EXTENSION OF A THEOREM OF BOCHNER ON EXPRESSING FUNCTIONALS AS RIEMANN INTEGRALS

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Introduction. S. Bochner[1] has shown that an additive homogeneous functional defined over a sufficiently large class C of functions can be realized as a Riemann integral with respect to a finitely additive measure V in the space X over which the functions are defined. His proof makes use of the fact that the constant function belongs to C, as a result, $V(X)$ is finite. It is the purpose of this note to show that a similar theorem holds even when $V(X)$ turns out to be infinite. A modification of Bochner's proof would suffice for this stronger theorem. We have chosen rather to treat it as a problem of extending the domain of definition of the given functional.

Throughout we have used the symbol $\rightarrow$ to be read as "implies." The equality $\equiv$ is used to denote an equality which holds by definition.

Notations. We consider a space X of points x, and real-valued point functions $f, g, \cdots$ over X. Given f, g, and real numbers a, b, we shall write

$$|f|, \quad af+bg, \quad fg, \quad f\wedge g, \quad f\vee g, \quad f^+, \quad f^-,$$

respectively, for those functions whose values for each x are given by

$$|f(x)|, \qquad af(x)+bg(x), \qquad f(x)g(x), \qquad \inf\,[f(x), g(x)],$$
$$\sup\,[f(x), g(x)], \qquad \sup\,[f(x), 0], \qquad \sup\,[-f(x), 0].$$

We shall write a for the constant function $f(x)=a$, and write $f\geqq g$ if for each x, $f(x)\geqq g(x)$. The function which coincides with f on a set A and is equal to 0 in $X-A$ will be denoted by f_A. In particular we write 1_A for the characteristic function of the set A. The symbol $\varnothing$ will denote the empty set.

It is clear that $f=f^+-f^-$, and that

$$(f_A)^+ = (f^+)_A, \qquad (f_A)^- = (f^-)_A.$$

1. R-measure.

1.1. By an R-measure in X we shall mean a set function $V(E)$ defined for sets E of a family $\mathbf{A}$ with the following properties:

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[1] S. Bochner, *Additive set functions on groups*, Ann. of Math. vol. 40 (1939) pp. 769–799. The theorem in question occurs in paragraph 4.

If E, E_1, $E_2 \in \mathbf{A}$, then
(1) $E_1 \cup E_2 \in \mathbf{A}$,
(2) $X - E \in \mathbf{A}$,
(3) $0 \leqq V(E) \leqq \infty$,
(4) $V(E) = 0$, and $B \subset E \rightarrow B \in \mathbf{A}$,
(5) $E_1 \cap E_2 = \varnothing \rightarrow V(E_1 \cup E_2) = V(E_1) + V(E_2)$.
Also
(6) there exists an $E \in \mathbf{A}$ with $0 < V(E) < \infty$.

1.2. *Remark.* (1), (2) imply $E_1 \cap E_2 \in \mathbf{A}$, $E_1 - E_2 \in \mathbf{A}$, $\varnothing \in \mathbf{A}$, $X \in \mathbf{A}$.

2. **The Riemann integral.** Let Δ be the class of all partitions δ of X into finitely many pairwise disjoint sets of $\mathbf{A}$. Given any $f \geqq 0$, bounded on $E \in \mathbf{A}$ with $V(E) < \infty$, we define

2.1
$$S_u(f, E, \delta) \equiv \sum_{D \in \delta} V(D \cap E)(\sup \{f_E(x) \mid x \in D\}),$$
$$S_l(f, E, \delta) \equiv \sum_{D \in \delta} V(D \cap E)(\inf \{f_E(x) \mid x \in D\}),$$

2.2
$$S_u(f, E) \equiv \inf \{S_u(f, E, \delta) \mid \delta \in \Delta\},$$
$$S_l(f, E) \equiv \sup \{S_l(f, E, \delta) \mid \delta \in \Delta\},$$

2.3
$$S_u(f) \equiv \sup \{S_u(f, E) \mid E \in \mathbf{A},\ V(E) < \infty\},$$
$$S_l(f) \equiv \sup \{S_l(f, E) \mid E \in \mathbf{A},\ V(E) < \infty\}.$$

