COLLECTIONS FILLING UP A SIMPLE PLANE WEB

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A web has been defined [2][1] by R. L. Moore. A compact plane continuum W is said [1] to be a simple plane web if there exist an upper semicontinuous collection G of mutually exclusive continua filling up W and another such collection H also filling up W such that (1) G is a dendron with respect to its elements and so is H and (2) if g and h are elements of G and H respectively, the common part of g and h exists and is totally disconnected. It has been shown [1] that we have an equivalent definition if we substitute for (1) in the above definition the condition that (1′) G is a dendron with respect to its elements and H is an arc with respect to its elements. The present paper shows that an equivalent definition is obtained if condition (1) is omitted.

This paper gives conditions under which the collections filling up a simple plane web are *non-equicontinuous*. A collection of mutually exclusive continua is said to be *equicontinuous* if for every positive number ϵ, there is a positive number δ_ϵ such that if P and Q are points of an element g of G at a distance apart of less than δ_ϵ, there is an arc from P to Q in g of diameter less than ϵ. If a collection is not equicontinuous, it is said to be *non-equicontinuous*. If there exists a point P belonging to the limiting set of G such that every subcollection of G whose limiting set contains P is non-equicontinuous, then G is said to be *hereditarily non-equicontinuous* at P.

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A simple plane web is a continuous curve [3]. In establishing some of the following theorems, we shall make use of the fact [1, Theorem 1] that a necessary and sufficient condition that a continuous curve be a simple plane web is that it remain connected and locally connected on the omission of any countable set of points.

THEOREM 1. *A necessary and sufficient condition that a compact plane continuum W be a simple plane web is that there exist an upper semicontinuous collection G of mutually exclusive continua filling up W and another such collection H filling up W such that if g and h are elements of G and H respectively, the common part of g and h exists and is totally disconnected.*

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[1] Numbers in brackets refer to the references cited at the end of the paper.

PROOF. The necessity of the above condition follows from the definition of a simple plane web. To demonstrate its sufficiency, we shall show that if Z is a countable subset of W, $W-Z$ is connected and locally connected. The argument is similar to that used in proving the necessity of Theorem 1 of [1].

Let M be the sum of all elements of G that do not intersect Z and let N be the sum of all elements of H that do not intersect Z. As $M+N$ is connected and dense in W, $W-Z$ is connected.

Assume that $W-Z$ is not locally connected. Let d be a domain containing a point P of $W-Z$ such that any domain d' containing P contains points of an infinitude of components of $(W-Z)\cdot d$. An infinitude of such components contain points of $M\cdot d'$. Let C_1, C_2, C_3 be circles in d such that C_1 encloses P, C_2 encloses C_1, and C_3 encloses C_2 but no element of G or H. Then C_1 encloses a sequence of points $P_1, P_2, \cdots$ of M such that (1) no two of them belong to the same component of $(W-Z)\cdot d$ and (2) if g_n is the component containing P_n of the common part of d and the element of G containing P_n, then $g_1, g_2, \cdots$ has a sequential limiting set. There exists a sequence of points $Q_1, Q_2, \cdots$ of N converging to a point of C_2 such that Q_n is a point of a continuum in g_m $(m \geqq n)$ irreducible from C_1 to C_3. Let h_n be the component containing Q_n of the common part of d and the element of H containing Q_n. It follows [2, pp. 391–392] that some nondegenerate continuum F is a subset of the common part of the limiting set of $g_1, g_2, \cdots$ and the limiting set of $h_1, h_2, \cdots$. As G and H are upper semicontinuous collections, F is a subset of the common part of an element of G and an element of H. This is contrary to the hypothesis of this theorem.

THEOREM 2. *If x is the closure of a connected set which is the sum of a countable number of points and complementary domains of a simple plane web W, the common part of W and x is a continuous curve.*[2]

PROOF. It has been shown [1, Theorem 6] that x is locally connected. As W is a continuous curve, for any positive number ϵ there are only a finite number of complementary domains of W of diameter more than ϵ. Since $W\cdot x$ is equal to x minus the sum of a set of complementary domains of W, then $W\cdot x$ is a continuous curve.

