A SECOND NOTE ON WEAK DIFFERENTIABILITY OF PETTIS INTEGRALS

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In a recent paper[1] the author proved that if Ω is any compact metric space containing non-denumerably many points and $C(\Omega)$ is the Banach space of all continuous functionals over Ω, then there is a Pettis integrable function from the unit interval to $C(\Omega)$ whose integral fails to be weakly differentiable on a set of positive measure. The purpose of this note is to obtain the same result, assuming only that Ω contains infinitely many points. This leads to a certain necessary and sufficient condition for weak differentiability of Pettis integrals.

The author's previous paper (cited above) will be referred to hereafter as Note I. It will be assumed that the reader is familiar with the notation, terminology, and results of that paper.

Let B be the non-dense perfect set described in Note I, and let $\overline{B}$ be its complement. Let the intervals of $\overline{B}$ be arranged in a sequence $I_1, I_2, I_3, \cdots$ in such a way that if the order of I_j is greater than the order of I_k, then $j>k$. For each positive integer k, let $n(k)$ be the order of I_k. As in Note I, we now define a function $\phi(x, t)$ over the unit square so that for each x, $\phi(x, t)$ is continuous in t. This will serve to define a function from the unit interval to the space C.

For $x \in B$, let $\phi(x, t)=0$; for $x\in I_1$, let $\phi(x, t)=0$; for $x\in I_k$ and $t=1/k$ $(k=2, 3, 4, \cdots)$, let $\phi(x, t)=2^{2n(k)}/n(k)$; for $x\in I_k$ and $t=1/(k\pm 1/2)$ $(k=2, 3, 4, \cdots)$, let $\phi(x, t)=0$; for $t=0$ or 1, let $\phi(x, t)=0$. Now for each x, let $\phi(x, t)$ be extended linearly as a function of t between successive points already determined. We now denote by $\phi(x)$ the function whose values are the elements of C determined by $\phi(x, t)$.

THEOREM 1. *$\phi(x)$ is Pettis integrable. Its integral is the element of C*

$$\Phi_E(t) = \int_E \phi(x, t)dx.$$

That $\Phi_E(t)$ is defined and continuous in t is obvious. For $t=1/k$, $\Phi_E(t) = |E\cdot I_k| 2^{2n(k)}/n(k) = |E\cdot I_k|/|I_k| n(k) \leq 1/n(k)$; for

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[1] M. E. Munroe, *A note on weak differentiability of Pettis integrals*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 167–174.

$t=1/(k\pm 1/2)$, $\Phi_E(t)=0$. $\Phi_E(t)$ is linear between successive points in this sequence, and $\Phi_E(0)=0$.

To show that $\Phi_E(t)$ is the integral of $\phi(x)$, we introduce the functions

$$\phi^{(k)}(x, t) = \begin{cases} \phi(x, t) & \text{for} \quad t \geqq 1/(k+1/2), \\ 0 & \text{otherwise.} \end{cases}$$

For each x, $\phi^{(k)}(x, t) \to \phi(x, t)$ uniformly in t. Furthermore, the functions $\phi^{(k)}(x)$ determined by $\phi^{(k)}(x, t)$ are step functions, hence Pettis integrable. Therefore,[2] their integrals are the elements of C

$$\Phi_E^{(k)}(t) = \int_E \phi^{(k)}(x, t)dx.$$

Now

$$\begin{aligned} \left| \Phi_E(t) - \Phi_E^{(k)}(t) \right| &= \int_E [\phi(x, t) - \phi^{(k)}(x, t)]dx \\ &= \begin{cases} \Phi_E(t) & \text{for} \quad t \leqq 1/(k+1/2) \\ 0 & \text{for} \quad t > 1/(k+1/2) \end{cases} \\ &\leqq 1/n(k) \quad \text{for all} \quad t. \end{aligned}$$

Thus $\Phi_E^{(k)}(t) \to \Phi_E(t)$ uniformly in t, and Theorem 1 follows from a theorem of Pettis.[3]

THEOREM 2. *The function*

$$\Phi(E) = \int_E \phi(x)dx$$

is not weakly differentiable at any point of B.

Let $x_0 \in B$, and let J_n be an interval having x_0 as its center and containing an interval of $\overline{B}$ of order n. This may be done with

$$|J_n| \leqq 2|G_n| = 2(1/2^n + 1/2^{2n-1}) < 1/2^{n-2}$$

where G_n is a gap of order n (see Note I). Let I_k be the interval of $\overline{B}$ of order n contained in J_n; then

$$\frac{\|\Phi(J_n)\|}{|J_n|} \geqq \frac{|\Phi_{J_n}(1/k)|}{|J_n|} \geqq \frac{1/n}{|J_n|} > \frac{2^{n-2}}{n}.$$

[2] See Note I, footnote 5.

[3] B. J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. vol. 44 (1938) pp. 277–304. See Theorem 4.1.

Thus the norm of the difference quotient is unbounded for each $x_0 \in B$, and the failure of weak differentiability follows from a theorem of Banach.[4]

THEOREM 3. *If Ω is a compact metric space containing infinitely many points and if $C(\Omega)$ is the space of all continuous functionals defined over Ω, then there is a Pettis integrable function from the unit interval to $C(\Omega)$ whose integral fails to be weakly differentiable on a set of positive measure.*

In this case Ω contains a homeomorph of the sequence $\{1/k\}$. The argument used in Theorem 4.1 of Note I now applies to the present theorem. We shall not bother to repeat it here.

THEOREM 4. *If M is an abstract M-space with unity,[5] a necessary and sufficient condition that every Pettis integral defined to M be almost everywhere weakly differentiable is that M be finite-dimensional.*

Obviously finite-dimensionality is sufficient. If M is infinite-dimensional, it contains an infinite-dimensional separable subspace which, by a theorem of Kakutani,[6] is equivalent to the space $C(\Omega)$ of Theorem 3. Hence, finite-dimensionality is also necessary.

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[4] S. Banach, *Théorie des opérations linéaires*, Monografje Matematyczne, vol. 1, Warsaw, 1932. See p. 224, Theorem 8.

[5] For a definition of this, see S. Kakutani, *Concrete representations of abstract (M)-spaces*, Ann. of Math. vol. 42 (1941) pp. 994–1024.

[6] Loc. cit. Theorem 2, p. 998.