ON THE HAMILTON DIFFERENTIAL

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1. Introduction. In the absolute geometric development of vector analysis Hamilton found it necessary to formulate a definition for the differential of a point function, since division by a vector is excluded in vector analysis. It is the purpose of this note to relate a restricted form of the Hamilton differential to that of Stolz and another modified form to a differential defined by Rainich.

2. The Hamilton differential. The definition of Hamilton for the differential ϕ' of a point function $\phi(P)$ may be expressed by

$$\phi'(P, dP) = \operatorname*{limit}_{\lambda \to 0} \frac{\phi(P + \lambda dP) - \phi(P)}{\lambda}. \tag{2.1}$$

This definition is not entirely satisfactory. For some functions it furnishes differentials which are usually considered as nonexistent. For example, $(dxdy)^{1/2}$ is the differential of $(xy)^{1/2}$ at the origin according to definition (2.1). That such situations arise is due to the fact that the differential here defined does not necessarily possess the linearity property which will be defined later. This defect was recognized by Rainich[1] who proposed another form of the Hamilton differential which possesses this desired property. This is essential if the differential of a tensor point function is to be again a tensor.

3. The Rainich differential. Instead of making a single point $P+\lambda dP$ approach P as λ goes to zero Rainich makes each of two points Q_λ and P_λ approach P as λ goes to zero. His definition may be formulated as follows:

DEFINITION (3.1).

(a) $$\phi'(P, dP) = \operatorname*{limit}_{\lambda \to 0} \frac{\phi(Q_\lambda) - \phi(P_\lambda)}{\lambda},$$

which limit must exist for all modes of approach of Q_λ and P_λ to P for which

(b) $$\operatorname*{limit}_{\lambda \to 0} \frac{Q_\lambda - P_\lambda}{\lambda} = dP.$$

Presented to the Society, April 28, 1945; received by the editors March 7, 1945, and, in revised form, July 5, 1945 and July 30, 1945.

[1] G. Y. Rainich, Amer. J. Math. vol. 46 (1924) p. 78.

4. The linearity property. The linearity property may be expressed by the formula

$$(4.1)\qquad \phi'(P, \tau dP + \sigma\delta P) = \tau\phi'(P, dP) + \sigma\phi'(P, \delta P)$$

where τ and σ are arbitrary scalars.

We wish to show that the Rainich differential possesses this property. In doing this we make a direct use of (3.1); we may thus write

$$\operatorname*{limit}_{\lambda\to 0} \frac{Q_\lambda - R_\lambda}{\tau\lambda} = dP \qquad \operatorname*{limit}_{\lambda\to 0} \frac{R_\lambda - P_\lambda}{\sigma\lambda} = \delta P,$$

so that

$$\operatorname*{limit}_{\lambda\to 0} \frac{Q_\lambda - P_\lambda}{\lambda} = \tau dP + \sigma\delta P.$$

But

$$(4.2)\qquad \begin{aligned} \operatorname*{limit}_{\lambda\to 0} \frac{\phi(Q_\lambda) - \phi(P_\lambda)}{\lambda} &= \operatorname*{limit}_{\lambda\to 0} \frac{\phi(Q_\lambda) - \phi(R_\lambda)}{\lambda} \\ &\quad + \operatorname*{limit}_{\lambda\to 0} \frac{\phi(R_\lambda) - \phi(P_\lambda)}{\lambda}, \end{aligned}$$

or

$$\phi'(P, \tau dP + \sigma\delta P) = \phi'(P, \tau dP) + \phi'(P, \sigma\delta P).$$

Also

$$\operatorname*{limit}_{\lambda\to 0} \frac{\phi(Q_\lambda) - \phi(R_\lambda)}{\tau\lambda} = \phi'(P, dP),$$

or multiplying by τ we obtain $\tau\phi'(P, dP)$, the first term in the right member of (4.2), while the second term is obtained in the same way. This establishes (4.1).

5. The modified Hamilton. The modified Hamilton differential will be defined to be *the Hamilton differential* (2.1) *upon which is imposed the linearity property* (4.1) *together with the continuity property*,

$$(5.1)\qquad \operatorname*{limit}_{\lambda\to 0} \phi'\left(R_\lambda, \frac{Q_\lambda - P_\lambda}{\lambda}\right) = \phi'(P, dP).$$

R_λ is a neighboring point to P which approaches P as $\lambda\to 0$.

THEOREM 1. *The existence of the modified Hamilton differential is a sufficient condition for the existence of the Rainich differential* (3.1).

