ON THE EXTENSION OF HOMEOMORPHISMS ON THE INTERIOR OF A TWO CELL

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The subject under discussion in this paper is the study of the existence and properties of extensions of homeomorphisms of the interior I of a two cell with boundary C onto a plane bounded region. Particular emphasis will be placed on the action of the extension on C. Application of the topological results will then be made to conformal maps on the interior of the unit circle.

The hypothesis that $f(I)=R$ is a homeomorphism of the interior I of a two cell with boundary C onto a plane bounded region R with boundary $F(R)$ will be assumed throughout the paper. The usual terminology of transformation theory will be used: the transformation $g(A)=B$ is said to be light if each $f^{-1}(x)$, $x\in B$, is totally disconnected, and non-alternating if for each x, $y\in B$, $f^{-1}(x)$ does not separate $f^{-1}(y)$.[1]

1. Action of extensions on the boundary.

THEOREM 1. *Suppose f is uniformly continuous. Then there exists a continuous extension g of f such that $g(\bar{I})=\bar{R}$ and $g=f$ on I. Moreover $g(C)=F(R)$ is a non-alternating transformation.*

PROOF. The existence of the extension is well known, since f is uniformly continuous. Moreover $g(C)=F(R)$. To prove this, we notice that $g(\bar{I})$ is compact and must contain $\bar{R}$. Since $g(I)=R$, then $g(C)\supset F(R)$. Suppose $g(C)\neq F(R)$; then there is a point $x\in C$ such that $g(x)\in R$. Let $(x_i)\to x$, $x_i\in I$; then $(f(x_i))\to g(x)$. Since $g(x)\in R$, then $(x_i)\to f^{-1}g(x)\in I$. This is a contradiction and $g(C)=F(R)$.

Suppose $g(C)=F(R)$ is not non-alternating; then there exist points x_1, x_2, y_1, $y_2\in C$ such that $g(x_1)=g(x_2)$, $g(y_1)=g(y_2)$, $g(x_1)\neq g(y_1)$, and x_1+x_2 separates y_1+y_2 on C. Let A_1 and A_2 be interiors of arcs x_1x_2 and y_1y_2 respectively, where $x_1x_2\subset I+x_1+x_2$, $y_1y_2\subset I+y_1+y_2$, $x_1x_2\cdot y_1y_2=p=A_1A_2$. Both $g(x_1x_2)$ and $g(y_1y_2)$ are simple closed curves and $g(x_1x_2)\cdot g(y_1y_2)=f(p)$. Moreover points of $g(y_1y_2)$ are contained both in the interior and exterior of $g(x_1x_2)$. For A_1 separates A_2 into two parts, one in each component of $I-A_1$; then $f(A_1)$ separates $f(A_2)$ into two parts, one in each component of $R-f(A_1)$. But one compo-

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[1] See G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, New York, 1942, pp. 127–129, 138–140, 165–170 for properties of non-alternating maps.

nent of $R-f(A_1)$ is contained in the interior of $f(A_1)+g(x)$, the other in the exterior. This furnishes a contradiction, since $f(p)$ would then be a cut point of $g(y_1y_2)$, and the theorem is proved.

DEFINITION. Let R be a plane region with boundary $F(R)$; we say that A is a *cut into* R if $A\subset R$ and if there exists a point $x\in F(R)$ such that $A+x$ is an arc. We shall call x the *end* of the cut A.

DEFINITION. We say that f possesses *property* P if

(1) A is a cut into R implies $f^{-1}(A)$ is a cut into I, and if

(2) A_1 and A_2 being cuts into R with distinct ends implies $f^{-1}(A_1)$ and $f^{-1}(A_2)$ are cuts into I with distinct ends.