[2] Indeed, $W\cdot x$ may be shown to be *hereditarily* a continuous curve. Furthermore, it may be shown that in order that the compact point set K should be the boundary of the closure of a connected point set which is the sum of a countable number of points and complementary domains of some simple plane web W it is necessary and sufficient that K should be a continuous curve such that (1) every nondegenerate cyclic element of K is a simple closed curve, (2) K does not have uncountably many cut points, and (3) no cyclic element of K encloses a point of K.

THEOREM 3. *If x is the closure of a connected set which is the sum of a countable number of points and complementary domains of a simple plane web W and R is a nondegenerate connected subset of the common part of W and x, then R is arcwise connected.*[3]

PROOF. Let A and B be two points of R and let AB be an arc in $W\cdot x$. If AB contains a point P not of R, there is an arc AOB in $W\cdot x-P$. There is a simple closed curve which is the sum of arcs $A'PB'$ and $A'QB'$ of AB and AOB respectively. It has been shown [1, Theorem 8] that each point of $W\cdot x$ is a limit point of the connected set $W-W\cdot x$. Therefore, no subset of $W\cdot x$ contains a simple closed curve plus an arc having only its end points on the simple closed curve. Using this fact and the fact that there is an arc from A to B in $W\cdot x$ minus a finite number of points not belonging to R, we find that $A'QB'$ is a subset of R. By replacing $A'PB'$ and other subarcs of AB by arcs which are subsets of R, we may obtain an arc from A to B in R.

THEOREM 4. *If x is the closure of a connected set which is the sum of a countable number of points and complementary domains of a simple plane web W and AB is an arc lying in the common part of W and x, then AB contains an arc lying on the boundary of some complementary domain of W.*

PROOF. Since AB is [1, Theorem 7] the sum of a countable number of points and closures of individual complementary domains of W, the boundary of some one of these domains contains a subarc of AB.

THEOREM 5. *Let G be an upper semicontinuous collection of mutually exclusive continua filling up a simple plane web W and let H be another such collection filling up W such that* (1) *H is a dendron with respect to its elements and* (2) *if g and h are elements of G and H respectively, the common part of g and h exists and is totally disconnected. Let g_1 and g_2 be two elements of G belonging to x_1 and x_2 respectively where x_i* $(i=1, 2)$ *is the closure of a connected set which is the sum of a countable number of points and complementary domains of W. Then g_1+g_2 is a subset of the boundary of a complementary domain of W.*

PROOF. By Theorems 3 and 4 there is a complementary domain d of W such that g_1 and the boundary of d have an arc in common. The boundary of d intersects two elements h_1 and h_2 of H and as g_2 intersects each of them, each element of H that separates h_1 from h_2 in W also disconnects both g_2 and $W\cdot x_2$.

[3] In fact, R is *locally* arcwise connected.

Let Z be a countable set of points such that Z plus the sum of a set of complementary domains of W is a connected set whose closure is x_2. Then each element of H that separates h_1 from h_2 intersects Z or contains a point of x_2 belonging to the boundary of d. As Z is countable and as there are uncountably many elements of H each of which separates h_1 from h_2 in W, the boundary of d contains uncountably many points of g_2.

Assume that g_i ($i=1$ or 2) is not a subset of the boundary of d. By Theorems 3 and 4 there is a complementary domain d' of W other than d such that some arc is a subset of the common part of g_i and the boundary of d'. Both g_1 and g_2 intersect the boundary of d' and by Theorem 3 there are two mutually exclusive arcs in $W\cdot x_1$ and $W\cdot x_2$ respectively such that each has an end point on the boundary of d and an end point on the boundary of d'. This is impossible, for no subset of $W\cdot x_1+W\cdot x_2$ plus the boundaries of d and d' is a simple closed curve plus an arc having only its end points on the simple closed curve.

THEOREM 6. *Suppose W is a simple plane web with two complementary domains. Suppose that G is an upper semicontinuous collection of mutually exclusive continua filling up W, that H is another such collection also filling up W, that G is a dendron with respect to its elements and so is H and finally that if g and h are elements of G and H respectively, then the common part of g and h exists and is totally disconnected. Then there is an element of G which does not separate W and which is not a subset of the closure of any connected set which is the sum of a countable number of points and complementary domains of W.*

PROOF. Those elements of G that do not separate W are called *end elements* of G. Assume that each end element of G is a subset of the closure of a connected set which is the sum of a countable number of points and complementary domains of W. Then by Theorem 5, if g_1 and g_2 are end elements of G, there is a complementary domain d of W whose boundary contains g_1 and g_2.