To prove this theorem we deduce the Rainich differential (3.1) from the modified Hamilton as follows: we define $f(\xi)$ by the equation

$$f(\xi) = \phi[P_\lambda + \xi(Q_\lambda - P_\lambda)] \tag{5.2}$$

where ξ is an arbitrary parameter. From (5.2) it follows that

$$\begin{aligned} f'(\xi) &= \lim_{\Delta\xi\to 0} \frac{\phi[P_\lambda + \xi(Q_\lambda - P_\lambda) + \Delta\xi(Q_\lambda - P_\lambda)] - \phi[P_\lambda + \xi(Q_\lambda - P_\lambda)]}{\Delta\xi} \\ &= \phi'[P_\lambda + \xi(Q_\lambda - P_\lambda), Q_\lambda - P_\lambda] = \phi'(R_\lambda, Q_\lambda - P_\lambda), \end{aligned}$$

where for $0 \leqq \xi \leqq 1$, R_λ plays the role of an intermediate point. This is by definition (2.1) the Hamilton differential. From (5.2), $f(1) = \phi(Q_\lambda)$ and $f(0) = \phi(P_\lambda)$ follow while the mean value theorem asserts the existence of a value ξ such that $f(1) - f(0) = f'(\xi) = \phi'(R_\lambda, Q_\lambda - P_\lambda)$. We then have

$$\lim_{\lambda\to 0} \frac{\phi(Q_\lambda) - \phi(P_\lambda)}{\lambda} = \lim_{\lambda\to 0} \phi'\left(R_\lambda, \frac{Q_\lambda - P_\lambda}{\lambda}\right) \tag{5.3}$$

(we have here used the linearity property) while if Q_λ and P_λ are made to go to P in any manner whatever provided that $\text{limit}_{\lambda\to 0}(Q_\lambda - P_\lambda)/\lambda = dP$ and if the Hamilton differential is continuous at P it follows that

$$\lim_{\lambda\to 0} \phi'\left(R_\lambda, \frac{Q_\lambda - P_\lambda}{\lambda}\right) = \phi'(P, dP).$$

6. **The restricted Hamilton.** By the restricted Hamilton differential we shall mean *the Hamilton differential* (2.1) *restricted by the linearity condition* (4.1) *only.*

In the sequel we shall write Q_λ for $P + \lambda dP$.

THEOREM 2. *For a point function $\phi(Q_\lambda)$ to possess a finite restricted Hamilton differential, it is necessary and sufficient that $\phi(Q_\lambda)$ can be represented in the form*

$$\phi(Q_\lambda) = \phi(P) + \phi'(P, Q_\lambda - P) + \eta \tag{6.1}$$

where η is an infinitesimal function of λ of higher order than the first, and ϕ' is linear in the second argument.

To prove the necessity of the condition we introduce the notation

$$\eta \equiv \phi(Q_\lambda) - \phi(P) - \phi'(P, Q_\lambda - P)$$

and we write, using the linearity of ϕ',

$$\frac{\eta}{\lambda} = \frac{\phi(Q_\lambda) - \phi(P)}{\lambda} - \phi'(P, dP) - \phi'\left(P, \frac{Q_\lambda - P}{\lambda} - dP\right).$$

It only remains to show that η/λ goes to zero with λ.

The difference of the first two terms goes to zero with λ, according to (2.1) of the definition, and the last term goes to zero because of (3.1b) and the fact that a linear function in a finite-dimensional vector space goes to zero with its argument; this means that η/λ goes to zero. The same relation may be used to prove the sufficiency by remarking that in this case it is given that η/λ goes to zero with λ.

In passing we may note that in euclidean 2-space (6.1) becomes

$$\phi(x', y') = \phi(x, y) + \phi'[P, (x' - x)i + (y' - y)j] \times \eta$$

where i and j are coordinate vectors. The linearity condition permits us to write this equation in the form

$$\phi(x', y') = \phi(x, y) + \phi'(P, i)(x' - x) + \phi'(P, j)(y' - y) + \eta \tag{6.2}$$

while it is easy to prove that $\phi'(P, i)$ and $\phi'(P, j)$ are the partial derivatives of ϕ at P. But this form of the function indicates that the restricted Hamilton is similar to a Stolz differential.[2] In fact when

$$(x', y') = (x + \lambda dx, y + \lambda dy)$$

is introduced into (6.2) and if the restricted Hamilton exists at P, it follows that for every $\epsilon > 0$ there exists a δ such that for $|\lambda| < \delta$, $|\eta/\lambda| < \epsilon$.

In conclusion it may be noted that the general notions arising here are not restricted to euclidean 3-space.

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[2] Otto Stolz, *Differential and Integral Rechnung*, vol. 1, p. 132.