THE MANIFOLDS OF LINEAR ELEMENTS OF AN n-SPHERE

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1. Introduction. The 3-manifolds of oriented and non-oriented linear elements of closed surfaces have been investigated by Nielsen,[1] Hotelling,[2] Threlfall,[3] van der Waerden and others.[4] In the present paper we take up the case of the space M of oriented linear elements, and the space M' of non-oriented linear elements, of an n-sphere, $n \geqq 1$. The chief tools in the present investigation are certain orthogonal transformations (§§3–4) and theorems on addition of complexes.[5] Our success in the determination of certain homology classes (§§7–8, 14) leads to complete determination of (integral) Betti groups of M and M'. Our results may be summarized as follows:

(M1) For $n>1$, M is an orientable $(2n-1)$-manifold. Its Betti groups, which are not the null groups, are the following: For even n, B^0 and $B^{2n-1} \approx G_0$ (AH, p. 556) and $B^{n-1} \approx G_2$; for odd n, B^0, B^{2n-1}, B^{n-1}, and $B^n \approx G_0$.

(M2) For $n=2$, M is the projective space. For $n>2$, its fundamental group is the identity.

(M3) For $n=1, 3, 7$, M is the topological product of an n-sphere and an $(n-1)$-sphere.

(M$'$1) For $n>1$, M' is an orientable or a non-orientable $(2n-1)$-manifold according as n is even or odd. Its Betti groups, which are not the null, are the following: For even n, B^0 and $B^{2n-1} \approx G_0$, $B^{n-1} \approx G_4$, and $B^r \approx G_2$, $r=1, 3, \cdots, n-3; n+1, n+3, \cdots, 2n-3$. For odd n, B^0 and $B^n \approx G_0$, and $B^r \approx G_2$, $r=1, 3, \cdots, n-2; n+1, n+3, \cdots, 2n-2$.

(M$'$2) For $n=2$, M' is the lens space (Linsenraum) (4,1).[6] For $n>2$, its fundamental group is the cyclic group of order 2.

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[1] J. Nielsen, *Untersuchungen zur Topologie der geschlossen zweiseitigen Flächen*, Acta. Math. vol. 50 (1927) pp. 302–306.

[2] H. Hotelling, *Three-dimensional manifolds of states of motions*, Trans. Amer. Math. Soc. vol. 27 (1925) pp. 329–344; *Multiple-sheeted spaces and manifolds of states of motions*, ibid. vol. 28 (1926) pp. 479–490.

[3] W. Threlfall, *Räume aus Linienelementen*, Jber. Deutschen Math. Verein. vol. 42 (1933) I, pp. 88–110.

[4] *Solutions of problem* 124 by B. L. van der Waerden, H. Kneser, H. Seifert, E. R. van Kampen, and W. Threlfall, Jber. Deutschen Math. Verein. vol. 42 (1933) II, pp. 112–117.

[5] Alexandroff-Hopf, *Topologie* I, Berlin (1935), pp. 287–293. This book will be referred to as AH.

[6] Seifert-Threlfall, *Lehrbuch der Topologie*, Leipzig, 1934, p. 210. This book will be referred to as ST.

(M′3) For $n=1, 3, 7$, M' is the topological product of an n-sphere and an $(n-1)$-dimensional projective space.

I. The manifold M of oriented linear elements

2. **The representation of M.** The n-sphere, $n\geqq 1$, may be represented in the Euclidean $(n+1)$-space X of the points $(x_0, x_1, \cdots, x_n)$ by the unit sphere: $x'x=x_0^2+x_1^2+\cdots+x_n^2=1$, where x denotes the matrix of one column of the coordinates of the point, and x' the transposed of the matrix x. Similarly we have y, y' and the space Y. The space M is then represented by the subspace of the topological product $R=X\times Y$, defined by the equations

$$M \qquad x'x = 1, \qquad x'y = 0, \qquad y'y = 1.$$

Evidently this subspace is a closed $(2n-1)$-manifold when it is connected (AH, p. 404). Henceforth we shall take this subspace as the space M, and speak of the points of M instead of oriented linear elements.

3. **The two "halves" of M.** The decomposition of the n-sphere $x'x=1$ into the two n-cells corresponding to $x_0\geqq 0$ and $x_0\leqq 0$ gives rise to that of M into the two halves:

$$M_1 \qquad x_0 \geqq 0, \qquad x'x = 1, \qquad x'y = 0, \qquad y'y = 1;$$

$$M_2 \qquad x_0 \leqq 0, \qquad x'x = 1, \qquad x'y = 0, \qquad y'y = 1.$$

Let e denote the matrix of one column of the $n+1$ elements $1, 0, \cdots, 0$. The matrix

$$A_1 = (x+e)(x+e)'/(x_0+1) - I,$$

where $x'x=1$, $x_0+1\neq 0$, and the matrix

$$A_2 = (x-e)(x-e)'/(x_0-1) + I,$$

where $x'x=1$, $x_0-1\neq 0$, are symmetric and orthogonal, and have x' and x as their first rows and columns.[7] Let $u^{(i)}=(u_0^{(i)}, u_1^{(i)}, \cdots, u_n^{(i)})$ and $v^{(i)}=(v_0^{(i)}, v_1^{(i)}, \cdots, v_n^{(i)})$ denote points in the Euclidean $(n+1)$-spaces U_i and V_i respectively, $i=1, 2$. Form the mappings[8]

$$t_i \qquad u^{(i)} = x, \qquad v^{(i)} = A_i y.$$

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[7] H. W. Turnbull and A. C. Aitken, *An introduction to the theory of canonical matrices*, London, 1932, Lemma II, p. 104.