We define the function classes

2.4
$$R_E \equiv \{f \mid S_u(f^+, E) = S_l(f^+, E) < \infty,\ S_u(f^-, E) = S_l(f^-, E) < \infty\},$$
$$R \equiv \{f \mid S_u(f^+), S_u(f^-) < \infty \quad \text{and} \quad (V(E) < \infty \rightarrow f \in R_E)\}.$$

Finally,

2.5
$$f \in R \rightarrow \int f \equiv S_u(f^+) - S_u(f^-).$$

It is easily shown that the supremum and infimum in 2.2 are in fact monotone limits over the directed set of partitions $\delta \in \Delta$, Δ being ordered by refinement. From this fact and from the definition it then follows that (when E, E_1, $E_2 \in \mathbf{A}$ and $V(E)$, $V(E_i) < \infty$)

2.6 $f \geqq 0$ and $E_1 \subset E_2 \rightarrow 0 \leqq S_u(f, E_1) \leqq S_u(f, E_2)$,

2.7 $f \geqq 0 \rightarrow S_u(f, E) = S_u(f_E)$, $\quad S_l(f, E) = S_l(f_E)$,

2.8 $f \in R_E \rightarrow (f_E \in R$ and $\int f_E = S_l(f^+, E) - S_l(f^-, E))$, 2.9 $f \in R$ and $f \geqq 0 \rightarrow \int f = \sup \left\{ \int f_E \,\middle|\, E \in A,\ V(E) < \infty \right\}$,

2.91 $f \in R \rightleftarrows f^+, f^- \in R$,

2.92 $\int af + bg = a \int f + b \int g$.

3. Modules.

3.1. A class C of real-valued functions over X, together with a real-valued linear functional L defined over C, is called a module if it satisfies conditions 3.1 (1)–(11) below. (f, g denote elements of C; a, a real number.)

(1) Each f in C is bounded.
(2) $f+g \in C$.
(3) $af \in C$.
(4) $f \wedge 0 \in C$.
(5) $f \wedge 1 \in C$.
(6) $|L(f)| < \infty$.
(7) $L(f+g) = L(f) + L(g)$.
(8) $L(af) = aL(f)$.
(9) $f \geqq 0 \rightarrow L(f) \geqq 0$.
(10) There exists an $f \in C$ with $L(f) > 0$.
(11) $\text{Inf}_{a>0} L(f \wedge a) \leqq 0$.

The main theorem of this paper is:

3.2. *If C is a module, there exists an R-measure $V(E)$ in X such that (1) $C \subset R$, (2) $f \in C \rightarrow L(f) = \int f$, (3) given $e > 0$ and $g \in R$, with $g \geqq 0$, there exists an $f \in C$ such that $0 \leqq f \leqq g$ and $L(f) \leqq \int g < L(f) + e$.*

Before constructing the R-measure we prove some elementary properties of a module C.

3.3 $f, g \in C \rightarrow f \vee g,\ f \wedge g \in C$. For example, $f \vee g = g - (g - f) \wedge 0$.

3.4 $f \in C,\ a > 0 \rightarrow f \wedge a \in C$, for $f \wedge a = a \cdot (1/a) f \wedge 1$.

3.5 $f, g \in C, f \geqq g \rightarrow L(f) \geqq L(g)$, for $L(f) - L(g) = L(f - g) \geqq 0$.

3.6 $f, 1_A \in C \rightarrow f_A = f \cdot 1_A \in C$, for $0 \leqq f(x) \leqq b \rightarrow f_A = f \wedge b 1_A$.

4. **Completion of a module.** In 4.1–4.5 below, f, h denote elements of a module C, while g may be any function.

4.1. $L_u(g) \equiv \inf \{ L(h) \mid h \geqq g \}$ (if there exists an h, such that $h \geqq g$).
4.2. $L_l(g) \equiv \sup \{ L(f) \mid f \leqq g \}$ (if there exists an f such that $f \leqq g$).

4.3. $C^* \equiv \{g \mid L_u(g) = L_l(g)\}$.

4.4. $L^*(g) \equiv L_u(g) = L_l(g)$ (for $g \in C^*$).

4.5. $C \subset C^*$ and $L^*(f) = L(f)$.

