ON THE ZEROS OF POLYNOMIALS WITH COMPLEX COEFFICIENTS[1]

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1. **Introduction.** Problems in dynamics very frequently have physically realizable solutions only if the determinantal equation of the system has all its roots in the negative half of the complex plane. It is therefore convenient to have a simple algorithm for testing whether this condition holds without actually computing the roots. Solutions to this problem have been considered by Cauchy [1],[2] Routh [6], and many others. Hurwitz [4] gave a method for polynomials with *real* coefficients of the form

$$(1.1) \qquad P(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n.$$

According to his rule, all of the roots lie in the half-plane $R(z) < 0$ if and only if all the determinants

$$D_p = \begin{vmatrix} a_1, & a_3, & a_5, & \cdots, & a_{2p-1} \\ 1, & a_2, & a_4, & \cdots, & a_{2p-2} \\ 0, & a_1, & a_3, & \cdots, & a_{2p-3} \\ 0, & 1, & a_2, & \cdots, & a_{2p-4} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_p \end{vmatrix}, \qquad p = 1, 2, \cdots, n, \; a_j = 0, \; j > n,$$

are positive.

Recently, Wall [8] formulated and proved this theorem by means of continued fractions. We extend his method to polynomials with *complex* coefficients,

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The referee has kindly called my attention to a recent article by Herbert Bilharz, *Bemerkung zu einem Satze von Hurwitz*, Zeitschrift für Angewandte Mathematik und Mechanik vol. 24 (1944) pp. 77–82 (lithoprinted by Edwards Brothers, Inc., Ann Arbor, Mich., 1945). There is presented in Bilharz' article an algorithm for the computation of determinants of type D_p similar to that given here in §2. Also Theorem 3.2 is essentially the same as the theorem stated and proved by Bilharz (p. 81), and Theorem 4.1 is equivalent to but approached differently from that stated by Bilharz without detailed proof.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

$$(1.2)\qquad P(z) = z^n + \alpha_1 z^{n-1} + \alpha_2 z^{n-2} + \cdots + \alpha_n,\ \alpha_k = p_k + iq_k,$$

$k=1, 2, \cdots, n$. Therefore, by a rotation and translation, the results can be applied in an *arbitrary* half-plane. We form

$$(1.3)\qquad Q(z) = p_1 z^{n-1} + iq_2 z^{n-2} + p_3 z^{n-3} + iq_4 z^{n-4} + \cdots$$

and the J-fraction

$$(1.4)\quad \frac{Q(z)}{P(z)} = \frac{1}{c_1 z + k_1 + 1} + \frac{1}{c_2 z + k_2} + \frac{1}{c_3 z + k_3} + \cdots + \frac{1}{c_n z + k_n},$$

where the c_p are real and the k_p are pure imaginary or zero. We find that all the zeros of $P(z)$ have negative real parts if and only if the expansion (1.4) exists and the c_p are positive (Theorem 3.1). Moreover, if this expansion exists with k of the c_p negative and $(n-k)$ positive, then k of the zeros of $P(z)$ have positive real parts and $(n-k)$ have negative real parts (Theorem 4.1). We find the proofs of these theorems carry over with no basic changes from those given by Wall [8] for the case of real polynomials, and at one step the proof of Theorem 3.1 is even simpler in the complex case.

We give in §2 some convenient formulas for expanding a rational function into a continued fraction of the form (1.4). This leads to formulations of the preceding theorems by means of determinants analogous to the Hurwitz determinants.

In §5, we give methods for modifying Theorem 4.1 in case the expansion (1.4) fails to exist.

In §6, we obtain formulas similar to those in [8] for finding bounds for the moduli of the zeros of (1.2).

2. **Expansion of a rational function into a J-fraction.** We consider here the following problem: If

$$(2.1)\qquad \begin{aligned} f_0 &= \alpha_{00} z^n + \alpha_{01} z^{n-1} + \cdots + \alpha_{0n}, \\ f_1 &= \alpha_{11} z^{n-1} + \alpha_{12} z^{n-2} + \cdots + \alpha_{1n} \end{aligned}$$

are two polynomials of degree n and $n-1$ respectively, *to determine conditions upon the coefficients* $\alpha_{00}, \cdots, \alpha_{0n}, \alpha_{11}, \cdots, \alpha_{1n}$ *which are necessary and sufficient in order that*

$$(2.2)\qquad \frac{f_1}{f_0} = \frac{1}{r_1 z + s_1} + \frac{1}{r_2 z + s_2} + \cdots + \frac{1}{r_n z + s_n},$$

where the r_p *and* s_p *are constants, the* r_p *different from zero.* This problem is equivalent to the problem of determining polynomials f_p of degree $n-p$, $p=2, 3, \cdots, n-1$, which are connected with f_0 and f_1

by the recurrence relations

$$
(2.3)\qquad \begin{aligned} &f_{p-1} = (r_p z + s_p) f_p + f_{p+1}, && p = 1, 2, \cdots, n, \\ &f_{n+1} = 0, \quad f_n = \alpha_{nn} \neq 0, && r_p \neq 0. \end{aligned}
$$

In other words, the expansion (2.2) exists if and only if the euclidean algorithm for the highest common factor of two polynomials, when applied to f_0 and f_1, gives a system of the form (2.3).

