

ON CERTAIN SYSTEMS WHICH ARE ALMOST GROUPS

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We consider systems \mathfrak{S} of elements A, B, \dots satisfying the following postulates.

I. To every pair of elements A, B in \mathfrak{S} there exists a uniquely determined product $A \cdot B$ in \mathfrak{S} .

II. $(AB)C = A(BC)$.

III. There exists in \mathfrak{S} an element E such that $EA = A$ for every A .

IV. To every A in \mathfrak{S} there exists an A' in \mathfrak{S} such that $AA' = E$.

The system \mathfrak{S} differs from a group only in that it contains a left unit and right inverse instead of a right unit and right inverse. We shall call such a system a left right system, abbreviated (l, r) system.

DEFINITION 1. An element F is called an idempotent if $F^2 = F$.

PROPOSITION 1. If F is idempotent then $FE = E$.

PROOF. From $FF = F$ we have $FFF^{-1} = FF^{-1}$, hence $FE = E$.

PROPOSITION 2. If $FE = E$ then F is idempotent.

For $FF = FEF = EF = F$.

PROPOSITION 3. The idempotents of a (l, r) system form a (l, r) system.

PROOF. If F and F' are idempotent then by Proposition 1

$$FF'E = FE = E,$$

hence FF' is idempotent and postulate I is fulfilled. Postulate II is fulfilled since it is fulfilled in \mathfrak{S} . E itself is idempotent and is by Proposition 1 a right inverse to every idempotent. Hence all the postulates of a (l, r) system are fulfilled.

PROPOSITION 4. If F is idempotent and A is any element then $FA = A$.

For $FA = FEA = EA = A$.

DEFINITION 2. A (l, r) system which contains only idempotents will be called an idempotent (l, r) system.

From Proposition 4 we see that in every idempotent (l, r) system every product is equal to its last factor. On the other hand it is easy to verify that every system of elements in which every product is

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equal to its last factor is an idempotent (l, r) system since every element of such a system can be considered the unit element and is then inverse to all elements. From the foregoing we have the following theorem.

THEOREM 1. *There exists one and only one idempotent (l, r) system of order n ($n = 1, 2, \dots$ ad inf.). It consists of the elements A_1, A_2, \dots, A_n with the law of composition $A_i A_j = A_j$.*

Remark. Similarly one and only one idempotent (l, r) system exists for every cardinal number.

PROPOSITION 5. *The elements of any (l, r) system which are of the form AF where F is a fixed idempotent form a group.*

PROOF. $(AF)(BF) = A(FB)F = ABF$ by Proposition 4. The composition is by hypothesis associative, F is a right unit and $A^{-1}F$ a right inverse. Since the system forms a group we must have $A^{-1}FAF = F$.

Let $\mathfrak{S} = (E, F_2, \dots)$ be an idempotent (l, r) system, $\mathfrak{G} = (\bar{E}, A_2, \dots)$ a group. We form a system \mathfrak{S} in the following way: \mathfrak{S} consists of all pairs $A_i F_j$ with the following rule of combination:

$$A_i F_j A_k F_l = A_i A_k F_l.$$

It is easy to verify that \mathfrak{S} is a (l, r) system.

DEFINITION 3. *The system \mathfrak{S} defined in the preceding paragraph is called the direct product of \mathfrak{S} and \mathfrak{G} .*

PROPOSITION 6. *Every element of a (l, r) system may be written in the form $(AE)F$ where F is idempotent.*

PROOF. We have $(A^{-1}A)(A^{-1}A) = A^{-1}(AA^{-1})A = A^{-1}(EA) = A^{-1}A$. Hence $A^{-1}A$ is idempotent. But $A = AA^{-1}A = A(EA^{-1}A) = AE(A^{-1}A)$.

PROPOSITION 7. *If $AEF = BEF'$ where F and F' are idempotents then $AE = BE$ and $F = F'$.*

PROOF. From $AEF = BEF'$ we have $AEFE = BEF'E$. Hence on account of Proposition 1, $AE = BE$. Since the elements of the form AE form a group, $A^{-1}E$ is a left inverse of AE and $F = F'$ follows.

From Propositions 5, 6 and 7 we have the following theorem.

THEOREM 2. *Every (l, r) system is the direct product of an idempotent (l, r) system and a group.*

PROPOSITION 8. *Every idempotent F of a (l, r) system \mathfrak{S} is a unit element of \mathfrak{S} .*

For by Proposition 4, $FA = A$ for every A and $A(A^{-1}F) = EF = F$.

Let S_1, \dots, S_n be n statements. Let A_i be the statement: "All the preceding statements are annulled but S_i is true." It is interesting to note that the statements A_i form an idempotent (I, r) system.

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A CONJECTURE IN ELEMENTARY NUMBER THEORY

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A well known conjecture of Catalan states that *if $f(n)$ is the sum of all divisors of n except n , then the sequence of iterates of $f(n)$ is either eventually periodic or ends at 1*. It not only seems impossible to prove this, but it is also very difficult to verify.¹

Another conjecture of Poulet,² which appears equally difficult to prove, has the doubtful merit that it is easy to verify. Let $\sigma(n)$ be the sum of all divisors of n , and let $\phi(n)$ be Euler's function. Then *for any integer n the sequence*

$$f_0(n) = n, \quad f_{2k+1}(n) = \sigma(f_{2k}(n)), \quad f_{2k}(n) = \phi(f_{2k-1}(n))$$

is eventually periodic.

We have verified this conjecture to $n = 10000$ (extending Poulet's verification) by using Glaisher's tables.³ The checking was facilitated by the following observation: if the conjecture is to be checked for all $n < x$, it is enough to find a member of the sequence other than the first which is less than x .

The longest cycle found was in the sequence $f_k(9216)$. It starts with $f_0(9216)$, and is: 34560, 122640, 27648, 81800, 30976, 67963, 54432, 183456, 48384, 163520, 55296, 163800, 34560. However our method of checking does not show that this is the largest cycle up to 10000, and in fact Poulet found that $f_k(1800)$ has the same length 12.

As a rule $\phi(\sigma(n))$ is less than n . In fact, it can be shown that *for every $\epsilon > 0$, $\phi(\sigma(n)) < \epsilon n$, except for a set of density 0*. The proof follows from the following two observations:

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¹ L. E. Dickson, *Theorems and tables on the sum of the divisors of a number*, Quart. J. Math. vol. 44 (1913) pp. 264-296, and P. Poulet, *La Chasse aux Nombres*, vol. 1, pp. 68-72, and vol. 2, p. 188.

² P. Poulet, *Nouvelles suites arithmétiques*, Sphinx vol. 2 (1932) pp. 53-54.

³ J. W. L. Glaisher, *Number-divisor tables*, British Association for the Advancement of Science, Mathematical Tables, vol. 8.