4.6. C^* is a module. We show (except for some obvious cases) that C^* has properties (1)–(11) of 3.1.

(3) and (8): Suppose $g \in C^*$ and, say, $a<0$. Then

$$\{f \mid f \leqq ag\} = \{ah \mid h \geqq g\}.$$

Hence

$$L_l(ag) = \sup \{L(ah) \mid h \geqq g\} = a \inf \{L(h) \mid h \geqq g\} = aL^*(g).$$

Similarly

$$L_u(ag) = aL^*(g).$$

(2) and (7): Suppose $g_1, g_2 \in C^*$. Then

$$\{f_1 + f_2 \mid f_i \leqq g_i\} \subset \{f \mid f \leqq g_1 + g_2\}.$$

Hence $L_l(g_1)+L_l(g_2) \leqq L_l(g_1+g_2)$ and, dually, $L_u(g_1+g_2) \leqq L_u(g_1) + L_u(g_2)$. (2) and (7) then follow from the fact that $L_l(g_1+g_2) \leqq L_u(g_1+g_2)$.

(4) and (5) follow from the inequality

$$h - f \geqq (h \wedge x) - (f \wedge x).$$

(11) follows from the fact that every $g \in C^*$ is covered by an $h \in C$, and that 3.5 does not depend on (11).

4.7. C^* is complete, in the sense that the process of extension described in 4.1–4.3 does not yield any new functions when applied to C^*.

PROOF. It follows from 4.2 and 4.4 that

$$\sup \{L^*(f) \mid f \in C^*, f \leqq g\} = \sup \{L(f) \mid f \in C, f \leqq g\},$$

and similarly for the approximations from above.

4.8. Let C be any module. Given $f \in C$ and a number $a>0$, let 1_a be the characteristic function of the set $\{x \mid f(x) \geqq a\}$. For each $f \in C$ there exists an everywhere dense set S of real numbers $a>0$ such that $a \in S \rightarrow 1_a \in C^*$, where C^* is the completion of C. Since C^* is a module and is its own completion we have as a corollary the same theorem with the weaker assumption $f \in C^*$.

PROOF. We shall prove the stronger result that there is at most a countable set $\{a_i\}$ of numbers $a_i>0$ such that 1_{a_i} is not in C^*. Given $e>0$, consider any $a \geqq e$ and numbers $b, c>0$ with $c \leqq e$. For any $d \geqq 0$ let $f^d = f \wedge d \in C$. Let $\phi(d) = L(f^d)$. We have

$$(1) \qquad c^{-1}\{f^a - f^{a-c}\} \geqq 1_a \geqq b^{-1}\{f^{a+b} - f^a\},$$

as may be seen by analyzing the three cases

$$f(x) \geqq a + b, \qquad a \leqq f(x) < a + b, \qquad f(x) < a.$$

Using 4.6 and (7), (8) of 3.1, we have from (1) that

$$(2) \quad c^{-1}[\phi(a) - \phi(a - c)] \geqq L_u(1_a) \geqq L_l(1_a) \geqq b^{-1}[\phi(a + b) - \phi(a)].$$

The outside inequalities imply that $\phi(a)$ is a convex function for $a \geqq e$. Taking limits in (2) as b, $c \to 0$, we have further that

$$D^-\phi(a) \geqq L_u(1_a) \geqq L_l(1_a) \geqq D^+\phi(a).$$

Since $\phi(a)$ is convex in the interval in question, $D^-\phi \neq D^+\phi$ at most at a countable number of points $\{a_i'\}$, $a_i' \geqq e$. Hence when $a \geqq e$ is not in $\{a_i'\}$, $D^-\phi = D^+\phi$ and $1_a \in C^*$ by 4.3. By taking successively $e = 1/n$, $n = 1, 2, \cdots$, we get at most a countable sum of countable sets—that is, at most a countable set $\{a_i\}$—in the interval $a > 0$ such that 1_{a_i} is not in C^*.

4.9. Let $\mathbf{A}^* = \{A \mid 1_A \in C^*\}$. Then

$$f \in C^* \to L^*(f) = \lim L^*(f_A),$$

where the limit is the limit taken on the directed system $\mathbf{A}^*$ ordered by $\supset$.