If we examine the long division process involved in the euclidean algorithm, we see that the numbers which contribute to the final result are only those contained in the following table.

$$
(2.4)\qquad \begin{array}{llll}
\alpha_{00} & \alpha_{01} & \alpha_{02} & \cdots \\
\alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots \\
\beta_{11}=\dfrac{\alpha_{11}\alpha_{01}-\alpha_{00}\alpha_{12}}{\alpha_{11}} & \beta_{12}=\dfrac{\alpha_{11}\alpha_{02}-\alpha_{00}\alpha_{13}}{\alpha_{11}} & \beta_{13}=\dfrac{\alpha_{11}\alpha_{03}-\alpha_{00}\alpha_{14}}{\alpha_{11}} & \cdots \\
\alpha_{22}=\dfrac{\alpha_{11}\beta_{12}-\beta_{11}\alpha_{12}}{\alpha_{11}} & \alpha_{23}=\dfrac{\alpha_{11}\beta_{13}-\beta_{11}\alpha_{13}}{\alpha_{11}} & \alpha_{24}=\dfrac{\alpha_{11}\beta_{14}-\beta_{11}\alpha_{14}}{\alpha_{11}} & \cdots \\
\beta_{22}=\dfrac{\alpha_{22}\alpha_{12}-\alpha_{11}\alpha_{23}}{\alpha_{22}} & \beta_{23}=\dfrac{\alpha_{22}\alpha_{13}-\alpha_{11}\alpha_{24}}{\alpha_{22}} & \beta_{24}=\dfrac{\alpha_{22}\alpha_{14}-\alpha_{11}\alpha_{25}}{\alpha_{22}} & \cdots \\
\alpha_{33}=\dfrac{\alpha_{22}\beta_{23}-\beta_{22}\alpha_{23}}{\alpha_{22}} & \alpha_{34}=\dfrac{\alpha_{22}\beta_{24}-\beta_{22}\alpha_{24}}{\alpha_{22}} & \alpha_{35}=\dfrac{\alpha_{22}\beta_{25}-\beta_{22}\alpha_{25}}{\alpha_{22}} & \cdots \\
\cdots & \cdots & \cdots & \cdots .
\end{array}
$$

The expansion (2.2) exists if and only if the numbers α_{00}, α_{11}, α_{22}, $\cdots$, α_{nn} are different from zero. When it exists, we have

$$
(2.5)\qquad r_p = \frac{\alpha_{p-1,p-1}}{\alpha_{p,p}}, \qquad s_p = \frac{\beta_{p,p}}{\alpha_{p,p}}, \qquad p = 1, 2, \cdots, n.
$$

Example. Let $f_0 = z^3 + (2+i)z^2 + (3+i)z + (2i+2)$, $f_1 = 2z^2 + iz + 2$. The table (2.4) in this case is

1	$2+i$	$3+i$	$2i+2$
2	i	2	
$2+i/2$	$2+i$	$2i+2$	
$9/4$	$3i/2$		
$-i/3$	2		
$16/9$			
$3i/2$			

Therefore,

$$r_1 = 1/2, \qquad r_2 = 8/9 \qquad r_3 = 81/64,$$
$$s_1 = 1 + i/4, \qquad s_2 = -\,4i/27, \qquad s_3 = 27i/32,$$

and the expansion (2.2) is

$$\frac{f_1}{f_0} = \frac{1}{z/2 + 1 + i/4 +} \; \frac{1}{8z/9 - 4i/27 +} \; \frac{1}{81z/64 + 27i/32}\,. \tag{2.6}$$

We now formulate the condition for the existence of (2.2) in terms of certain determinants.

THEOREM 2.1. *The quotient f_1/f_0 of two polynomials* (2.1) *can be expressed in the form* (2.2) *if and only if*

$$D_p \neq 0, \qquad p = 0, 1, \cdots, n, \tag{2.7}$$

where $D_0 = \alpha_{00}$ and $D_1, D_2, \cdots, D_n$ are the first n principal minors of odd order (blocked off by lines) in the array

$$\begin{array}{c|cc|cc|cc} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} & \cdots \\ \alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \cdots \\ 0 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \cdots \\ 0 & \alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \cdots \\ 0 & 0 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \cdots \\ 0 & 0 & \alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \tag{2.8}$$

where $\alpha_{0p} = \alpha_{1p} = 0$ if $p > n$.

