A NOTE ON HYPERGEODESICS AND CANONICAL LINES

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In this note we introduce two families of hypergeodesics on a non-ruled surface in ordinary projective space. Consideration of the properties of these hypergeodesics leads to certain geometrical constructions which yield canonical lines of the first kind from a given canonical line of the second kind.

We shall assume that the differential equations of a non-ruled surface S are written in the Fubini canonical form[1]

$$(1)\quad x_{uu} = px + \theta_u x_u + \beta x_v, \qquad x_{vv} = qx + \gamma x_u + \theta_v x_v \quad (\theta = \log \beta\gamma).$$

We select an ordinary point P_x of the surface S as one vertex of the usual local tetrahedron of reference. When a curve C_λ through the point P_x is regarded as being imbedded in the one-parameter family of curves represented on S by the equation

$$(2)\qquad dv - \lambda(u, v)du = 0,$$

the osculating plane at the point P_x of the curve C_λ has the local equation

$$(3)\qquad 2\lambda(\lambda x_2 - x_3) + (\lambda' + \beta - \theta_u\lambda + \theta_v\lambda^2 - \gamma\lambda^3)x_4 = 0,$$

in which we have placed $\lambda' = \lambda_u + \lambda\lambda_v$.

It will be recalled that two lines $l_1(a, b)$, $l_2(a, b)$ are reciprocal lines[2] at a point P_x of a surface if the line $l_1(a, b)$ joins the point P_x and the point y defined by placing

$$y = -ax_u - bx_v + x_{uv}$$

and the line $l_2(a, b)$ joins the points ρ, σ defined by

$$\rho = x_u - bx, \qquad \sigma = x_v - ax,$$

where a, b are functions of u, v. As the point P_x varies over the surface S, the lines $l_1(a, b)$, $l_2(a, b)$ generate two reciprocal congruences Γ_1, Γ_2, respectively.

The two reciprocal lines $l_1(a, b)$, $l_2(a, b)$ are canonical lines $l_1(k)$, $l_2(k)$ of the first and second kind respectively in case

$$a = -k\psi, \qquad b = -k\phi,$$

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[1] E. P. Lane, *Projective differential geometry of curves and surfaces*, Chicago, 1932, p. 69.

[2] E. P. Lane, *loc. cit.*, pp. 82–85.

where k is a constant and ϕ, ψ are defined by

$$\phi = (\log \beta\gamma^2)_u, \qquad \psi = (\log \beta^2\gamma)_v.$$

Canonical lines of the first kind lie in the canonical plane whose local equation is

$$\phi x_2 - \psi x_3 = 0. \tag{4}$$

The equation of the plane $\pi(a, b)$ which is the harmonic conjugate of the tangent plane of the surface at the point P_x with respect to the two focal planes of a general line $l_2(a, b)$ of the congruence Γ_2 is given by

$$x_1 + bx_2 + ax_3 + [2^{-1}(a_u + b_v) + ab]x_4 = 0. \tag{5}$$

For a general canonical line $l_2(k)$, the local coordinates $\xi_1, \cdots, \xi_4$ of the plane $\pi(k)$ are given by

$$\xi_1 = 1, \quad \xi_2 = -k\phi, \quad \xi_3 = -k\psi, \quad \xi_4 = -3k\theta_{uv}/2 + k^2\phi\psi. \tag{6}$$

The equations, in plane coordinates, of the two asymptotic osculating quadrics Q_u, Q_v of a curve C_λ are respectively

$$\begin{aligned} &2\lambda^3(\xi_2\xi_3 - \xi_1\xi_4) - 2\beta\xi_1(\beta\xi_1 - \lambda\xi_2 + \lambda^2\xi_3) - C\xi_1^2 = 0, \\ &2(\xi_2\xi_3 - \xi_1\xi_4) - 2\gamma\lambda\xi_1(\gamma\lambda\xi_1^2 + \xi_2 - \lambda\xi_3) - D\xi_1^2 = 0, \end{aligned} \tag{7}$$

where we have placed

$$C = \beta[\lambda' - \beta + (\phi - \theta_u)\lambda - (2\psi - \theta_v)\lambda^2] - (\beta\gamma + \theta_{uv})\lambda^3,$$
$$D = \gamma[-\lambda' - \gamma\lambda^3 + (\psi - \theta_v)\lambda^2 - (2\phi - \theta_u)\lambda] - (\beta\gamma + \theta_{uv}).$$