Proof. It is sufficient to prove 4.9 for $f \geqq 0$. By 4.8 there exists a sequence $a_n \downarrow 0$ such that the characteristic functions 1_n of the sets $\{x \mid f(x) \geqq a_n\}$ are all in C^*. Put $g_n = f - (f \wedge a_n)$. Then $g_n \in C^*$, $g_n \leqq f$, and $g_n \cdot 1_n = g_n$. Hence

$$L^*(g_n) \leqq L^*(f \cdot 1_n) \leqq L^*(f).$$

But $\inf L^*(f - g_n) = 0$ by 3.1, (11), 4.9 for $f \geqq 0$ now follows, since if $f \geqq 0$

$$\begin{aligned} L^*(f) \geqq \lim_A L^*(f_A) &= \sup \{L^*(f_A) \mid A \in \mathbf{A}^*\} \\ &\geqq \sup_n L^*(f \cdot 1_n) \geqq \sup_n L^*(g_n) = L^*(f). \end{aligned}$$

5. Extension of L^* to "unbounded" functions.

5.1. $C^{**} \equiv \{f \mid A \in \mathbf{A}^* \to f_A \in C^*\}$.

5.2(a). $L^{**}(f) \equiv \lim_A L^*(f_A)$ (for $f \in C^{**}$, $f \geqq 0$). Here the limit is taken as in 4.9.

5.2. $L^{**}(f) \equiv L^{**}(f^+) - L^{**}(f^-) = \lim_A L^*(f_A^+) - \lim_A L^*(f_A^-) = \lim_A L^*(f_A)$ (for $f \in C^{**}$ and $L^{**}(f^+), L^{**}(f^-) < \infty$). Thus $|L^{**}(f)| < \infty$, except possibly if $f > 0$.

5.3. $C^* \subset C^{**}$ and $f \in C^* \to L^{**}(f) = L^*(f)$ (3.6, 4.9).

5.4. $f \geqq 0 \to L^{**}(f) \geqq 0$.

5.5. $f, g \in C^{**} \to af + bg \in C^{**}$ and (if $L^{**}(f), L^{**}(g) < \infty$) $L^{**}(af+bg) = aL^{**}(f) + bL^{**}(g)$, since $(af+bg) \cdot 1_A = a \cdot f1_A + b \cdot g1_A$, and L^{**} is defined as a limit on the directed set $\mathbf{A}^*$.

5.6. $f \leqq g \to L^{**}(f) \leqq L^{**}(g)$ (5.4, 5.5).

5.7. $0 \leqq f \leqq g$ and $g \in C^{**}$ and $L^{**}(g) = 0 \to f \in C^{**}$ and $L^{**}(f) = 0$. For $f_A \in C^*$ by 4.7. Hence by 5.4, 5.6

$$f \in C^{**}, \quad \text{and} \quad L^{**}(f) = 0$$

5.8. f, $1_E \in C^{**} \to f_E \in C^{**}$. For $1_A \in C^* \to (f_E)1_A = (f_A)(1_E 1_A) \in C^*$ by 3.6.

5.9. $f \in C^{**}$, $f \geqq 0 \to L^{**}(f) = \sup \{L^{**}(f_A) \mid 1_A \in C\}$ (5.6, 4.9, 5.3).

5.10. $1_X = 1 \in C^{**}$ (5.1).

Actually 5.8 is a special case of the following theorem, which however will not be needed for this paper:

5.11. $f, g \in C^{**} \to f \cdot g \in C^{**}$.

PROOF. Assume $0 \leqq h$, $i \in C^*$ and $1_A \in C^*$. It follows from 4.8 that we may subdivide X into a finite number of sets E_ν such that $1_{E_\nu} \cdot 1_A \in C^*$ and that the oscillation of h and i on each set is less than e. Denoting by a_ν'', b_ν'', a_ν', b_ν' the maximum and minimum of h and i on E_ν we have

$$\sum a_\nu' \cdot b_\nu \cdot 1_{E_\nu} \cdot 1_A \leqq h \cdot i \cdot 1_A \leqq \sum a_\nu'' \cdot b_\nu'' \cdot 1_{E_\nu} \cdot 1_A.$$

Hence by the completeness of $C^*: h \cdot i \cdot 1_A \in C^*$. The theorem now follows since putting $f_A = h$ and $g_h = i$ we have that $f \cdot g \cdot 1_A = f_A \cdot g_A \in C^*$ for every $1_A \in C^*$.

6. The R-measure defined by L^{**}.

6.1(a). $\mathbf{A} \equiv \{E \mid 1_E \in C^{**}\}$.

6.1(b). $V(E) \equiv L^{**}(1_E)$ (for $E \in \mathbf{A}$).

6.1(c). $\mathbf{A}' \equiv \{E \mid E \in \mathbf{A}$ and $V(E) < \infty\}$.