PROOF. We suppose first that the expansion (2.2) exists with $r_p \neq 0$, $p = 1, 2, \cdots, n$, so that the numbers α_{pp}, $p = 0, 1, 2, \cdots$, of (2.4) are not zero. Consider the determinant D_p of order $2p-1$, $2 \leqq p \leqq n$. If we subtract α_{00}/α_{11} times the $(2k-1)$th row from the $2k$th row, for $k = 1, 2, \cdots, p-1$, we find with the aid of (2.4) that

$$D_p = \alpha_{11} \begin{vmatrix} \beta_{11}, & \beta_{12} & \beta_{13}, & \cdots \\ \alpha_{11}, & \alpha_{12}, & \alpha_{13}, & \cdots \\ 0, & \beta_{11}, & \beta_{12}, & \cdots \\ 0, & \alpha_{11}, & \alpha_{12}, & \cdots \\ 0, & 0, & \beta_{11}, & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

where the new determinant is of order $2p-2$. On subtracting β_{11}/α_{11} times the $2k$th row from the $(2k-1)$th row, for $k=1, 2, \cdots, p-1$, and making use of (2.4), we readily obtain

$$(2.9) \qquad D_p = (-1)^{p-1}\alpha_{11}^2 D_{p-1}^{(1)}, \qquad p = 2, 3, \cdots, n,$$

where $D_r^{(k)}$ denotes the determinant D_r with both the subscripts of all its elements increased by k. From (2.9) we then find immediately that

$$(2.10) \quad D_p = (-1)^{p(p-1)/2}\alpha_{11}^2\alpha_{22}^2 \cdots \alpha_{p-1,p-1}^2\alpha_{pp}, \quad p = 2, 3, \cdots, n.$$

Since $\alpha_{pp} \neq 0$, $p=0, 1, \cdots, n$, it follows from (2.10) that (2.7) holds.

We suppose now, conversely, that (2.7) holds. Then, $\alpha_{00} \neq 0$, $\alpha_{11} \neq 0$, since, by definition, $D_0 = \alpha_{00}$, $D_1 = \alpha_{11}$. Since $\alpha_{11} \neq 0$, then (2.10) holds for $p=2$, so that $D_2 = -\alpha_{11}^2\alpha_{22} \neq 0$, or $\alpha_{22} \neq 0$. This guarantees that (2.10) holds for $p=3$, so that $D_3 = -\alpha_{11}^2\alpha_{22}^2\alpha_{33} \neq 0$, or $\alpha_{33} \neq 0$. On continuing this argument, we finally arrive at $\alpha_{nn} \neq 0$, and the proof of Theorem 2.1 is complete.

We observe that if $f_1 = c_0z^{n-1} + c_1z^{n-2} + \cdots + c_{n-1}$, $f_0 = z^n$, then the condition of Theorem 2.1 reduces to

$$D_p = \begin{vmatrix} c_0, & c_1, & \cdots, & c_p \\ c_1, & c_2, & \cdots, & c_{p+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_p, & c_{p+1}, & \cdots, & c_{2p} \end{vmatrix} \neq 0,$$

$$p = 0, 1, \cdots, n-1 \ (c_p = 0 \text{ for } p > n-1).$$

This leads to the well known condition for a power series

$$\frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \cdots + \frac{c_{n-1}}{z^n} + \cdots$$

to have a J-fraction expansion. We may obtain this expansion from formulas (2.4), (2.5) if we take $\alpha_{00}=1$, $\alpha_{0p}=0$, $p \geqq 1$; $\alpha_{1p} = c_{p-1}$.

Analogous considerations show that there exists a Stieltjes expansion [7] of the form[3]

$$(2.11) \quad \frac{f_1}{f_0} = \frac{1}{d_1z} + \frac{1}{d_2} + \frac{1}{d_3z} + \cdots + \frac{1}{T}, \quad T = \begin{cases} d_{2n-1}z & \text{if } f_0(z) = 0, \\ d_{2n} & \text{if } f_0(0) \neq 0, \end{cases}$$

[3] Expansions for rational functions of the form (2.2) and (2.11) find application in certain problems in electrical network theory (cf. [2, 3]).