Suppose that g_3 is an end element of G other than g_1 or g_2. As the boundaries of two complementary domains of a simple plane web have no more than a point in common, it follows by Theorem 5 that g_3 is a subset of the boundary of d. However, if it were a subset of the boundary of d, no two elements of H could intersect each of the sets g_1, g_2, and g_3.

Suppose that g_1 and g_2 are the only end elements of G. If d' is a complementary domain of W other than d, its boundary is a subset of an element of H, for no three mutually exclusive continua in W could intersect the boundary of d' as well as both g_1 and g_2. Likewise, the boundary of d' is a subset of an element of G, for no three mutually exclusive continua in W could intersect both the boundary of d and the boundary of d'. This is impossible as no element of G intersects an element of H in a nondegenerate continuum.

THEOREM 7. *Let W be a simple plane web with two complementary domains. Let G be an upper semicontinuous collection of mutually exclusive continua filling up W and let H be another such collection filling up W such that* (1) *G is a dendron with respect to its elements and so is H and* (2) *if g and h are elements of G and H respectively, the common part of g and h exists and is totally disconnected. Then the collection G is non-equicontinuous.*

PROOF. Let X be a set such that x is an element of X only if x is maximal with respect to being the closure of a connected set which is the sum of a countable number of points and complementary domains of W. By Theorem 6 there is an end element g_e of G which is not a subset of an element of X. There is [1, Theorem 8] an arc AB in W which contains no point of an element of X and which has only its end points in common with g_e.

Let d be one of the complementary domains of g_e+AB which has AB as a subset of its boundary. If F is a point of W in d and E is a point of W belonging to neither d, AB, nor g_e, then there is an arc EF in W which contains no point of AB. Let C be the first point of the closure of d on EF in the order from E to F. If ρ is one-half the distance from C to AB, we shall show that for any positive number ϵ there exist points P and Q belonging to an element g of G such that the distance from P to Q is less than ϵ and the diameter of any subcontinuum of g containing P and Q is of diameter more than ρ.

There exist points P_1 and P_2 such that each can be joined to C by an arc in W of diameter less than the minimum of ρ and $\epsilon/2$ and such that P_1 belongs to d and P_2 belongs to neither d nor g_e. As g_e is an end element of G, there is an element g of G which separates g_e from P_1+P_2 in W. There exist on g points P and Q at a distance from C of less than the minimum of ρ and $\epsilon/2$ such that P belongs to d and Q does not. Every subcontinuum of g that contains P and Q intersects AB and therefore is of diameter more than ρ.

THEOREM 8. *Let W be a simple plane web with two complementary domains. Let G be an upper semicontinuous collection of mutually exclusive continua filling up W and let H be another such collection filling up W such that* (1) *G is an arc with respect to its elements*, (2) *H is a* *dendron with respect to its elements, and* (3) *if g and h are elements of G and H respectively, the common part of g and h exists and is totally disconnected. Then W contains a point at which G is hereditarily non-equicontinuous.*

PROOF. Obtain g_e, AB, C, ρ, and g as in Theorem 7. Of every countable sequence of different elements of G having a subset of g as a limiting set, all but a finite number separate g from g_e. Hence G is hereditarily non-equicontinuous at C.

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DIMENSIONAL TYPES

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Let H and S be topological spaces. We say that H is of *dimensional type* S (symbol: D_S) if for each closed set X and mapping $f:X\rightarrow S$ there exists an extension $\bar{f}:H\rightarrow S$.

It is clear that (from a result due to Hurewicz [1, p. 83]) when H is separable metric and S is an n-sphere, then H can be of dimensional type S if and only if dim $H\leqq n$. For simplicity we write D_n for D_S when S is an n-sphere. It is, of course, possible to define dim H as the least integer n for which H is of type D_n even when H is not separable metric. But this seems to be open to objection except in certain cases (cf. (d) below).

It is at once clear that we have:

(a) *If H is of type D_S then so also is any closed subset.*

(b) *If the closed sets H_1 and H_2 are of type D_S then so also is the set H_1+H_2.*

As a matter of notation we may suppose that $H=H_1+H_2$. Let $f:X\rightarrow S$. Several cases may arise of which we shall consider only the

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