From these definitions it follows:

6.2. V is an R-measure as defined in §1.

PROOF. The properties (1)–(5) are obvious from §5. As for (6), we have from 3.1 (10) an $f \in C$ with $L(f) > 0$. We can assume $0 \leqq f(x) \leqq 1$. If $L^*(f1_A) = 0$ for all $1_A \in C^*$, then $L(f) = 0$ by 4.9, 4.5. Hence for one 1_A, $L^*(f_A) > 0$. But $1_A \geqq f_A$. Hence $L^*(1_A) = V(A) > 0$.

7. Comparison of $L^{**}(f)$ and $\int f(x)dV$.

7.1. $f \in C^*$, $f = f_E \geqq 0$, $E \in \mathbf{A}' \to f \in R_E$ and $\int f = L^*(f)$.

PROOF. (a) If $V(E) = 0$, then $S_u(f_E) = S_l(f_E) = 0 = \int f$, since for some

$a\neq 0$, $0\leqq af^+\leqq 1_E$ and $0\leqq af^-\leqq 1_E$, we have $L^*(af)=0$ by 5.3, 5.7. Hence $L^*(f)=0=\int f$.

(b) If $0<V(E)<\infty$, given $e>0$, by 3.1 (1) and 4.8, there exists a partition δ of X into sets $A_0, \cdots, A_n\in \mathbf{A}$ such that

$$\sup \{f(x)-f(y) \mid x, y\in A_i\} < e[V(E)]^{-1}, \qquad i=0,\cdots,n.$$

Let $E_i=EA_i$. Then $1_{E_i}=1_i\in C^{**}$ and hence $f_i=f1_i\in C^{**}$ by 5.8. Letting $b_i=\sup \{f(x)\mid x\in E_i\}$, we have

$$L^*(f) = L^{**}(f_E) = \sum_i L^{**}(f1_i)$$

and

$$S_u(f, E, \delta) = \sum_i V(E_i)b_i = \sum_i L^{**}(1_i\cdot b_i).$$

Hence $|L^*(f)-S_u(f,E,\delta)|<e$ and $L^*(f)=S_u(f,E)$. Similarly $L^*(f)=S(f,E)$.

7.2. If $f\geqq 0$, if $f\in R_A$ for $A\in A^*$, and if f is bounded on any $E\in\mathbf{A}$, then

(a) $f\in R_E$ for every $E\in\mathbf{A}'$,

(b) $\sup \{\int f_E\mid E\in \mathbf{A}'\}=\sup \{\int f_A\mid A\in\mathbf{A}^*\}$.

PROOF. Since f is bounded on E and $L^{**}(1_E)<\infty$, $S_u(f,E)-S_u(f,A)<e/2$ for some $1_{A_1}\in C^*$, $1_{A_1}\leqq 1_E$. Dually $S_l(f,E)-S_l(f,A_2)<e/2$ ($1_{A_2}\in C^*$, $1_{A_2}\leqq 1_E$). The inequalities still hold if we replace A_1 and A_2 by $A=A_1\cup A_2$. Since $S_u(f,A)=S_l(f,A)$ we have $S_u(f,E)-S_l(f,E)<e$ for any e. Hence $f\in R_E$ and $\int f_E-\int f_A<e$ from which (b) follows.

7.3. $f\in C^*\rightarrow f\in R$ and $\int f=L^*(f)$.

PROOF. Assume $f\geqq 0$. By 7.1, $f_A\in R_A$, that is, $f\in R_A$ for any $A\in\mathbf{A}^*$ and $\int f_A=L^*(f_A)$. Since f is bounded, $f\in R_E$ ($E\in\mathbf{A}'$) by 7.2(a). From 7.2(b) and 4.9 it follows that $\sup \{\int f_E\mid E\in\mathbf{A}'\}$ is equal to $L^*(f)$, hence finite, and equal to $\int f$ by 2.9. For any $f\in C^*$, 7.3 then follows by 2.91, 2.92.

7.4. $f\in R_A$ and $A\in\mathbf{A}^*\rightarrow f\in C^*$.

PROOF. $f\in R_A$ means that f can be approximated from above and below by functions $\sum a_\nu 1_{A_\nu}$ where $A_\nu\in A^*$. Hence $f\in C^*$ by 4.7.

7.5. $f\in R\rightarrow f\in C^{**}$ and $L^{**}(f)=\int f$.