[8] The author wishes to express his gratitude to his colleague Professor P. L. Hsu for the construction of the matrices B_{n+1} and the mapping t_0 in §10 and for the reference in footnote 7. The construction of t_i is then immediate.

t_1 and t_2 are topological mappings of M_1 and M_2 respectively on

$$K_1 \qquad u_0^{(1)} \geqq 0, \quad u^{(1)\prime}u^{(1)} = 1, \quad v_0^{(1)} = 0, \quad v^{(1)\prime}v^{(1)} = 1$$

in $R_1 = U_1 \times V_1$ and

$$K_2 \qquad u_0^{(2)} \leqq 0, \quad u^{(2)\prime}u^{(2)} = 1, \quad v_0^{(2)} = 0, \quad v^{(2)\prime}v^{(2)} = 1$$

in $R_2 = U_2 \times V_2$. Let E_1 be the n-cell: $u_0^{(1)} \geqq 0$, $u^{(1)\prime}u^{(1)} = 1$ in U_1, E_2 the n-cell: $u_0^{(2)} \leqq 0$, $u^{(2)\prime}u^{(2)} = 1$ in U_2, and S_i the $(n-1)$-sphere: $v_0^{(i)} = 0$, $v^{(i)\prime}v^{(i)} = 1$ in V_i. Evidently $K_i = E_i \times S_i$.

4. M as sum. The case $n=1$. Let us denote the subspace: $x_0 = 0$, $x'x = 1$, $x'y = 0$, $y'y = 1$ in R by $\text{Bd}(M_1) = \text{Bd}(M_2)$, and the subspace: $u_0^{(i)} = 0$, $u^{(i)\prime}u^{(i)} = 1$ in U_i by $\text{Bd}(E_i)$, and finally set $\text{Bd}(K_i) = \text{Bd}(E_i) \times S_i$, the topological product of two $(n-1)$-spheres. Let P be any point of $\text{Bd}(M_i)$. Then $t_i(P) = P_i$ of $\text{Bd}(K_i)$, and $P_2 = t_2 t_1^{-1}(P_1)$ is a topological mapping of $\text{Bd}(K_1)$ on $\text{Bd}(K_2)$. Since the symmetric and orthogonal matrix $A_1 = A_1' = A_1^{-1}$, the topological mapping $t = t_2 t_1^{-1}$ of $\text{Bd}(K_1)$ on $\text{Bd}(K_2)$ is given by $A_2 A_1$, with x in the two factor matrices replaced by $u^{(1)}$. Let $\bar{u}^{(1)}$ denote the matrix of one column of the $n+1$ elements: $0, u_1^{(1)}, \cdots, u_n^{(2)}$. Then evidently

$$t \qquad u^{(2)} = u^{(1)}, \qquad v^{(2)} = \left\{2(\bar{u}^{(1)}\bar{u}^{(1)\prime} + ee') - I\right\}v^{(1)}.$$

Through identification of all pairs of points of $\text{Bd}(K_1)$ and $\text{Bd}(K_2)$, corresponding under t, there results from K_1 and K_2 a *sum*[5] $K_1 + K_2$. Since M is homeomorphic with $K_1 + K_2$, we shall write $M = K_1 + K_2$.

For $n=1$, K_i consists of two semicircles. By means of t, the two end points of each semicircle of K_1 or K_2 are identified with the two end points of one and only one semicircle of K_2 or K_1 respectively. Hence M is the topological product of a 1-sphere and a 0-sphere.

5. The fundamental group. The case $n=2$. For $n > 2$, all the fundamental groups of the topological products K_i and $\text{Bd}(K_i)$ are the identity (ST, p. 156), and therefore that of M is also the identity (ST, p. 179).

For general n, let ϵ denote the point $(0, \cdots, 0, 1)$ in a Euclidean $(n+1)$-space. Let m_i and b_i denote respectively the $(n-1)$-spheres $\text{Bd}(E_i) \times \epsilon$ and $\epsilon \times S_i$ on $\text{Bd}(K_i)$, and be called the *meridian* and *latitude* of $\text{Bd}(K_i)$.

In the remaining part of this section we confine ourselves exclusively to the case $n=2$. Denote by O_i the point $(\epsilon; \epsilon)$ of the torus $\text{Bd}(K_i)$ and take it as the initial point of closed oriented curves on K_i. Obviously $t(O_1) = O_2$. The meridian and latitude circles m_i and b_i will be regarded as with definite orientations, fixed as follows. First, take the senses of increasing θ and ϕ in the parametric representations

$$m_1 \quad \overset{(1)}{u_0} = 0,\ \overset{(1)}{u_1} = \sin\theta,\ \overset{(1)}{u_2} = \cos\theta;\ \overset{(1)}{v_0} = 0,\ \overset{(1)}{v_1} = 0,\ \overset{(1)}{v_2} = 1,$$

$$b_1 \quad \overset{(1)}{u_0} = 0,\ \overset{(1)}{u_1} = 0,\ \overset{(1)}{u_2} = 1;\ \overset{(1)}{v_0} = 0,\ \overset{(1)}{v_1} = -\sin\phi,\ \overset{(1)}{v_2} = \cos\phi$$