LEMMA 1. *A necessary and sufficient condition that the boundary $F(R)$ of a bounded simply connected plane region R be locally connected is that for each sequence (x_i), $x_i\in R$, such that (x_i) converges to x, $x\in F(R)$, there exists an infinite subsequence (y_j) of (x_i) and an arc a_1x, $a_1x\subset R+x$, such that $a_1x\supset(y_j)$.*

PROOF. Let R be a bounded simply connected plane region with locally connected boundary $F(R)$. To show the necessity portion of the lemma, we proceed to show two preliminary statements.

(1) There exists a sequence (R_i) of regions contained in R such that $R_{i+1}\subset R_i$, each R_i possesses property S,[2] each R_i contains an infinite number of points of (x_i), and the diameter of R_i goes to 0 with $1/i$.

To prove (1), we note first that R possesses property S, since $F(R)$ is locally connected.[3] Then R may be decomposed into a finite number of regions, each with property S and each of diameter less than 1/2.[4] One of these, say R_1, contains an infinite number of points of (x_i) Then R_1 possesses property S and hence may be decomposed into a finite number of regions possessing property S and of diameter less than 1/4. One of these, say R_2, contains an infinite number of points of (x_i). By continuing the process, we obtain the required (R_i).

(2) Let $A=pq-p$ be a cut into a region R_i of (1), $p\in R\cdot F(R_i)$, $q\in R_i$. Let $R_j\subset R_i$ be such that $\overline{R}_j\cdot A=0$, and let $r\in R_j$. Then there exists an arc qr, $qr\subset(R_i-A)+q$, such that $qr\cdot F(R_j)$ consists of a single point, where $F(R_n)$ is used to denote the boundary of R_n.

To prove (2) we note first that R_i-A is connected, since a cut into a region does not disconnect it. Furthermore q is accessible from R_i-A since a sufficiently small neighborhood of q is contained in R_i-A except for points of the arc pq. Let qz be an arc contained in

[2] R. L. Moore, *Concerning connectedness im kleinen and a related property*, Fund. Math. vol. 3 (1922) pp. 232–237.

[3] R. L. Moore, loc. cit. p. 235.

[4] G. T. Whyburn, loc. cit. p. 21.

$(R_i-A)+q$, where $z\in R_j$. Let y be the first point of $qz\cdot F(R_j)$ in the order q, z. Then $qy\subset(R_i-R_i\cdot\overline{R}_j)+y$, $y\in F(R_j)$. Since R_j possesses property S, each boundary point is accessible from R.[5] Then let yr be an arc, $yr\subset R_j+y$. Then $qr=qy+yr$ is the desired arc, and (2) is justified.

We now proceed to show the necessity portion of the lemma. Let $y_1\in R_1\cdot(x_i)$. Let R_2' be such that R_2' is a member of the sequence (R_i) and $y_1\notin\overline{R}_2'$. Such an R_2' exists since R_i is arbitrarily close to $x\in F(R)$ for i sufficiently large. Let $y_2\in R_2'\cdot(x_i)$. Let $y_1y_2=A$ be an arc contained in R and such that $A\cdot F(R_2')$ consists of a single point. Such an arc may be constructed by an argument similar to the one used in (2). Let R_3' be such that R_3' is a member of (R_i) and $y_1y_2\cdot\overline{R}_3'=0$; let $y_3\in R_3'\cdot(x_i)$. Since $R_2'\cdot y_1y_2$ is a cut into R_2' we may use (2) to obtain an arc y_2y_3 contained in $(R_2'-R_2'\cdot y_1y_2)+y_2$ and intersecting $F(R_3')$ in a single point. Let R_4' be a member of (R_i) such that $\overline{R}_4'\cdot(y_1y_2+y_2y_3)=0$, let $y_4\in R_4'\cdot(x_i)$, and continue the process indefinitely. We thus obtain a sequence $(y_{i-1}y_i)$ of arcs, $y_{i-1}y_i\subset R$, $y_i\in(x_i)$, $y_{i-1}y_i$ intersects $y_1y_2+y_2y_3+\cdots+y_{i-2}y_{i-1}$ in the single point y_{i-1}, and $y_{i-1}y_i$ approaches x as a limit. Let $A=\sum_{i=2}^{\infty}y_{i-1}y_i+x$. Then every point of A except y_1 and x are cut points. Moreover, A is compact, since the arcs $y_{i-1}y_i$ approach x as limit. Then A is an arc satisfying all the conditions, and the necessity is shown.