Let us now suppose that, at each point P_x of the curve C_λ, the asymptotic osculating quadric Q_u of C_λ is tangent to the plane $\pi(k)$. Then we find that the function λ satisfies the differential equation

$$\lambda' = A_1 + B_1\lambda + C_1\lambda^2 + D_1\lambda^3, \tag{8}$$

where the coefficients A_1, B_1, C_1, D_1 are given by

$$\begin{aligned} A_1 &= -\beta, & B_1 &= \theta_u - (1 + 2k)\phi, \\ C_1 &= 2(1 + k)\psi - \theta_v, & D_1 &= (1/\beta)[\beta\gamma + (1 + 3k)\theta_{uv}], \end{aligned} \tag{9}$$

in which, for the present, we assume $k \neq -1/3$, so that the canonical line $l_2(k)$ is not the second axis a_2 of Čech.

Similarly, the asymptotic osculating quadric Q_v of C_λ is tangent to the plane $\pi(k)$ if, and only if,

$$\lambda' = A_2 + B_2\lambda + C_2\lambda^2 + D_2\lambda^3, \tag{10}$$

where the coefficients A_2, B_2, C_2, D_2 are given by

$$(11)\quad \begin{aligned} A_2 &= -(1/\gamma)[\beta\gamma + (1+3k)\theta_{uv}], & B_2 &= \theta_u - 2(1+k)\phi, \\ C_2 &= (1+2k)\psi - \theta_v, & D_2 &= \gamma, \end{aligned}$$

in which $k \neq -1/3$. Thus we reach the following conclusion:

At each point P_x of a curve C_λ, each of the asymptotic osculating quadrics Q_u and Q_v of C_λ is tangent to the plane $\pi(k)$ which is the harmonic conjugate of the tangent plane of the surface at the point P_x with respect to the two focal planes of any canonical line $l_2(k)$, except the second axis a_2 of Čech, if, and only if, C_λ is an integral curve (hypergeodesic) of the respective differential equations (8), (10).

By means of equation (3), together with equation (8), we find that the osculating plane at a point P_x of a hypergeodesic of the family (8) has the local equation

$$(12)\quad \begin{aligned} &2\beta(\lambda x_2 - x_3) \\ &\qquad + [-(1+2k)\beta\phi + 2(1+k)\beta\psi\lambda + (1+3k)\theta_{uv}\lambda^2]x_4 = 0. \end{aligned}$$

The envelope of the osculating planes at the point P_x of all the hypergeodesics of the family (8) is found from equation (12) to be the nondegenerate quadric cone whose equation is

$$(13)\quad \beta[x_2 + (1+k)\psi x_4]^2 + (1+3k)\theta_{uv}[2x_3 + (1+2k)\phi x_4]x_4 = 0.$$

Similarly, the envelope of the osculating planes at the point P_x of the hypergeodesics (10) is the quadric cone

$$(14)\quad \gamma[x_3 + (1+k)\phi x_4]^2 + (1+3k)\theta_{uv}[2x_2 + (1+2k)\psi x_4]x_4 = 0.$$

The vertex of each of the cones (13), (14) is, of course, the point P_x. Furthermore, the cone (13) is tangent to the tangent plane of the surface at the point P_x along the asymptotic v-tangent at P_x, while the cone (14) has the asymptotic u-tangent of S at P_x for its line of contact with the tangent plane. The polar plane of any point on the u-tangent with respect to the cone (13) intersects this cone, besides in the v-tangent at P_x, also in a generator which is the line $l_1(a, b)$ with a and b defined by the formulas

$$(15)\quad a = (1+k)\psi, \qquad b = 2^{-1}(1+2k)\phi.$$

We may also regard this line as being determined by the planes whose equations are

$$(16)\quad x_2 + (1+k)\psi x_4 = 0, \qquad x_3 + 2^{-1}(1+2k)\phi x_4 = 0.$$

Similarly, the polar plane of any point on the v-tangent with respect to the cone (14) intersects this cone in the u-tangent at P_x and in the line $l_1(a, b)$ for which