where $d_p \neq 0$, $p=1, 2, 3, \cdots$, if and only if, in addition to condition (2.7), it is required that the principal minors of *even* order in the array (2.8) are different from zero up to and including the one of order $2n-2$ or $2n$, according as $f_0(0)=0$ or $f_0(0)\neq 0$, respectively. The coefficients d_p in (2.11) can be computed by forming the table

$$
\begin{array}{llll}
\alpha_0 & \alpha_{01} & \alpha_{02} & \cdots \\
\alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots \\
\alpha_{22}=\dfrac{\alpha_{11}\alpha_{01}-\alpha_{00}\alpha_{12}}{\alpha_{11}} & \alpha_{23}=\dfrac{\alpha_{11}\alpha_{02}-\alpha_{00}\alpha_{13}}{\alpha_{11}} & \alpha_{24}=\dfrac{\alpha_{11}\alpha_{03}-\alpha_{00}\alpha_{14}}{\alpha_{11}} & \cdots \\
\alpha_{33}=\dfrac{\alpha_{22}\alpha_{12}-\alpha_{11}\alpha_{23}}{\alpha_{22}} & \alpha_{34}=\dfrac{\alpha_{22}\alpha_{13}-\alpha_{11}\alpha_{24}}{\alpha_{22}} & \alpha_{35}=\dfrac{\alpha_{22}\alpha_{14}-\alpha_{11}\alpha_{25}}{\alpha_{22}} & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{array} \tag{2.12}
$$

Then

$$d_p = \frac{\alpha_{p-1,p-1}}{\alpha_{pp}}, \qquad p = 1, 2, 3, \cdots. \tag{2.13}$$

This may be shown if we apply Theorem 2.1 to the function $zf_1(z^2)/f_0(z^2)$. Since this is an *odd* function, its expansion (2.2) will have $s_p=0$, $p=1, 2, 3, \cdots$. From this, (2.11) can be obtained by simple transformations.

3. Conditions for the zeros of a polynomial to lie in a half-plane.

There is no loss in generality if we assume that the given half-plane is $R(z)<0$ since any half-plane can be reduced to this by a rotation and translation.

THEOREM 3.1. *Let $P(z)$ be a polynomial with complex coefficients (1.2), and form $Q(z)$ (1.3). The zeros of $P(z)$ all lie in the half-plane $R(z)<0$ if and only if*

$$\frac{Q(z)}{P(z)} = \frac{a_0}{z+a_0+b_1} + \frac{a_1}{z+b_2} + \frac{a_2}{z+b_3} + \cdots + \frac{a_{n-1}}{z+b_n}, \tag{3.1}$$

where the a_p are real and positive and the b_p are pure imaginary or zero.

PROOF. If the expansion (3.1) holds, one may regard the continued fraction as generated by the transformations

$$t = \frac{a_0}{z+a_0+b_1+w_1}, \quad w_1 = \frac{a_1}{z+b_2+w_2}, \cdots, w_{n-1} = \frac{a_{n-1}}{z+b_n+w_n},$$

and show exactly as in [8] that $Q(z)/P(z)$ is irreducible, and

$$\left| \frac{Q(z)}{P(z)} - \frac{1}{2} \right| \leqq \frac{1}{2} \qquad \text{for } R(z) \geqq 0. \tag{3.2}$$

Hence $P(z) \neq 0$ for $R(z) \geqq 0$.

Conversely, let $P(z)$ be a given polynomial whose zeros are all in the half-plane $R(z) < 0$. Let $\bar{P}(z)$ denote the polynomial obtained from $P(z)$ by replacing its *coefficients* by their complex conjugates. The polynomial $Q(z)$ of (1.3) is then $[P(z) + \bar{P}(-z)]/2$ or $[P(z) - \bar{P}(-z)]/2$ according as the degree n of $P(z)$ is odd or even, respectively. We note that the set of zeros of $P(z)$ is symmetrical to the set of zeros of $\bar{P}(-z)$, with respect to the imaginary axis. Hence the geometrical argument used in [8] can be applied to show that all the zeros of $Q(z)$ lie on the axis of imaginaries, and that (3.2) holds. Since the zeros of $P(z)$ are in the half-plane $R(z) < 0$ while those of $Q(z)$ are on the line $R(z) = 0$, it follows that $Q(z)/P(z)$ is irreducible.

By division we now get

$$\frac{Q(z)}{P(z)} = \frac{a_0}{z + a_0 + b_1 + [C(z)/Q(z)]}, \tag{3.3}$$

where a_0 is the negative of the sum of the real parts of the zeros of $P(z)$ and is therefore positive, b_1 is pure imaginary or zero, and $C(z)/Q(z)$ is an irreducible rational fraction in which the denominator is of degree $n-1$ and the degree of the numerator is less than $n-1$. Just as in [8] it follows that $R[C(z)/Q(z)] \geqq 0$ for $R(z) \geqq 0$, and hence that there is a partial fraction expansion of the form

