ON APPROXIMATE ISOMETRIES

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In a previous paper, a problem of mathematical "stability" for the case of the linear functional equation was studied.[1] It was shown that if a transformation $f(x)$ of a vector space E_1 into a Banach space E_2 satisfies the inequality $\|f(x+y)-f(x)-f(y)\|<\epsilon$ for some $\epsilon>0$ and all x and y in E_1, then there exists an additive transformation $\phi(x)$ of E_1 into E_2 such that $\|f(x)-\phi(x)\|<\epsilon$.

In the present paper we consider a stability problem for isometries. By an ϵ-isometry of one metric space E into another E' is meant a transformation $T(x)$ which changes distances by at most ϵ, where ϵ is some positive number; that is, $|\rho(x, y)-\rho(T(x), T(y))|<\epsilon$ for all x and y in E. Given an ϵ-isometry $T(x)$, our object is to establish the existence of a true isometry $U(x)$ which approximates $T(x)$; more precisely, to establish the existence of a constant $k>0$ depending only on the metric spaces E and E' such that $\rho(T(x), U(x))<k\epsilon$ for all x in E. In this paper this result will be proved for the case in which $E=E'$, where E is n-dimensional Euclidean space or Hilbert space (not necessarily separable). The case in which E is the space C of continuous functions will be treated in another paper.

The above problem of ϵ-isometries is related to the problem of constructing space models for sets in which distances between points are given only with a certain degree of exactness (measurements are possible only with a certain degree of precision). The question of the uniqueness of the idealized model corresponding to the given measurements and the extrapolation from the measurements to the model could be looked upon as a problem in determining a strict isometry from an approximate isometry.

In the case of certain simple metric spaces, for example the surface of the Euclidean sphere, this question can be answered in the affirmative, but it may be more difficult for other bounded manifolds. A simple but interesting example showing a case where the answer is *negative* has been worked out by R. Swain.

THEOREM 1. *Let E be a complete abstract Euclidean vector space.*[2]

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[1] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A. vol. 27 (1941) pp. 222–224.

[2] A complete Euclidean vector space is a Banach space whose norm is generated by an inner product, (x, y). It includes real Hilbert space and n-dimensional Euclidean spaces as special cases.

Let $T(x)$ be an ϵ-isometry of E into itself such that $T(0)=0$. The limit $U(x)=\lim_{n\to\infty}(T(2^nx)/2^n)$ exists for every x in E and $U(x)$ is an isometric transformation.

PROOF. Put $r=\|x\|$. Then $|\|T(x)\|-r|<\epsilon$ and $|\|T(x)-T(2x)\|-r|<\epsilon$. Put also $y_0=T(2x)/2$, so that $|r-\|y_0\||<\epsilon/2$. Consider the intersection of the two spheres: $S_1=[y;\ \|y\|<r+\epsilon]$, $S_2=[y;\ \|y-2y_0\|<r+\epsilon]$. Now $T(x)$ belongs to this intersection, and for any point y of $S_1\cap S_2$ we have

$$2\|y-y_0\|^2=2\|y\|^2+2\|y_0\|^2-4(y,y_0);$$
$$\|y-2y_0\|^2=\|y\|^2+4\|y_0\|^2-4(y,y_0)<(r+\epsilon)^2$$

and $\|y\|^2<(r+\epsilon)^2$. It follows that

$$\begin{aligned}2\|y-y_0\|^2&<(r+\epsilon)^2+\|y\|^2-2\|y_0\|^2<2(r+\epsilon)^2-2\|y_0\|^2\\&<2(r+\epsilon)^2-2(r-\epsilon/2)^2=6\epsilon r+3\epsilon^2/2.\end{aligned}$$

Hence, $\|T(x)-T(2x)/2\|<2(\epsilon\|x\|)^{1/2}$ if $\|x\|\geqq\epsilon$, and $\|T(x)-T(2x)/2\|<2\epsilon$ in the contrary case.

Therefore, for all x in E the inequality

$$\|T(x/2)-T(x)/2\|<2^{-1/2}k(\|x\|)^{1/2}+2\epsilon \tag{1}$$

is satisfied, where $k=2\epsilon^{1/2}$. Now let us make the inductive assumption

$$\|T(2^{-n}x)-2^{-n}T(x)\|<2^{-n/2}k(\|x\|)^{1/2}\left(\sum_{i=0}^{n-1}2^{-i/2}\right)+(1-2^{-n})\cdot 4\epsilon. \tag{2}$$

The inequality (2) is true for $n=1$. Assuming it true for any particular value of n we shall prove it for $n+1$.