as the orientations of m_i and b_i respectively. Now

$$t(m_1) \quad \begin{array}{lll} \overset{(2)}{u_0} = 0, & \overset{(2)}{u_1} = \sin\theta, & \overset{(2)}{u_2} = \cos\theta; \\ \overset{(2)}{v_0} = 0, & \overset{(2)}{v_1} = \sin 2\theta, & \overset{(2)}{v_2} = \cos 2\theta, \end{array}$$

$$t(b_1) \quad \begin{array}{lll} \overset{(2)}{u_0} = 0, & \overset{(2)}{u_1} = 0, & \overset{(2)}{u_2} = 1; \\ \overset{(2)}{v_0} = 0, & \overset{(2)}{v_1} = \sin\phi, & \overset{(2)}{v_2} = \cos\phi. \end{array}$$

Let $(u^{(i)}; v^{(i)})$ denote a variable point of a subspace Γ. We shall call the subspace of $(u^{(i)}; \epsilon)$ and that of $(\epsilon; v^{(i)})$ the *projections* $g_1(\Gamma)$ and $g_2(\Gamma)$. g_i are evidently continuous mappings of Γ. The projections $g_1t(m_1)$ of $t(m_1)$ and $g_2t(b_1)$ of $t(b_1)$ are respectively m_2 and b_2. Moreover g_1t is a topological mapping of m_1 on m_2, and $g_2t=t$ a topological mapping of b_1 on b_2. Now fix the orientations of m_2 and b_2 by demanding that g_1t maps the oriented m_1 on the oriented m_2 and that g_2t maps the oriented b_1 on the oriented b_2.

Let $[m_i]$, $[b_i]$ denote the classes of closed oriented curves through O_i on K_i, homotopically deformable into m_i and b_i when the point O_i is kept fixed. The fundamental group of K_i is given by the two generators $[m_i]$, $[b_i]$ and the defining relation $[m_i]=1$. From the representation of $t(m_1)$ it is obvious that, when a variable point P_1 starts from O_1 and describes the oriented m_1 once, the point $t(P_1)$ starts from O_2 and describes a closed oriented curve on $\mathrm{Bd}(K_2)$ in such a way that its projection $g_1t(P_1)$ describes the oriented m_2 once and its projection $g_2t(P_1)$ the oriented b_2 twice. Furthermore t maps the oriented b_1 on the oriented b_2. From the usual consideration of the Euclidean plane as the universal covering of $\mathrm{Bd}(K_2)$, we conclude that in the fundamental group of K_2,

$$[t(m_1)] = [m_2]\,[b_2^2], \qquad [t(b_1)] = [b_2].$$

Hence the fundamental group of M is given by the single generator $[b_1]$ and the defining relation $[b_1^2]=1$ (ST, pp. 177–178), or is the cyclic group of order 2.

In fact, K_1, K_2 and the mapping t of $\mathrm{Bd}(K_1)$ on $\mathrm{Bd}(K_2)$ define the Heegaard diagram of the 3-manifold M. From the enumeration of all the 3-manifolds whose Heegaard diagrams lie on a torus (ST, pp. 210, 220), M is the 3-dimensional projective space. This is a new proof of a well known result.[9]

6. **The projections of** $t(m_1)$. These projections were used in the preceding section for the determination of the fundamental group of M for $n=2$, and will be investigated here and in the next two sections for the determination in §9 of $B^{n-1}(M)$ and $B^n(M)$ for $n\geqq 2$. Henceforth suppose always $n\geqq 2$. Now

$$(tm_1)\quad \begin{aligned} &u_0^{(2)} = 0,\quad u_1^{(2)} = u_1^{(1)},\cdots, u_n^{(2)} = u_n^{(1)};\\ &v_0^{(2)} = 0,\quad v_1^{(2)} = 2u_1^{(1)}u_n^{(1)},\cdots, v_{n-1}^{(2)} = 2u_{n-1}^{(1)}u_n^{(1)},\\ &v_n^{(2)} = 2(u_n^{(1)})^2 - 1,\end{aligned}$$

$$t(b_i)\quad \begin{aligned} &u_0^{(2)} = 0,\quad u_1^{(2)} = 0,\cdots, u_{n-1}^{(2)} = 0,\quad u_n^{(2)} = 1;\\ &v_0^{(2)} = 0,\quad v_1^{(2)} = -v_1^{(1)},\cdots, v_{n-1}^{(2)} = -v_{n-1}^{(1)},\quad v_n^{(2)} = v_n^{(1)}.\end{aligned}$$