$$\frac{C(z)}{Q(z)} = \sum_{p=1}^{n-1} \frac{L_p}{z + ix_p}, \tag{3.4}$$

where the x_p are real and distinct, and the $L_p > 0$. Then

$$\frac{-iC(-iz)}{Q(-iz)} = \sum_{p=1}^{n-1} \frac{L_p}{z - x_p},$$

so that

$$\frac{-iC(-iz)}{Q(-iz)} = \frac{a_1}{z + b_2 i} - \frac{a_2}{z + b_3 i} - \cdots - \frac{a_{n-1}}{z + b_n i},$$

where the a_p are real and positive and the b_p are pure imaginary or zero. On replacing z by iz and dividing both members by $-i$, we get

$$\frac{C(z)}{Q(z)} = \frac{a_1}{z + b_2} + \frac{a_2}{z + b_3} + \cdots + \frac{a_{n-1}}{z + b_n}.$$

On substituting this expression into (3.3), we obtain (3.1), and the proof of the theorem is complete.[4] If we put

$$a_0 = \frac{1}{c_1}, \quad a_p = \frac{1}{c_p c_{p+1}}, \quad p = 1, 2, \cdots, n-1, \quad p = 1, 2, \cdots, n, \tag{3.5}$$
$$k_p = b_p c_p,$$

then (3.1) takes the form (1.4). The c_p are evidently positive if and only if the a_p are positive.

The expansion (1.4) may be conveniently obtained by the method of §2. For example, if $P(z) = z^3 + (2+i)z^2 + (3+i)z + (2i+2)$, then $Q(z) = 2z^2 + iz + 2$, and $Q(z)/P(z)$ is the fraction (2.6). Therefore, $c_1 = 1/2$, $c_2 = 8/9$, $c_3 = 81/64$, so that the zeros of $P(z)$ are all in the left half-plane. Here the zeros are actually $-1-i$, $(-1-i\,7^{1/2})/2$, $(-1+i\,7^{1/2})/2$.

From the formulas (2.5) and (2.10), we may formulate the condition for the zeros of $P(z)$ to lie in the half-plane $R(z) < 0$ by means of certain determinants analogous to the Hurwitz determinants [4]. In fact, we conclude at once that the numbers c_p of (1.4) are positive if and only if $(-1)^{p(p-1)/2} D_p > 0$, $p = 0, 1, \cdots, n$, where D_p is the determinant of Theorem 2.1 formed with $f_0 = P(z)$, $f_1 = Q(z)$. By simple transformations of these determinants, one may formulate this result as the following theorem.

THEOREM 3.2. *The polynomial $P(z)$ of Theorem 3.1 has all its zeros in the half-plane $R(z) < 0$ if and only if the determinants*

$$\Delta_1 = p_1,$$
$$\Delta_k = (-1)^{k(k-1)/2} D_k$$
$$= \begin{vmatrix} p_1, & p_3, & p_5, & \cdots, & p_{2k-1}, & -q_2, & -q_4, & \cdots, & -q_{2k-2} \\ 1, & p_2, & p_4, & \cdots, & p_{2k-2}, & -q_1, & -q_3, & \cdots, & -q_{2k-3} \\ & & \cdots & & & & \cdots & & \\ 0, & & & \cdots, & p_k, & 0, & & \cdots, & -q_{k-1} \\ 0, & q_2, & q_4, & \cdots, & q_{2k-2}, & p_1, & p_3, & \cdots, & p_{2k-3} \\ 0, & q_1, & q_3, & \cdots, & q_{2k-3}, & 1, & p_2, & \cdots, & p_{2k-4} \\ & & \cdots & & & & \cdots & & \\ 0, & & & \cdots, & q_k, & 0, & & \cdots, & p_{k-1} \end{vmatrix}, \tag{3.6}$$

$$k = 2, 3, \cdots, n \ (p_r = q_r = 0 \text{ for } r > n),$$

[4] An additional step is needed in the case of real polynomials (cf. §1), namely, to show that the $b_p = 0$.

are all positive.

If the q_p are zero, this reduces to the theorem of Hurwitz [4].

4. Determination of the number of zeros of $P(z)$ in each of the half-planes $R(z)<0$, $R(z)>0$. We suppose that $P(z)$ is a polynomial with complex coefficients of the form (1.2). We assume that (1.4) exists and that the $c_p \neq 0$. We then have the following theorem.