Dividing the inequality (2) by 2 we have

$$\begin{aligned}&\|T(2^{-n}x)/2-2^{-n-1}T(x)\|\\&\qquad<2^{-(n+1)/2}k(\|x\|)^{1/2}\left(\sum_{i=1}^{n}2^{-i/2}\right)+(1/2-2^{-n-1})\cdot 4\epsilon.\end{aligned}$$

Replacing x by $2^{-n}x$ in the inequality (1) we get

$$\|T(2^{-n-1}x)-T(2^{-n}x)/2\|<2^{-(n+1)/2}k(\|x\|)^{1/2}+2\epsilon.$$

On adding the last two inequalities we obtain

$$\begin{aligned}&\|T(2^{-n-1}x)-2^{-n-1}T(x)\|\\&\qquad<2^{-(n+1)/2}k(\|x\|)^{1/2}\left(\sum_{i=0}^{n}2^{-i/2}\right)+(1-2^{-n-1})\cdot 4\epsilon.\end{aligned}$$

This proves the induction. Therefore inequality (2) is true for all x

in E and for $n=1, 2, 3, \cdots$. If we put $a=k\sum_{i=0}^{\infty}2^{-i/2}$, we have

$$\|T(2^{-n}x) - 2^{-n}T(x)\| < 2^{-n/2}a(\|x\|)^{1/2} + 4\epsilon.$$

Hence, if m and p are any positive integers,

$$\begin{aligned}&\|2^{-m}T(2^m x) - 2^{-m-p}T(2^{m+p}x)\| \\ &\qquad = 2^{-m}\|T(2^{m+p}x/2^p) - 2^{-p}T(2^{m+p}x)\| < 2^{-m/2}a(\|x\|)^{1/2} + 2^{2-m}\epsilon,\end{aligned}$$

for all x in E.

Therefore since E is a complete space, the limit $U(x) = \lim_{n\to\infty} (T(2^n x)/2^n)$ exists for all x in E.

To prove that $U(x)$ is an isometry, let x and y be any two points of E. Divide the inequality

$$|\,\|T(2^n x) - T(2^n y)\| - 2^n\|x - y\|\,| < \epsilon$$

by 2^n and take the limit as $n\to\infty$. The result is $\|U(x) - U(y)\| = \|x-y\|$. This completes the proof of Theorem 1.

THEOREM 2. *Let T satisfy the hypotheses of Theorem* 1 *and let u and x be any points of E such that $\|u\|=1$ and $(x, u)=0$. Then $|(T(x), U(u))| \leqq 3\epsilon$, where $U(x)$ is defined as in the statement of Theorem* 1.

PROOF. For an arbitrary integer n put $z=2^n u$. Let y denote an arbitrary point of the sphere S_n of radius 2^n and center at z. Then $\|y-z\| = \|z\|$ and it follows that $(y, u) = 2^{-n-1}(y, y)$. Since T is an ϵ-isometry, $\|T(y) - T(z)\| = \eta(y, z) + \|T(z)\|$ where $|\eta(y, z)| < 2\epsilon$.

The last equality may be written

$$2(T(y), T(z)) = (T(y), T(y)) - 2\eta\|T(z)\| - \eta^2.$$

Dividing by 2^{n+1} and remembering that $z=2^n u$, we obtain the equality

$$(3)\quad (T(y), 2^{-n}T(2^n u)) = \frac{1}{2^{n+1}}\left[(T(y), T(y)) - \eta^2\right] - \eta\left\|\frac{T(2^n u)}{2^n}\right\|.$$

Now let x be any point of the hyperplane $(x, u)=0$.

Then $y=x+ru$, where $r=2^n-(2^{2n}-\|x\|^2)^{1/2}$, is a point of the sphere S_n. For,

$$\begin{aligned}\|y - z\|^2 &= (y, y) - 2(y, z) + (z, z) \\ &= (x, x) + r^2 - 2(x, z) - 2(u, z)\cdot r + (z, z) \\ &= r^2 - 2^{n+1}r + \|x\|^2 + \|z\|^2 = \|z\|^2.\end{aligned}$$

Moreover, $\|y-x\| = r \to 0$ as $n \to \infty$. By Theorem 1, $t = \lim_{n\to\infty} 2^{-n}T(2^n u)$ exists and is a unit vector. Finally, for an arbitrary positive δ and n sufficiently large, one can easily establish the following inequalities by means of equality (3) and the above remarks:

$$\begin{aligned} |(T(x), t)| &\leqq \left|\left(T(x), t - \frac{T(2^n u)}{2^n}\right)\right| + \left|\left(T(y), \frac{T(2^n u)}{2^n}\right)\right| \\ &\quad + \left|\left(T(x) - T(y), \frac{T(2^n u)}{2^n}\right)\right| \\ &< \|T(x)\| \cdot \left\| t - \frac{T(2^n u)}{2^n}\right\| + \frac{\delta}{2} + 2\epsilon \left\|\frac{T(2^n u)}{2^n}\right\| \\ &\quad + \|T(x) - T(y)\| \cdot \left\|\frac{T(2^n u)}{2^n}\right\| < \delta + 3\epsilon(1+\delta). \end{aligned}$$

