak{A}_0$ and $\mathfrak{B}$, we have $\mathfrak{A}_0\mathfrak{B} = \mathfrak{B}\mathfrak{A}_0 = \{0\}$ (since $\mathfrak{A}_0 \cap \mathfrak{B} = \{0\}$), so that it only remains for us to show that $\mathfrak{B}$ has a unit element. In the decomposition $A = \mathfrak{A}_0 + \mathfrak{B}$, we write $a' = \alpha + \beta$, $\alpha \in \mathfrak{A}_0$, $\beta \in \mathfrak{B}$. We have already seen that $aa' = -n'a$ for each $a \in A$. For any $b \in \mathfrak{B}$, we have $-n'b = ba' = b(\alpha+\beta) = b\beta$. Since $\beta \in \mathfrak{B}$, $e_1 \equiv -(1/n')\beta$ is defined, and is such that $be_1 = b$ for every $b \in \mathfrak{B}$. Exactly as in Lemma II, we have $be_1 = e_1 b$, so that e_1 is the desired unit element for $\mathfrak{B}$. It is clear that $e_0 + e_1$ is the unit element for the entire ring A.

Having established the existence of a unit element in A, we can easily derive that A satisfies the descending chain condition. Again we write the semi-simple module $\mathfrak{A}^L$ as the direct sum, $\sum_{\lambda \in L} \mathfrak{A}_\lambda$, of simple left ideals. The unit element then decomposes in the form $1 = \sum_{\lambda \in L} 1_\lambda$, with $1_\lambda = 0$ for all $\lambda \neq \lambda_i$, $1 \leqq i \leqq N$. But then $a = a \cdot 1 = \sum_{\lambda \in L} a \cdot 1_\lambda = \sum_{\lambda \in L} a_\lambda$, $a_\lambda = a \cdot 1_\lambda$, so that $a_\lambda = 0$ for all $\lambda \neq \lambda_i$ and all $a \in A$. This shows that $\mathfrak{A}_\lambda = \{o\}$ for $\lambda \neq \lambda_i$ so that $\mathfrak{A}^L$ is the direct sum of a finite number of simple left ideals, giving immediately the descending chain condition. The module $\mathfrak{A}^L$ is semi-simple. Because $1 \in \mathfrak{A}^L$, the annihilator of $\mathfrak{A}^L$ reduces to $\{0\}$ or $\mathfrak{A}^L$ is faithful. By Theorem II the ring A is semi-simple. This concludes the proof of Theorem III.

It is well known[6] that every left ideal in a semi-simple ring with the descending chain condition is principally generated by an idempotent. The method of proof of Theorem III enables us to prove the converse:

THEOREM IV. *If every left ideal in a ring A is principally generated by an idempotent, A is a semi-simple ring with the descending chain condition.*

We shall first show that A has a unit element. A being its own left ideal, there is an idempotent e such that $Ae = A$. Thus, given any $a \in A$, there is a $b \in A$ such that $a = be$. But then $ae = be^2 = be = a$, since e is idempotent. Thus e is a right unit element in A. Let now $\mathfrak{B}$ be

[6] N. Jacobson, *The theory of rings*, Mathematical Surveys, vol. 2, 1943, p. 65.

the set of all elements of the form $ea-a$. Since $be=b$, $\mathfrak{B}$ is clearly a left ideal so that there is an $f\in A$, with $f^2=f$ and $\mathfrak{B}=Af$. Since f is idempotent, it is clear that $f\in\mathfrak{B}$. There exists then an element $g\in A$ with $f=eg-g$. But $f=f^2=ff=f(eg-g)=0$ or $\mathfrak{B}=\{0\}$. Thus $ea=a$ or e is a unit element for A.

From the condition of the theorem, and from the existence of a unit element in A, it follows that $\mathfrak{A}^L$ is semi-simple. For, if $\mathfrak{A}$ is any left ideal, there is an idempotent f with $Af=\mathfrak{A}$. But then $\mathfrak{A}^L$ is the direct sum of Af and $A(e-f)$ so that $\mathfrak{A}^L$ is semi-simple. By the argument in the last paragraph of the proof of Theorem III, it follows that A is a semi-simple ring with the descending chain condition.

PRINCETON UNIVERSITY