THEOREM 4.1. *The polynomial* (1.2) *has k zeros with positive real parts and $(n-k)$ zeros with negative real parts if, in the expansion* (1.4), *k of the coefficients c_p are negative and the remaining $(n-k)$ are positive.*

PROOF. Since the expansion (1.4) exists, $Q(z)/P(z)$ is irreducible. It follows that $P(z)$ cannot have a zero on the imaginary axis. For, if $P(ir)=0$, r real, then (cf. §3) $Q(ir)=[P(ir)\pm\overline{P}(-ir)]/2 = [P(ir)\pm\overline{P}(i\bar{r})]/2=0$, which is impossible since $Q(z)/P(z)$ is irreducible. Thus the zeros of $P(z)$ have their real parts different from zero, so that, for $R(z)=0$, we can write $P(z)=re^{i\pi\theta}$, where $r>0$. If we consider $P(z)$ as the product of the vectors from its zeros to the point z, then we see at once that, as z ranges along the axis of imaginaries from $i\cdot\infty$ to $-i\cdot\infty$, then θ decreases by the integral amount $\Delta=N-P$, where N and P are the numbers of zeros of $P(z)$ with negative and positive real parts, respectively. The same evidently holds if, instead of $P(z)$, we consider $i^nP(-iz)=re^{i\pi\theta}$, and let z range along the *real* axis from $-\infty$ to $+\infty$. Now

$$\begin{aligned} i^nP(-iz) &= (z^n - q_1z^{n-1} - p_2z^{n-2} + q_3z^{n-3} + p_4z^{n-4} + \cdots) \\ &\quad + i(p_1z^{n-1} - q_2z^{n-2} - p_3z^{n-3} + q_4z^{n-4} + \cdots) \\ &= U(z) + iV(z), \end{aligned} \tag{4.1}$$

where

$$\frac{V(z)}{U(z)} = \frac{1}{c_1z+ik_1} - \frac{1}{c_2z+ik_2} - \cdots - \frac{1}{c_nz+ik_n}, \tag{4.2}$$

which is real when z is real. This may be seen as follows:

The pth denominator of (1.4) can be written in the form

$$B_1(z) = c_1z + k_1 + 1,$$

$$B_p(z) = \begin{vmatrix} c_1z+k_1+1, & -1, & 0, & 0,\cdots, & 0 \\ 1, & c_2z+k_2, & -1, & 0,\cdots, & 0 \\ 0, & 1, & c_3z+k_3, & -1,\cdots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & c_pz+k_p \end{vmatrix},$$

$p=2, 3, \cdots, n$. Since $c_1c_2 \cdots c_nP(z)=B_n(z)$, we then have

$$
\begin{aligned}
&c_1c_2\cdots c_nP(z)\\
&= \begin{vmatrix} c_1z+k_1, & -1, & 0, & 0,\cdots, & 0\\ 1, & c_2z+k_2, & -1, & 0,\cdots, & 0\\ 0, & 1, & c_3z+k_3, & -1,\cdots, & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & c_nz+k_n \end{vmatrix}\\
(4.3)\quad &+ \begin{vmatrix} c_2z+k_2, & -1, & 0,\cdots, & 0\\ 1, & c_3z+k_3, & -1,\cdots, & 0\\ 0 & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & c_nz+k_n \end{vmatrix}\\
&= H_n(z)+G_n(z)
\end{aligned}
$$

and

$$
(4.4)\qquad \frac{G_n(z)}{H_n(z)}=\frac{1}{c_1z+k_1+}\,\frac{1}{c_2z+k_2+}\cdots+\frac{1}{c_nz+k_n}.
$$

If we replace z by $-iz$ in (4.3) and (4.4) and make some simple transformations [5, p. 194], (4.4) becomes (4.2) and (4.3) becomes (4.1).

From this point on, the proof runs almost exactly the same as in [8]. The number Δ is the net decrease in

$$
\theta=\frac{1}{\pi}\arctan\frac{V(z)}{U(z)}
$$

as z increases through real values from $-\infty$ to $+\infty$. Using (4.2), we form the sequence $f_0=1, f_1=c_nz+ik_n, \cdots, f_n$, defined by the recurrence formula $f_{p+1}=(c_{n-p}z+ik_{n-p})f_p-f_{p-1}$, $p=1, 2, \cdots, n-1$. These form a Sturm's sequence, and we find that $\Delta=n-2k$, where k is the number of negative terms in the sequence $c_1, c_2, \cdots, c_n$. Therefore, $N-P=n-2k$, $N+P=n$, so that $P=k$, $N=n-k$, as was to be proved.

By means of formulas (2.5) and (3.6), Theorem 4.1 can be formulated in terms of the numbers α_{pp} in the first column of table (2.4), or in terms of the determinants Δ_p of (3.6). In this way the methods of Routh and Hurwitz, respectively, are extended to polynomials with complex coefficients.

The method of Theorem 4.1 applies, save in the exceptional case where some determinant Δ_p vanishes and the expansion (1.4) fails to exist. In the next section, we give simple methods for taking care of this exceptional case.