Again obviously g_1t is a topological mapping of m_1 on m_2 and $g_2t=t$ a topological mapping of b_1 on b_2. The most important projection $g_2t(m_1)$ is given by

$$g_2t(m_1)\quad \begin{aligned} &u_0^{(2)} = 0,\quad u_1^{(2)} = 0,\cdots, u_{n-1}^{(2)} = 0,\quad u_n^{(2)} = 1,\\ &v_0^{(2)} = 0,\quad v_1^{(2)} = 2u_1^{(1)}u_n^{(1)},\cdots, v_{n-1}^{(2)} = 2u_{n-1}^{(1)}u_n^{(1)},\\ &v_n^{(2)} = 2(u_n^{(1)})^2 - 1.\end{aligned}$$

g_2t maps m_1 continuously in b_2. It maps all the points $(0, u_1^{(1)},\cdots, u_{n-1}^{(1)}, 0)\times\epsilon$ of m_1 on the single point $\epsilon\times(0,\cdots,0,-1)$ of b_2. When $v_n^{(2)}\neq -1$, that is, when $u_n^{(1)}\neq 0$, the last n equations above can be solved for $\bar{u}^{(1)}$ in terms of $v_1^{(2)},\cdots, v_n^{(2)}$. Hence g_2t maps m_1 on b_2. Moreover, on the same point $\epsilon\times(0, v_1^{(2)},\cdots, v_n^{(2)})$ of b_2, g_2t maps the pair of points $(0, \pm u_1^{(1)},\cdots, \pm u_n^{(1)})\times\epsilon$ of m_1, but no other point of m_1 when $v_n^{(2)}\neq -1$. In fact, g_2t maps continuously each of the closed halves, m_1' and m_1'', of m_1 corresponding to $u_n^{(1)}\geqq 0$ and $u_n^{(1)}\leqq 0$, on b_2; and it maps topologically each of the open halves of m_1, corresponding to $u_n^{(1)}>0$ and $u_n^{(1)}<0$, on b_2 minus the point corresponding to $v_n^{(2)}=-1$.

Take a simplicial decomposition of the meridian m_1, symmetric with respect to the center and to the topological product of the hyperplane $u_n^{(1)}=0$ and the point ϵ. Take a coherent orientation of this simplicial m_1. Denote respectively again by m_1, m_1', m_1'' the $(n-1)$-cycle and the $(n-1)$-complexes, which are the sums of all the

[9] See, for example, Seifert's *Solution of problem* 124, mentioned in footnote 4; and C. Weber's *Solution of problem* 84, ibid. pp. 5–6.

$(n-1)$-simplexes on m_1, m_1', m_1'', so oriented. By means of the mapping g_2t, the coherent orientation of m_1' induces an orientation of b_2. Take any simplicial decomposition of b_2 and a coherent orientation determined by this induced orientation. Denote again by b_2 the $(n-1)$-cycle which is the sum of all the $(n-1)$-simplexes on b_2 so oriented. Then, from the last paragraph, on $\mathrm{Bd}(K_2)$ the *homology class* of the singular cycle (ST, p. 97) $g_2t(m_1')$ and that of the cycle b_2 are the same: $(g_2t(m_1'))^* = b_2^*$ on $\mathrm{Bd}(K_2)$.

Now let $Q_0Q_1 \cdots Q_{n-1}$ be an oriented simplex in the coherent orientation of the simplicial m_1. Let Q_i' be the diametrically opposite point of Q_i on m_1. Then either the oriented simplex $Q_0'Q_1' \cdots Q_{n-1}'$ or the oppositely oriented simplex $-Q_0'Q_1' \cdots Q_{n-1}'$ is in the coherent orientation of m_1 according as n is even or odd. Hence, on $\mathrm{Bd}(K_2)$, $(g_2t(m_1''))^* = b_2^*$ for even n, but $= -b_2^*$ for odd n. Hence, on $\mathrm{Bd}(K_2)$, and therefore also on K_2, $(g_2t(m_1))^* = 2b_2^*$ for even n, but $=0$ for odd n.

7. The homology classes of $t(m_1)$ and $t(b_1)$ on K_2. To determine the homology class of $t(m_1)$ on K_2, let us regard the $2n+2$ equations of $t(m_1)$ in §6 as parametric equations of $t(m_1)$ with the n parameters $\bar{u}^{(1)}$, which are the coordinates of a variable point of the $(n-1)$-sphere S: $\bar{u}^{(1)\prime}\bar{u}^{(1)}=1$. Introduce as another parameter the point τ of the interval T: $0 \leqq \tau \leqq 1$. After proper simplicial decomposition and coherent orientation of $D = T \times S$, we have as boundary $\mathrm{Bd}(D)$ of D: $\mathrm{Bd}(D) = S_0' - S_1'$, where S_τ' denotes the topological product of the point τ and S. Now map D continuously in R_2 by the mapping:

$$f_1 \qquad \begin{aligned} &u_0^{(2)} = -(1-\tau^2)^{1/2}, \quad u_1^{(2)} = \tau u_1^{(1)}, \cdots, u_n^{(2)} = \tau u_n^{(1)}; \\ &v_0^{(2)} = 0, \quad v_1^{(2)} = 2u_1^{(1)}u_n^{(1)}, \cdots, v_{n-1}^{(2)} = 2u_{n-1}^{(1)}u_n^{(1)}, \\ &\qquad\qquad v_n^{(2)} = 2(u_n^{(1)})^2 - 1. \end{aligned}$$