$$a = 2^{-1}(1 + 2k)\psi, \qquad b = (1 + k)\phi. \tag{17}$$

This line may also be regarded as determined by the planes

$$x_2 + 2^{-1}(1 + 2k)\psi x_4 = 0, \qquad x_3 + (1 + k)\phi x_4 = 0. \tag{18}$$

The locus of the line (16) for all canonical lines $l_2(k)$ is found by eliminating k from equations (16) to be the plane

$$\phi x_2 - \psi x_3 + 2^{-1}\phi\psi x_4 = 0. \tag{19}$$

Similarly, the locus of the line (18) is the plane

$$\phi x_2 - \psi x_3 - 2^{-1}\phi\psi x_4 = 0. \tag{20}$$

Thus we find that *the tangent plane of the surface at the point P_x and the canonical plane* (4) *separate the planes defined by equations* (19), (20) *harmonically.*

We now describe simple geometrical constructions which yield canonical lines of the first kind from a given canonical line $l_2(k)$ of the second kind, except the second axis a_2 of Čech. In the first place, the plane determined by the two lines (16), (18) has the equation

$$\phi x_2 + \psi x_3 + 2^{-1}(3 + 4k)\phi\psi x_4 = 0, \tag{21}$$

and is found to intersect the canonical plane (4) in the canonical line $l_1(k_1)$ for which

$$k_1 = -\,4^{-1}(3 + 4k). \tag{22}$$

The polar plane of any point on the u-tangent at P_x with respect to the cone (13) intersects the polar plane of any point on the v-tangent at P_x with respect to the cone (14) in the canonical line $l_1(k_2)$ for which

$$k_2 = -\,(1 + k). \tag{23}$$

Furthermore, the plane which is tangent to the cone (13) along the line (16) intersects the plane which is tangent to the cone (14) along the line (18) in the canonical line $l_1(k_3)$ for which

$$k_3 = -\,2^{-1}(1 + 2k). \tag{24}$$

We remark that if the line $l_2(k)$ is the second edge e_2 of Green, for which $k = -1/4$, then the three canonical lines $l_1(k_1)$, $l_1(k_2)$, and $l_1(k_3)$ obtained by the preceding constructions are respectively the first directrix d_1 of Wilczynski, the first canonical line for which $k=-3/4$, and the first edge e_1 of Green.

It may be seen from the formulas (22), (23), and (24) that a given canonical line $l_2(k)$ will yield in turn each of the canonical lines $l_1(k_1)$, $l_1(k_2)$, and $l_1(k_3)$ for its reciprocal if, and only if, the given canonical line $l_2(k)$ is the second canonical line for which $k=-3/8$, the second directrix d_2 of Wilczynski, and the second edge e_2 of Green.

Finally, if the given canonical line $l_2(k)$ is the second axis a_2 of Čech for which $k=-1/3$, *the asymptotic osculating quadric Q_u of C_λ is tangent to the plane $\pi(-1/3)$ if, and only if,*

$$\lambda' = -\beta + (\theta_u - \phi/3)\lambda + (4\psi/3 - \theta_v)\lambda^2 + \gamma\lambda^3, \tag{25}$$

so that the curve C_λ is a union curve of the congruence Γ_1 of lines $l_1(a, b)$ for which

$$a = 2\psi/3, \qquad b = \phi/6.$$

Similarly, the asymptotic osculating quadric Q_v of C_λ is tangent to the plane $\pi(-1/3)$ if, and only if,

$$\lambda' = -\beta + (\theta_u - 4\phi/3)\lambda + (\psi/3 - \theta_v)\lambda^2 + \gamma\lambda^3, \tag{26}$$

in which case C_λ is a union curve of the congruence Γ_1 of lines $l_1(a, b)$ for which

$$a = \psi/6, \qquad b = 2\phi/3.$$

The plane determined by the two lines thus defined at the point P_x intersects the canonical plane in the canonical line $l_1(k)$ for which $k=-5/12$, namely, the first axis of Bompiani.

We conclude with the statement that if at the point P_x both of the asymptotic osculating quadrics Q_u and Q_v of C_λ are tangent to the plane $\pi(-1/3)$, then $\phi+\psi\lambda=0$, so that the curve C_λ is tangent at P_x to the second canonical tangent t_2.

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