Suppose for each sequence the condition of the lemma is satisfied and suppose $F(R)$ is not locally connected. Then there is a point $x\in F(R)$ such that x is not regularly accessible from R.[6] That is, for some $e>0$ there exists a sequence (x_i), $x_i\in R$ and $(x_i)\rightarrow x$, such that x_i may not be joined to x by an arc of diameter less than e in $R+x$. By hypothesis, there exists an arc $a_1x\subset R+x$ such that a_1x contains an infinite subsequence of (x_i). This arc may be supposed to be of diameter less than e. This is a contradiction and the theorem is proved.

LEMMA 2. *If $F(R)$ is locally connected and f possesses property* P, *then f is uniformly continuous.*

PROOF. Suppose f is not uniformly continuous. Then for some $e>0$ and each d_i of a sequence $(d_i)\rightarrow 0$, there exist x_i, $y_i\in I$ such that $\rho(x_i, y_i)<d_i$ and $\rho(f(x_i), f(y_i))\geqq e$. We may suppose that $(x_i)\rightarrow x$; then $(y_i)\rightarrow x$ and $x\in C$. We may also suppose that $(f(x_i))\rightarrow z_1$ and $(f(y_i))\rightarrow z_2$. Then $z_1\neq z_2$. Moreover $z_1\in F(R)$ and $z_2\in F(R)$. For suppose $z_1\in R$; then $(x_i)\rightarrow f^{-1}(z_1)\in I$. But this is impossible; thus z_1, $z_2\in F(R)$. By Lemma 1 there exist cuts A_1 and A_2 which con-

[5] G. T. Whyburn, loc. cit. p. 111.

[6] G. T. Whyburn, loc. cit. p. 112.

tain infinite subsequences of $(f(x_i))$ and $(f(y_i))$ respectively and which have z_1 and z_2 as ends. By property P, $f^{-1}(A_1)$ and $f^{-1}(A_2)$ are cuts into I with different ends. Since each of them must have x as an end, this is a contradiction and the lemma is established.

THEOREM 2. *Let $F(R)$ be locally connected. Then a necessary and sufficient condition that f have property* P *is that there exist an extension g of f, where $g(\bar{I})=\bar{R}$, $g=f$ on I, and $g(C)=F(R)$ is light and non-alternating.*

PROOF. Suppose f has property P. Then by Lemma 2 and Theorem 1, g exists and $g(C)=F(R)$ is non-alternating. Suppose $g(C)=F(R)$ is not light; let $x\in F(R)$ be such that $g^{-1}(x)$ is not totally disconnected. Let M be a nondegenerate component of $g^{-1}(x)$. Since $I+C$ is a two cell, there exists a set $N\subset I$ which is homeomorphic to the graph of the equation $y=\sin 1/x$, $0<x\leqq 1$, and such that $\bar{N}=N+M$. Then $f(N)$ is a cut into R with end x. Since $f^{-1}f(N)=N$ is not a cut into I, then f does not possess property P, and we have a contradiction. This shows the necessity of the theorem.

Suppose f may be extended to g, where $g(\bar{I})=\bar{R}$, $g=f$ on I, and $g(C)=F(R)$ is light and non-alternating. Let A be a cut into R with end x. Suppose $A=a_1x-x$. Let $(x_i)\to x$, where $x_i\in A$ and where x_i precedes x_{i+1} on the arc a_1x ordered from a_1 to x. We may also suppose that $(f^{-1}(x_i))$ converges to a point y. Let $A_i=x_ix_{i+1}$ be the subarc of A joining x_i to x_{i+1}. Then $\limsup f^{-1}(A_i)\subset g^{-1}(x)$ by the continuity of g. Since $\liminf f^{-1}(A_i)\supset y$, then $\liminf f^{-1}(A_i)\neq 0$ and $\limsup f^{-1}(A_i)$ is connected.[7] Let $B=f^{-1}(A)$. Then $\limsup f^{-1}(A_i) =\bar{B}-B$. Then $\bar{B}-B$ is both connected and totally disconnected, and hence is a single point. Then $B=f^{-1}(A)$ is a cut into I.