Obviously $f_1(D)$ is on K_2, and $f_1 = t$ on $S_1' = m_1$. Hence

$$t(m_1) \sim f_1(S_0') \quad \text{on} \quad K_2.$$

Again after proper simplicial decomposition and coherent orientation of $H = \Sigma \times S$, where Σ is the interval: $0 \leqq \sigma \geqq 1$, we have $\mathrm{Bd}(H) = S_0'' - S_1''$ where S_σ'' denotes the topological product of the point σ and S. Map H in R_2 by the continuous mapping

$$f_2 \quad \begin{aligned} &u_0^{(2)} = -\cos(\sigma\pi/2), \quad u_1^{(2)} = 0, \cdots, u_{n-1}^{(2)} = 0, \quad u_n^{(2)} = \sin(\sigma\pi/2); \\ &v_0^{(2)} = 0, \quad v_1^{(2)} = 2u_1^{(1)}u_n^{(1)}, \cdots, v_{n-1}^{(2)} = 2u_{n-1}^{(1)}u_n^{(1)}, \quad v_n^{(2)} = 2(u_n^{(1)})^2 - 1. \end{aligned}$$

Obviously $f_2(H)$ is on K_2, $f_2=f_1$ on $S_0''=S_0'$, and $f_2(S_1'')=g_2t(m_1)$. Hence

$$f_1(S_0') = f_2(S_0'') \sim f_2(S_1'') = g_2t(m_1) \quad \text{on} \quad K_2.$$

This homology together with the preceding one gives

$$t(m_1) \sim g_2t(m_1) \quad \text{on} \quad K_2.$$

From the result at the end of §6, we have finally that, on K_2, $(t(m_1))^* = 2b_2^*$ for even n, but $=0$ for odd n.

Obviously $(t(b_1))^*=b_2^*$ on K_2.

On the sum K_1+K_2, formed by means of the mapping t, $m_1^*=2b_2^*$ for even n, but $=0$ for odd n, and $b_1^*=b_2^*$.

8. **The homology classes of $t(m_1)$ and $t(b_1)$ on Bd(K_2).** From Künneth's theorem on the Betti groups of topological product of complexes (AH, p. 308), the Betti groups of K_i and Bd(K_i) together with their bases can be easily determined. All K_i and Bd(K_i) have no torsion coefficients. Their only Betti numbers, which are not zero, are $p^0(K_i) = p^{n-1}(K_i) = 1$, and $p^0(\text{Bd}(K_i)) = p^{2n-2}(\text{Bd}(K_i)) = 1$, $p^{n-1}(\text{Bd}(K_i))=2$.

Let m_i^* and b_i^* denote respectively the homology classes of the $(n-1)$-cycles on K_i or Bd(K_i), which are the sums of all the coherently oriented $(n-1)$-simplexes on certain simplicial decompositions of m_i and b_i.[10] Then the $(n-1)$-dimensional homology basis of K_i consists of b_i^* only, while that of Bd(K_i), of m_i^* and b_i^*.

The $(n-1)$-dimensional Betti group $B^{n-1}(\text{Bd}(K_i))$ is the free Abelian group with the two free generators m_i^* and b_i^*. The topological mapping t of Bd(K_1) on Bd(K_2) induces an isomorphism of $B^{n-1}(\text{Bd}(K_1))$ on $B^{n-1}(\text{Bd}(K_2))$ (ST, p. 98). This isomorphism must be given by

$$(t(m_1))^* = \alpha m_2^* + \beta b_2^*, \qquad (t(b_1))^* = \gamma m_2^* + \delta b_2^*,$$

on Bd(K_2), where the coefficients are integers and $\alpha\delta-\beta\gamma=\pm 1$.[11] From the equations of $t(b_1)$ in §6, $\gamma=0$; and therefore $\delta=\pm 1$. Hence $\alpha=\pm 1$. Take the simplicial decompositions and coherent orientations of m_1 and b_2 as given in §6. We can then, and shall, assign coherent orientations to m_2 and b_1 such that $\delta=1$ and $\alpha=1$. Hence $(t(m_1))^* = m_2^*+\beta b_2^*$, and $(t(b_1))^*=b_2^*$ on Bd(K_2). The first equation implies

[10] The two interpretations of the symbols m_i^* and b_i^* (namely, on K_i or on Bd(K_i)) will give rise to no confusion, as we shall always state explicitly the complex on which the homology classes are considered.

[11] K. Reidemeister, *Einführung in die kombinatorische Topologie*, Braunschweig, 1932, p. 95.

$(t(m_1))^* = \beta b_2^*$ on K_2, since $m_2^* = 0$ on K_2. From the result at the end of §7, $\beta = 2$ for even n, but $= 0$ for odd n. Hence, on $\mathrm{Bd}(K_2)$, $(t(m_1))^* = m_2^* + 2b_2^*$ for even n, but $= m_2^*$ for odd n.

Let $K_1 \cdot K_2$ denote either of the isomorphic sub-complexes of K_1 and K_2, which are to be identified in forming the sum $K_1 + K_2$: $K_1 \cdot K_2 = \mathrm{Bd}(K_1) = \mathrm{Bd}(K_2)$. On $K_1 \cdot K_2$, $m_1^* = m_2^* + 2b_2^*$ for even n, but $= m_2^*$ for odd n, and $b_1^* = b_2^*$.

9. **The Betti groups of M.** It is obvious that $B^0(M) \approx G_0$.

Let $N^r(K_1 \cdot K_2)$ be the subgroup of $B^r(K_1 \cdot K_2)$, whose elements are the homology classes of those r-cycles, which are null-homologous on both K_1 and K_2; $S^r(K_1 + K_2)$ the sub-group of $B^r(K_1 + K_2)$, whose elements are the homology classes of those r-cycles, each of which is the sum of one r-cycle on K_1 and one r-cycle on K_2. We shall determine as follows $B^r(M)$ by means of the theorem:

$$B^r(M) - S^r(K_1 + K_2) \approx N^{r-1}(K_1 \cdot K_2)$$

for every $r \geqq 1$ (AH, p. 293).