It follows that

$$|(T(x), U(u))| = |(T(x), t)| \leqq 3\epsilon.$$

THEOREM 3. *Let $T(x)$ satisfy again the hypotheses of Theorem* 1, *and let it take E into the whole of E. Then the transformation $U(x)$ also takes E into the whole of E.*

PROOF. For each point z of E, let $T^{-1}(z)$ denote any point whose T-image is z. Then $T^{-1}(z)$ is an ϵ-isometry of E. By Theorem 1, the limit $U^*(z) = \lim_{n\to\infty} (T^{-1}(2^n z)/2^n)$ exists, and U^* is an isometry of E. Now clearly

$$\begin{aligned} \|2^n z - T(2^n U^*(z))\| &= \left\| T\left(2^n \frac{T^{-1}(2^n z)}{2^n}\right) - T(2^n U^*(z)) \right\| \\ &< 2^n \left\| \frac{T^{-1}(2^n z)}{2^n} - U^*(z) \right\| + \epsilon. \end{aligned}$$

On dividing by 2^n and letting $n \to \infty$, we see, for each point z of E, that $z = UU^*(z)$. Therefore $U(E) = E$.

THEOREM 4. *Let E be a complete abstract Euclidean vector space. If $T(x)$ is an ϵ-isometry which takes E into the whole of E such that $T(0)=0$, then the transformation $U(x) = \lim_{n\to\infty} (T(2^n x)/2^n)$ is an isometry of E into the whole of itself, and the inequality $\|T(x) - U(x)\| < 10\epsilon$ is satisfied for all x in E.*

PROOF. For a given point $x \neq 0$ let M denote the linear manifold orthogonal to x. By Theorem 3, U is an isometric transformation which takes E into the whole of E. Hence $U(M)$ is the linear manifold orthogonal to $U(x)$. Let w be the projection of $T(x)$ on $U(M)$. If $w=0$ put $t=0$. Otherwise put $t=w/\|w\|$. In either case (cf. Theorem 2), the inequality $|(T(x),t)|\leqq 3\epsilon$ is satisfied. Put $v=(1/\|x\|)U(x)$. Then v is a unit vector orthogonal to t and is coplanar with $T(x)$ and t.

Hence, by the pythagorean theorem we have the identity:

$$\|T(x) - U(x)\|^2 = (T(x), t)^2 + [\|x\| - (T(x), v)]^2. \tag{4}$$

Let $z_n=2^n x$ and if the projection w_n of $T(z_n)$ on $U(M)$ is not zero, put $t_n=w_n/\|w_n\|$. Otherwise we shall put $t_n=0$. In either case $(t_n, v)=0$, and $|(T(z_n), t_n)|\leqq 3\epsilon$. If $\|T(z_n)\|<3\epsilon$, it is obvious that $\|T(z_n)\|-|(T(z_n), v)|\leqq 3\epsilon$. If $\|T(z_n)\|\geqq 3\epsilon$, we have $0\leqq\|T(z_n)\| - |(T(z_n), v)| = \|T(z_n)\| - (\|T(z_n)\|^2-(T(z_n), t_n)^2)^{1/2}\leqq 3\epsilon$.

Hence the inequality:

$$\big|\, \|z_n\| - |(T(z_n), v)| \,\big| < 4\epsilon \tag{5}$$

is satisfied, since $\|z_n\|<\|T(z_n)\|+\epsilon$.

Two cases arise. If $(T(x), v)\geqq 0$, we put $n=0$ in the inequality (5) and use the identity (4) to obtain the inequality $\|T(x)-U(x)\|<5\epsilon$. If $(T(x), v)<0$, then for some integer $m\geqq 0$ we must have $(T(z_m), v)<0$ and $((T(2z_m), v)\geqq 0$, since $(U(x), v)$ is positive and $U(x)=\lim_{n\to\infty}(T(z_n)/2^n)$. Hence, by inequality (2),

$$\|T(2z_m) - T(z_m)\| \geqq (T(2z_m), v) - (T(z_m), v) > 3\|z_m\| - 8\epsilon.$$

But we know that $\|T(2z_m)-T(z_m)\|<\|z_m\|+\epsilon$. Therefore,

$$\|x\| \leqq \|z_m\| < (9/2)\epsilon, \quad \text{and} \quad \|T(x) - U(x)\| < 2\|x\| + \epsilon \leqq 10\epsilon.$$

In order to prove the above theorem we had to assume that $T(x)$ takes E into itself. We now show that the theorem is not always true for ϵ-isometric transformations of one Euclidean space into *part* of another. Consider the transformation $T(x)$ of the real axis into a subset of the plane defined as follows: the coordinates x, y of $T(x)$ are $(x, 0)$ for $x\leqq 1$, and $(x, c\cdot\log x)$ for $x>1$. It is easy to verify that T will be an ϵ-isometry if we choose c in such a way that $\epsilon>c^2 \max_{x>1}((\log x)^2/(2x-2))$.

On the other hand, $T(x)$ obviously cannot approximate an isometry in the sense of our theorem.

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