Suppose A_1 and A_2 are cuts into R with ends x_1 and x_2 respectively, $x_1\neq x_2$, while $f^{-1}(A_1)$ and $f^{-1}(A_2)$ are cuts into I, both with end y. Then $g(f^{-1}(A_1))=A_1$ and $g(f^{-1}(A_2))=A_2$ are cuts into I, both with end $g(y)$. This is a contradiction and the theorem is established.

2. Application to conformal maps.

THEOREM 3. *Let I be the interior of the unit circle, and let f be a one-to-one conformal map. Then f possesses property* P.[8]

COROLLARY 3.1. *Let $f(I)=R$ be a one-to-one conformal map of the interior I of the unit circle C onto a bounded plane region R with bound-*

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[7] G. T. Whyburn, loc. cit. p. 14.

[8] For proof, see C. Carathéodory, *Conformal representation*, London, 1932, pp. 82–85.

ary $F(R)$. Then a necessary and sufficient condition that f be extensible to g on $\bar{I}$ is that $F(R)$ be locally connected. In case $F(R)$ is locally connected, and if g denotes the extension, then the mapping $g(C) = F(R)$ is light and non-alternating.

PROOF. The sufficiency follows from Theorem 3 and Theorem 2. If f is extensible to g on $\bar{I}$, then $g(C) = F(R)$ gives a continuous map of the locally connected continuum C onto $F(R)$. Then $F(R)$ itself must be locally connected.

COROLLARY 3.2. *Let $f(I) = R$ be a one-to-one conformal map of the interior I of the unit circle onto a bounded plane region R with boundary $F(R)$. Then a necessary and sufficient condition that f be uniformly continuous is that $F(R)$ be locally connected.*

PROOF. If f is uniformly continuous, then f is extensible to g on $\bar{I}$ and $F(R)$ is locally connected by 3.1. If $F(R)$ is locally connected, then f is extensible to g on $\bar{I}$ by 3.1. Then g is uniformly continuous and so is f.

COROLLARY 3.3. THE OSGOOD-CARATHÉODORY THEOREM. *Let $f(I) = R$ be a one-to-one conformal map of the interior I of the unit circle C onto the interior R of a simple closed curve J. Then f may be extended to g on $\bar{I}$ such that $g(\bar{I}) = \bar{R}$ is a homeomorphism.*

PROOF. The mapping $g(C) = J$ is light and non-alternating; the only light and non-alternating transformation of one simple closed curve onto another is a homeomorphism.[9] Hence $g(\bar{I}) = \bar{R}$ is a homeomorphism.

Note. Another example of a map with property P is the inverse of the relative distance transformation[10] on a plane bounded region with property S. The following topological theorem is usually proven by means of this transformation:[11] *if B is any boundary curve, there exists a light and non-alternating transformation $g(C) = B$, where C is a simple closed curve.* If B is restricted to a boundary curve which contains at least one simple closed curve, we may prove the theorem by means of conformal maps as follows. We may suppose that B is the boundary of a bounded plane region R. Then there exists a 1-1 conformal map f of the interior of the unit circle onto R, since R is simply connected. Then f is extensible to g on $\bar{I}$ and $g(C) = B$ is light and non-alternating.

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[9] G. T. Whyburn, loc. cit. p. 165.
[10] G. T. Whyburn, loc. cit. pp. 155–162.
[11] G. T. Whyburn, loc. cit. p. 166.