Let n be even. We say that $S^{n-1}(K_1 + K_2) \approx G_2$, $N^{2n-2}(K_1 \cdot K_2) \approx G_0$, and $S^r(K_1 + K_2)$, $N^{r-1}(K_1 \cdot K_2)$ are the null for all other values of $r \geqq 1$. Let us prove first for example $S^{n-1}(K_1 + K_2) \approx G_2$. Any $(n-1)$-cycle on K_i is homologous on K_i to an integral multiple of b_i. Hence any $(n-1)$-cycle on $K_1 + K_2$, which is a sum of one $(n-1)$-cycle on K_1 and one $(n-1)$-cycle on K_2, is homologous on $K_1 + K_2$ to a linear combination of b_1 and b_2 with integral coefficients. From the results in §7, $2b_2^* = 0$ and $b_1^* = b_2^*$ on $K_1 + K_2$. Moreover, $b_1^* \neq 0$ on $K_1 + K_2$. For, if otherwise, let C be a complex on $K_1 + K_2$, such that $\mathrm{Bd}(C) = b_1$. Let $C = C_1 + C_2$, where C_i is on K_i. Then $b_1 - \mathrm{Bd}(C_1) = \mathrm{Bd}(C_2)$. This shows that the right member would be on K_2, while the left member would be on K_1. Hence $\mathrm{Bd}(C_2)$ and $b_1 - \mathrm{Bd}(C_1)$, and therefore $\mathrm{Bd}(C_1)$, would be all on $K_1 \cdot K_2$. Being on $K_1 \cdot K_2$ and null-homologous on K_i, $\mathrm{Bd}(C_i)$ would be homologous on $K_1 \cdot K_2$ to an integral multiple of m_i, say $\lambda_i m_i$. Hence $b_1^* = \lambda_1 m_1^* + \lambda_2 m_2^*$ on $K_1 \cdot K_2$, which is impossible from §8. Hence $b_1^* \neq 0$ on $K_1 + K_2$, and $S^{n-1}(K_1 + K_2) \approx G_2$. Next $N^{n-1}(K_1 \cdot K_2) \approx G_1$ follows from §8. The other isomorphisms stated are obvious, and we shall omit the proof.

Hence for every $r \geqq 1$, at least one of the two groups $S^r(K_1 + K_2)$ and $N^{r-1}(K_1 \cdot K_2)$ is the null, and, from the theorem stated above, $B^r(M)$ is isomorphic with the other group. Consequently $B^{n-1}(M) \approx G_2$, $B^{2n-1}(M) \approx G_0$, and $B^r(M) \approx G_1$ for all other values of $r \geqq 1$.

Similarly, when n is odd, $S^{n-1}(K_1 + K_2)$, $N^{n-1}(K_1 \cdot K_2)$, $N^{2n-2}(K_1 \cdot K_2)$ all $\approx G_0$, and $S^r(K_1 + K_2)$, $N^{r-1}(K_1 \cdot K_2) \approx G_1$ for all other values of $r\geqq 1$. Consequently $B^{n-1}(M)$, $B^n(M)$ and $B^{2n-1}(M)$ all $\approx G_0$, and $B^r(M)\approx G_1$ for all other values of $r\geqq 1$.

Thus the proof of the statements in (M1) and (M2) in §1 is completed.

10. **The cases $n=1, 3, 7$.**[8] Let

$$B_8 = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ x_1 & -x_0 & x_3 & -x_2 & x_5 & -x_4 & -x_7 & x_6 \\ x_2 & -x_3 & -x_0 & x_1 & x_6 & x_7 & -x_4 & -x_5 \\ x_3 & x_2 & -x_1 & -x_0 & x_7 & -x_6 & x_5 & -x_4 \\ x_4 & -x_5 & -x_6 & -x_7 & -x_0 & x_1 & x_2 & x_3 \\ x_5 & x_4 & -x_7 & x_6 & -x_1 & -x_0 & -x_3 & x_2 \\ x_6 & x_7 & x_4 & -x_5 & -x_2 & x_3 & -x_0 & -x_1 \\ x_7 & -x_6 & x_5 & x_4 & -x_3 & -x_2 & x_1 & -x_0 \end{pmatrix}.$$

Let B_4 be the four-rowed square matrix at the upper left corner of B_8, and B_2 the two-rowed square matrix at the upper left of B_4. For $n=1, 3, 7$, B_{n+1} has x' and x as the first row and column, and is orthogonal. The mapping

$$t_0 \qquad u = x, \qquad v = B_{n+1}y$$

is a topological mapping of the *whole* M, for $n=1, 3, 7$, on

$$u'u = 1, \qquad v_0 = 0, \qquad v'v = 1,$$

the topological product of an n-sphere and an $(n-1)$-sphere. Hence we have our statement (M3) in §1.