5. **The case where some $\Delta_p = 0$.** We assume $Q(z)/P(z)$ is irreducible. This is no restriction since the common factor can be removed by the euclidean algorithm. We shall show that the method of §4 may be extended to find the number of roots in each half-plane even if some $\Delta_p = 0$.

THEOREM 5.1. *There exists a number $\delta > 0$ such that for all numbers η in the interval $-\delta < \eta < 0$, the expansion* (1.4) *exists for the quotient $Q(z+\eta)/P(z+\eta)$.*

PROOF. Form the determinants Δ_p of (3.6) for the polynomial

$$P(z+\eta) = z^n + ({}_nC_1\cdot\eta + \alpha_1)z^{n-1} + ({}_nC_2\cdot\eta^2 + \alpha_1\,{}_{n-1}C_1\cdot\eta + \alpha_2)z^{n-2} + \cdots + \alpha_n. \tag{5.1}$$

Then one may readily verify that the Δ_p are polynomials in η of degree p^2, in which the coefficient of the highest power of η is

$$\begin{vmatrix} {}_nC_1, & {}_nC_3, & {}_nC_5, & \cdots, & {}_nC_{2p-1} \\ 1, & {}_nC_2, & {}_nC_4, & \cdots, & {}_nC_{2p-2} \\ 0, & {}_nC_1, & {}_nC_3, & \cdots, & {}_nC_{2p-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & {}_nC_p \end{vmatrix} \cdot \begin{vmatrix} {}_nC_1, & {}_nC_3, & \cdots, & {}_nC_{2p-3} \\ 1, & {}_nC_2, & \cdots, & {}_nC_{2p-4} \\ 0, & {}_nC_1, & \cdots, & {}_nC_{2p-5} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & {}_nC_{p-1} \end{vmatrix}.$$

These determinants are always positive, for they are the determinants (3.6) formed for the polynomial $(z+1)^n = z^n + {}_nC_1 z^{n-1} + {}_nC_2 z^{n-2} + \cdots + {}_nC_n$, which has its only zero in the half-plane $R(z) < 0$. Then the Δ_p for (5.1) are polynomials in η which are not identically zero. Hence there must exist a constant $\delta > 0$ such that, for $-\delta < \eta < 0$, none of the Δ_p can vanish, and thus the $c_p \neq 0$ since they are quotients of the Δ_p.

Example. We shall apply this theorem to find the number of zeros in each half-plane for the polynomial $P(z) = z^3 + (2+i)z^2 + (-3/2+i)z + (-5/2 - 5i/2)$. Here $\Delta_2 = 0$ so that the expansion (1.4) cannot be formed. However, if we expand $Q(z+\eta)/P(z+\eta)$ into a J-fraction and *retain at each step only the powers of η which dominate* as η approaches zero, we find that, when η is near zero, c_1, c_2, and c_3 are positive, negative, and positive, respectively. Therefore, $P(z+\eta)$ has two zeros with negative real parts and one zero with positive real part for all $|\eta|$ sufficiently small. Hence $P(z)$ must also have two zeros with negative real parts and one zero with positive real part. Here the zeros are actually $(-1\pm(11)^{1/2})/2$, $-1-i$.

We now describe two methods for finding the number of roots in each half-plane, which are based on the fact that we may alter the coefficients of $P(z)$ by a very small amount without displacing the zeros of $P(z)$ more than a very small amount. Therefore, since we have assumed that $P(z)$ has no zeros on the imaginary axis, the altered polynomial $P'(z)$ will have the same number of zeros on each side of the imaginary axis as the original polynomial $P(z)$.

In the first method, we actually increase one or more of the coefficients by a small positive amount ϵ, and obtain the J-fraction expansion (1.4) for the altered polynomial. We then count the signs of the c_p as before. The following example illustrates the method.

Example. Let $P(z)=z^5-3z^4-20z^3+60z^2-z-78$. We find that the expansion (1.4) does not exist for $Q(z)/P(z)$. We therefore form $P'(z)=z^5-3z^4+(-20+\epsilon)z^3+60z^2-z-78$, $Q'(z)=Q(z)=-3z^4+60z^2-78$, and the J-fraction expansion (1.4) for $Q'(z)/P'(z)$. In this expansion we find, for all ϵ sufficiently small, c_1, c_2, c_3 are negative and c_4, c_5 are positive, so that there are three roots of $P'(z)$ and hence of $P(z)$ with positive real parts and two with negative real parts. The zeros of $P(z)$ are approximately -1.0, -4.4, $+4.7$, $+1.86\pm.5i$.