PROOF. Assume $f\geqq 0$, then $f\in R\rightarrow f_A\in R_A$ for every $A\in\mathbf{A}^*$. By 7.4, $f_A\in C^*$ and hence $f\in C^{**}$.

Furthermore $\int f=\sup \{\int f_A\mid A\in\mathbf{A}^*\}=\sup \{L^*(f_A)\mid A\in\mathbf{A}'\}=L^{**}(f)$ (2.9, 7.2, 7.1). For any $f\in R$, 7.5 follows from its truth for f^+, f^-.

The proof of our main theorem, 3.2, is now complete: That V is an R-measure was shown in §6, (1) and (2) follow from 7.3, (3) from 7.5 and the definition of L^{**}.

8. Some special cases.

A. 8.1. Assume that $f(x)=1\in C$ and $L(1)=1$. (This makes 3.1 (5), (10) and (11) redundant, (11) follows from the fact that $L(f\wedge a) \leqq L(a\cdot 1)=a$.) In this case $V(X)=1$ and 3.2 reduces to Bochner's theorem.

B. 8.2. DEFINITION. By an L-extension of V we shall mean a countably additive, complete measure U defined for sets of a countably additive, complemented family $\mathbf{B}$ such that $\mathbf{B}\supset\mathbf{A}$ and for $E\in\mathbf{A}$, $U(E)=V(E)$.

8.3. Replace 3.1 (11) by 3.1 (12): $\{f_n,\ g\in C,\ 0\leqq f_n\leqq g,\ \lim_n f_n(x)=0$ for all $x\}\rightarrow\lim_n L(f_n)=0$. Then (a) Theorem 3.2 still holds and in addition (b) V possesses an L-extension U such that (c) 3.2 (3) holds when "$g\in R$" is replaced by "g is measurable and integrable (U)."

PROOF. (a) 3.1 (12) implies 3.1 (11). For $f\in C$ put $f_n=\inf(f,1/n)$. Then $\inf_{a>0} L(f\wedge a)\leqq\lim L(f_n^+)=0$.

(b) It is known[2] that any V with properties 1.1 (1)–(5) possesses an L-extension if and only if V has the additional property: $\{E_n\in\mathbf{A},\ E_n\geqq E_{n+1},\ V(E_1)<\infty,\ \bigcap_n E_n=\varnothing\}\rightarrow\lim_n V(E_n)=0$. That V has this property follows from

8.4. $\{f_n\in C^{**},\ f_n\geqq f_{n+1},$ for each $x\ \inf f_n(x)=0,\ L^{**}(f_1)<\infty\}\rightarrow\lim L^{**}(f_n)=0$.

PROOF. Suppose that for all n, $L^{**}(f_n)\geqq e>0$. Since $L^{**}(f_n)<\infty$ there exists an $A\in\mathbf{A}^*$ such that $L^{**}(f_1-f_{1A})\leqq e/2$. Since $(1-1_A)f_n \leqq(1-1_A)f_1$ we also have $L^{**}(f_n-f_{nA})\leqq e/2$, and $L^{**}(f_{nA})\geqq e/2$ for all n. Hence we can find a $g_n\in C$ such that $0\leqq g_n\leqq f_{nA}$ and $L(g_n)\geqq e/3$. But evidently for each x, $\lim g_n(x)=0$. Since there exists a $g\in C$ such that $g\geqq f_1\cdot 1_A>g_n$, this contradicts 3.1 (12).

(c) To show that the analogue of 3.2 (3) holds for g measurable and integrable (U), we point out that (a) is sufficient to show that 3.2 (3) holds when g is the characteristic function of a set B measurable (U) with $U(B)<\infty$ and (b) if U is an L-extension of V then, given $e>0$, V contains an $E\in\mathbf{A}$ such that $V(E)>U(E)-e$. The result then follows from 3.2 (3).

(a) is a consequence of the ordinary Lebesgue theory while (b) results from the manner in which U is defined as an extension of V.[2]

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[2] This theorem is proved by Kolmogoroff (A. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Berlin, 1933) for the case $V(X)=1$. When X is the sum of countably many sets of finite measure, the proof given by Jessen (B. Jessen, *Abstrakt maal- og integraltheorie*, 1, Matematisk Tidsskrift (B) (1934) p. 78) applies. The proof in the general case follows by a modification of that of Jessen.