II. The manifold M' of non-oriented linear elements

11. **M' as sum. The cases $n=1, 3, 7$.** By the symbol $(x; y)\rightleftarrows(x; -y)$ we mean that the two points $(x; y)$ and $(x; -y)$ are to be identified. The space M' of non-oriented linear elements of an n-sphere is then represented by the equations of M and the additional condition $(x; y)\rightleftarrows(x; -y)$. In this sense we shall say for simplicity that M' is M *after diametrical identification* $(x; y)\rightleftarrows(x; -y)$. From the symmetry of x and y in the equations of M, M' is homeomorphic with the space given by the equations of M and the additional condition $(x; y)\rightleftarrows(-x; y)$, and therefore is also the space of oriented linear elements of an n-dimensional projective space. M' is obviously a closed $(2n-1)$-manifold if it is connected.

Let M' be decomposed also into two halves M_i', $i=1, 2$, namely the M_i after diametrical identification $(x;\ y)\rightleftarrows(x;\ -y)$. From the nature of the equations of the topological mapping t_i on M_i, t_i carries $(x;\ y)$ and $(x^*;\ y^*)$ respectively into $(u^{(i)};\ v^{(i)})$ and $(u^{(i)};\ -v^{(i)})$ when and only when $(x^*;\ y^*)=(x;\ -y)$. Hence t_i is a topological mapping of M_i' on K_i', namely the K_i after diametrical identification $(u^{(i)};\ v^{(i)})\rightleftarrows(u^{(i)};\ -v^{(i)})$. Let π_i be the S_i in V_i (§3) after diametrical identification $v^{(i)}\rightleftarrows -v^{(i)}$, and $\mathrm{Bd}(M_i')$, $\mathrm{Bd}(K_i')$ the $\mathrm{Bd}(M_i)$, $\mathrm{Bd}(K_i)$ in §4 after diametrical identifications. Then we have here $K_i'=E_i\times\pi_i$, $\mathrm{Bd}(M_1')=\mathrm{Bd}(M_2')$ and $\mathrm{Bd}(K_i')=\mathrm{Bd}(E_i)\times\pi_i$. The mapping t is a topological mapping of $\mathrm{Bd}(K_1')$ on $\mathrm{Bd}(K_2')$. Thus, in the sense of homeomorphism, $M'=K_1'+K_2'$, the sum being defined by means of t.

In the particular cases $n=1, 3, 7$, from the mapping t_0 in §10, topological on the whole M', we conclude that M' is the topological product of an $(n-1)$-sphere and an $(n-1)$-dimensional projective space.

12. **The fundamental group. The case** $n=2$. Let us denote by π_i^r the r-dimensional projective space: $v_0^{(i)}=0$, $v^{(i)\prime}v^{(i)}=1$, $v_{r+2}^{(i)}=\cdots=v_n^{(i)}=0$, $v^{(i)}\rightleftarrows -v^{(i)}$. In particular, π_i^{n-1} is our π_i previously defined. Let $\sigma_i^r=\epsilon\times\pi_i^r$. For $n>2$, the fundamental group of K_i' or $\mathrm{Bd}(K_i')$ is the cyclic group of order 2 with the class of the oriented σ_i^1 as the generator (ST, p. 156). From the equations of $t(b_1)$ in §6, we know that t maps the oriented σ_1^1 into properly oriented σ_2^1. Therefore the fundamental group of M' is the cyclic group of order 2 (ST, pp. 177–178).

When $n=2$, $\mathrm{Bd}(K_i')$ is the torus. From §5 together with a necessary change of notations, we know that the fundamental group of K_i' is given by two generators $[m_i]$ and $[\sigma_i]$ and the defining relation $[m_i]=1$, and that

$$([t(m_1)]) = [m_2]([\sigma_2^1])^4, \qquad ([t(\sigma_1^1)]) = [\sigma_2^1].$$

Hence the fundamental group of M' is the cyclic group of order 4 (ST, pp. 177–178), in agreement with Threlfall's result,[8] and M' is the Linsenraum (4, 1) (ST, pp. 219–220, 210, 215).

13. **The Betti groups of** K_i' **and** $\mathrm{Bd}(K_i')$. Henceforth we always assume $n\geqq 2$. For simplicity, let our σ_i^{n-1} previously defined be denoted by σ_i.

The Betti groups of the factors of K_i' and $\mathrm{Bd}(K_i')$ together with their bases are well known. The Betti groups of K_i' and $\mathrm{Bd}(K_i')$ together with their bases can be easily determined from Künneth's theorem. The Betti groups B^0 of K_i' and $\mathrm{Bd}(K_i')$ are all $\approx G_0$, and the others are listed as follows.

When n is even, $B^r(K_i') \approx G_2$, $r=1, 3, \cdots, n-3$, with the basis $(\sigma_i^r)^*$, $B^{n-1}(K_i') \approx G_0$ with the basis σ_i^*, $B^r(\mathrm{Bd}(K_i'))$ and $B^{n-1+r}(\mathrm{Bd}(K_i')) \approx G_2$ with the respective bases $(\sigma_i^r)^*$ and $(\mathrm{Bd}(E_i)\times\pi_i^r)^*$, $B^{n-1}(\mathrm{Bd}(K_i')) \approx G_0+G_0$ with the basis m_i^* and σ_i^*, $B^{2n-2}(\mathrm{Bd}(K_i')) \approx G_0$ with the basis $(\mathrm{Bd}(E_i)\times\pi_i)^*$, and all the rest are the null.