A second method consists in the formation of the continued fraction expansion for $V(z)/U(z)$ by the euclidean algorithm, that is, we form

$$\frac{V(z)}{U(z)}=\frac{1}{q_1(z)}+\frac{1}{q_2(z)}+\cdots+\frac{1}{q_j(z)},\qquad j<n, \tag{5.2}$$

where the q_p are certain uniquely determined polynomials. Let us imagine that we have formed a polynomial

$$P'(z)=z^n+\alpha_1'z^{n-1}+\alpha_2'z^{n-2}+\cdots+\alpha_n', \tag{5.3}$$

whose coefficients differ by very small amounts from the coefficients of $P(z)$, and the corresponding expansion

$$\frac{V'(z)}{U'(z)}=\frac{1}{c_1z+ik_1}-\frac{1}{c_2z+ik_2}-\cdots-\frac{1}{c_nz+ik_n}, \tag{5.4}$$

and compare this with (5.2). The method can best be explained by examples.

Examples. Consider again $P(z)=z^5-3z^4-20z^3+60z^2-z-78$. Here

$$\frac{V(z)}{U(z)}=\frac{1}{-z/3}+\frac{1}{(z^3+20z)/9}+\frac{1}{27z/78}. \tag{5.5}$$

We form the Sturm's functions for $P'(z), f_0=1, f_1=c_1z, \cdots$, defined by the recurrence relation $f_{p+1}=c_{n-p}z\cdot f_p-f_{p-1}$, $p=1, 2, \cdots$, where the c_p are undetermined. We then compare (5.4) and (5.5), and require that $c_1=27/78$, $c_5=-1/3$, and $f_4=V(z)$. Therefore, $c_2c_3c_4=1/9$, $-c_3c_4-27c_4/78-27c_2/78=60/78$. Since $f_5=U(z)-c_3z(27c_2z^2/78-1)$, we see that if we choose $c_3=-\epsilon$, a small negative number, f_5 and $U(z)$ will differ by a very small amount. Then $c_2=(-29\pm(296+36/\epsilon)^{1/2})/18$, $c_4=-20/9-c_2-26/81c_2$, so that two of the c_2, c_3, c_4 have negative signs and one has a positive sign. These, together with the values of c_1 and c_5, give three negative and two positive c_p for $P'(z)$, so that we find the same result as by the first method.

For the polynomial $P(z)=z^5+(i-5)z^4-10iz^3+(10+50i)z^2-16z+(-16i+80)$ we find by the same method that in the J-fraction expansion of the form (5.4) for $V'(z)/U'(z)$ three of the c_p have negative signs and two have positive. Hence $P'(z)$ and $P(z)$ have three zeros in $R(z)>0$ and two zeros in $R(z)<0$. Here the roots are actually $1+i$, $-1-i$, $2+2i$, $-2-2i$, $5-i$.

Still another method, which, however, may not always be successful, is to replace z by $1/z$ and consider the polynomial $z^nP(1/z)$.

6. **Bounds for the moduli of the zeros of $P(z)$.** We extend the method in [8] to obtain bounds for the moduli. If we set

$$h_1=\frac{1}{(c_1z+k_1+1)(c_2z+k_2)},\quad h_2=\frac{1}{(c_2z+k_2)(c_3z+k_3)},\cdots,$$
$$h_{n-1}=\frac{1}{(c_{n-1}z+k_{n-1})(c_nz+k_n)},$$

then (1.4) becomes

$$\frac{Q(z)}{P(z)}=\frac{(c_1z+k_1+1)^{-1}}{1}\;\frac{h_1}{+\,1}\;\frac{h_2}{+\,1}\;+\cdots+\;\frac{h_{n-1}}{1}.$$

If $g_1, g_2, \cdots, g_{n-1}$ are numbers such that $0<g_p<1$, $p=1, 2, \cdots, n-1$, then $P(z)\neq 0$ if z satisfies the inequalities $|h_1|\leqq g_1$, $|h_2|\leqq(1-g_1)g_2$, $|h_3|\leqq(1-g_2)g_3, \cdots$, $|h_{n-1}|\leqq(1-g_{n-2})g_{n-1}$. This sequence gives the bound $|z|\geqq c$, where c is the largest of the numbers

$$(6.1)\qquad \begin{aligned}&\frac{1}{2}\left(\left|\frac{k_1+1}{c_1}\right|+\left|\frac{k_2}{c_2}\right|+\left[\left(\left|\frac{k_1+1}{c_1}\right|-\left|\frac{k_2}{c_2}\right|\right)^2+\frac{4}{g_1|c_1c_2|}\right]^{1/2}\right),\\&\frac{1}{2}\left(\left|\frac{k_2}{c_2}\right|+\left|\frac{k_3}{c_3}\right|+\left[\left(\left|\frac{k_2}{c_2}\right|-\left|\frac{k_3}{c_3}\right|\right)^2+\frac{4}{g_2(1-g_1)|c_2c_3|}\right]^{1/2}\right),\cdots,\end{aligned}$$