When n is odd, $B^r(K_i') \approx G_2$, $r=1, 3, \cdots, n-2$, with the basis $(\sigma_i^r)^*$, $B^r(\mathrm{Bd}(K_i'))$ and $B^{n-1+r}(\mathrm{Bd}(K_i')) \approx G_2$ with the respective bases $(\sigma_i^r)^*$ and $(\mathrm{Bd}(E_i)\times\pi_i^r)^*$, $B^{n-1}(\mathrm{Bd}(K_i')) \approx G_0$ with the basis m_i^*, and all the rest are the null.

14. **The homology classes of $t(m_1)$, $t(\sigma_1)$ and $t(\mathrm{Bd}(E_1)\times\pi_1^r)$ on** $\mathrm{Bd}(K_2')$. We shall assume as in §6 that, for even n, σ_i has a simplicial decomposition and a proper orientation. For odd n, the projective space σ_i is non-orientable and there is no longer any non-null (integral) $(n-1)$-cycle on it. We assume only that σ_i in this case has a simplicial decomposition and that all its $(n-1)$-simplexes are oriented so that t carries the integral complex σ_1 into the integral complex σ_2. Regarding σ_i as b_i after diametrical identification, we can apply to M' the discussions in §§6–8 with only slight modification in arguments and with necessary change of notations. From these sections we have evidently the following results: On K_i', $(t(m_1))^*=4\sigma_2^*$ for even n, but $=0$ for odd n, and $(t(\sigma_1))^*=\sigma_2^*$ for even n. Hence on the sum $M'=K_1'+K_2'$, $m_1^*=4\sigma_2^*$ for even n, but $=0$ for odd n, and $\sigma_1^*=\sigma_2^*$ for even n. On $\mathrm{Bd}(K_2')$, $(t(m_1))^*=m_2^*+4\sigma_2^*$ for even n, but $=m_2^*$ for odd n, and $(t(\sigma_1))^*=\sigma_2^*$ for even n. Hence on $K_1'\cdot K_2'$, $m_1^*=m_2^*+4\sigma_2^*$ for even n, but $=m_2^*$ for odd n, and $\sigma_1^*=\sigma_2^*$ for even n.

Suppose that σ_i^r and $\mathrm{Bd}(E_i)\times\pi_i^r$, $r=1, 3, \cdots, n-3, n-1$ for even n and $r=1, 3, \cdots, n-2$ for odd n, has simplicial decompositions and proper coherent orientations. From the nature of the equations of $t(b_1)$ in §6,

$$(t(\overset{r}{\sigma_1}))^* = \pm (\overset{r}{\sigma_2})^* ;$$

and since t is a topological mapping of $\mathrm{Bd}(K_1')$ on $\mathrm{Bd}(K_2')$, it induces an isomorphism of $B^{n-1+r}(\mathrm{Bd}(K_1'))$ on $B^{n-1+r}(\mathrm{Bd}(K_2'))$:

$$(t(\mathrm{Bd}\ (E_1)\times\overset{r}{\pi_1}))^* = \pm (\mathrm{Bd}\ (E_2)\times\overset{r}{\pi_2})^*.$$

We can, and shall, assume that the coherent orientations of σ_i^r and $\mathrm{Bd}(E_i)\times\pi_i^r$ have been so chosen that the signs in the above equations are all positive. Hence on $K_1'\cdot K_2'$, $(\sigma_1^r)^*=(\sigma_2^r)^*$, $(\mathrm{Bd}(E_1)\times\pi_1^r)^*=(\mathrm{Bd}(E_2)\times\pi_2^r)^*$.

15. **The Betti groups of M'.** Obviously $B^0(M')$ is the infinite cyclic group, and M' is connected.

Let n be even. We say that $S^r(K_1'+K_2')\approx G_2$, $r=1, 3, \cdots, n-3$, $S^{n-1}(K_1'+K_2')\approx G_4$, $N^{r-1}(K_1'\cdot K_2')\approx G_2$, $r=n+1, n+3, \cdots, 2n-3$, $N^{2n-2}(K_1'\cdot K_2')\approx G_0$, and $S^r(K_1'+K_2')$, $N^{r-1}(K_1'\cdot K_2')$ are the null for all other values of $r\geqq 1$. The proof of these isomorphisms is based on the results in §§13–14 and an argument similar to that used in §9. Then, as in §9, we have $B^r(M')\approx G_2$, $r=1, 3, \cdots, n-3; n+1, n+3, \cdots, 2n-3$, $B^{n-1}(M')\approx G_4$, $B^{2n-1}(M')\approx G_0$, and $B^r(M')$ are the null for all other values of $r\geqq 1$. M' is therefore orientable, when n is even.

Similarly, when n is odd, $S^r(K_1'+K_2')\approx G_2$, $r=1, 3, \cdots, n-2$, $N^{r-1}(K_1'\cdot K_2')\approx G_2$, $r=n+1, n+3, \cdots, 2n-2$, $N^{n-1}(K_1'\cdot K_2')\approx G_0$, and $S^r(K_1'+K_2')$, $N^{r-1}(K_1'\cdot K_2')$ are the null for all other values of $r\geqq 1$. Hence $B^r(M')\approx G_2, r=1,3, \cdots, n-2; n+1, n+3, \cdots, 2n-2$, and $B^n(M')\approx G_0$, and $B^r(M')$ are the null for all other values of $r\geqq 1$. M' is therefore non-orientable when n is odd.

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