ON MINIMUM CIRCUMSCRIBED POLYGONS

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This paper contains the proofs of two theorems on the n-gon M_n of minimum area circumscribed about a convex region R in the plane. Theorem 1 shows that the area of M_n is a convex function of n and Theorem 3 shows that if R is symmetric about a point there exists an M_{2n} which is also symmetric. The corresponding theorems on inscribed polygons are also given. These theorems were conjectured by R. B. Kershner.

The symbols a and b with subscripts will be used to represent the sides of circumscribed polygons or the vertices of inscribed polygons. It will be convenient to replace the circular order of the sides (vertices) $a_0, a_1, \cdots, a_{n-1}$ of a polygon by an artificial linear ordering $a_0 < a_1 < \cdots < a_{n-1} < a_n$, where a_n represents the same side (vertex) as a_0. The order of the sides (vertices) of circumscribed (inscribed) polygons is established by the order of the contact points (vertices) on the boundary of the convex region.

LEMMA 1. *Let* $A_n = a_0a_1 \cdots a_{n-1}$ *and* $B_m = b_0b_1 \cdots b_{m-1}$ *be two polygons circumscribed about the convex region* R *and let either* (1) $a_0 \leqq b_0 < b_1 < a_1 \leqq a_{r-1} \leqq b_{s-1} < b_s < a_r$ *or* (2) $a_0 \leqq b_0 < b_1 < a_1 \leqq b_{s-1} \leqq a_{r-1} < a_r < b_s$. *Let* $C_{n-r+s} = a_0b_1b_2 \cdots b_{s-1}a_r \cdots a_{n-1}$ *and* $D_{m-s+r} = b_0a_1a_2 \cdots a_{r-1}b_s \cdots b_{m-1}$. *Then the areas satisfy the inequality* $A+B \geqq C+D$ *and there is equality if and only if* $a_0 = b_0$ *and* $a_{r-1} = b_{s-1}$.

PROOF. The common part of A and B is the common part of C and D. The remaining part of $A+B$ is the remaining part of $C+D$ together with the areas of the quadrilaterals $a_0a_1b_0b_1$ and $a_{r-1}a_rb_{s-1}b_s$. Equality holds if and only if both these areas are zero, that is, if $a_0 = b_0$ and $a_{r-1} = b_{s-1}$.

LEMMA 2. *If* M_n *is an* n*-gon of maximum area inscribed in the convex region* R, *the vertices of* M_n *are contact points of the sides of a circumscribed polygon.*

PROOF. If $M_n = a_0a_1 \cdots a_{n-1}$, the line through a_1 parallel to the line a_0a_2 is a supporting line of R, for otherwise M_n would not have maximum area. Supporting lines determined similarly at all vertices of M_n are seen to form a circumscribed polygon.

LEMMA 3. *Let* $A_n = a_0a_1 \cdots a_n$ *and* $B_m = b_0b_1 \cdots b_m$ *be two polygons*

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inscribed in the convex region R, each having as vertices points of contact of the sides of some circumscribed polygon, and let either (1) $a_0 \leqq b_0 < b_1 < a_1 \leqq a_{r-1} \leqq b_{s-1} < b_s < a_r$ *or* (2) $a_0 \leqq b_0 < b_1 < a_1 \leqq b_{s-1} \leqq a_{r-1} < a_r < b_s$. *Let* $C_{n-r+s} = a_0 b_1 b_2 \cdots b_{s-1} a_r \cdots a_{n-1}$ *and* $D_{m-s+r} = b_0 a_1 a_2 \cdots a_{r-1} b_s \cdots b_{m-1}$. *Then the areas satisfy the inequality* $A + B \leqq C + D$ *and there is equality if and only if* $a_0 = b_0$ *and* $a_{r-1} = b_{s-1}$.

PROOF. The vertices b_0 and b_1 are in or on the triangle consisting of $a_0 a_1$ and the adjacent sides of the circumscribed polygon corresponding to a_0 and a_1. Hence (if $a_0 \neq b_0$) $a_0 b_0$ produced meets $a_1 b_1$ produced. Hence $\Delta a_0 b_0 b_1 \leqq \Delta a_0 b_0 a_1$, with equality only if $a_0 = b_0$. Let $a_0 b_1$ intersect $a_1 b_0$ at p. Then $\Delta a_0 p a_1 \geqq \Delta b_0 p b_1$, with equality only if $a_0 = b_0 = p$.

Let $a_{r-1} b_s$ intersect $a_r b_{s-1}$ at q. Then similarly, if $a_{r-1} < b_{s-1} < b_s < a_r$, $\Delta a_{r-1} q a_r > \Delta b_{s-1} q b_s$, and if $b_{s-1} < a_{r-1} < a_r < b_s$, $\Delta b_{s-1} q b_s > \Delta a_{r-1} q a_r$.

If $a_{r-1} \leqq b_{s-1} < b_s < a_r$, $C + D - A - B = \Delta a_0 p a_1 - \Delta b_0 p b_1 + \Delta a_{r-1} q a_r - \Delta b_{s-1} q b_s \geqq 0$ and equality holds only if $a_0 = p = b_0$ and $a_{r-1} = q = b_{s-1}$. If $b_{s-1} \leqq a_{r-1} < a_r < b_s$, $C + D - A - B = \Delta a_0 p a_1 - \Delta b_0 p b_1 + \Delta b_{s-1} q b_s - \Delta a_{r-1} q a_r \geqq 0$ and equality holds only if $a_0 = p = b_0$ and $a_{r-1} = q = b_{s-1}$. Thus in either case $A + B \leqq C + D$ with equality if and only if $a_0 = b_0$ and $a_{r-1} = b_{s-1}$.

THEOREM 1. *If M_n is an n-gon of minimum area circumscribed about the convex region R, the areas satisfy the inequality* $M_n + M_{n+2} \geqq 2M_{n+1}$.

PROOF. Let $M_n = a_0 a_1 \cdots a_{n-1}$, $M_{n+2} = b_0 b_1 \cdots b_{n+1}$.

Case 1. Let there be a sequence of sides $a_0 \leqq b_0 < b_1 < b_2 < a_1$. In this case, if $C_{n+1} = a_0 b_1 a_1 \cdots a_{n-1}$, $D_{n+1} = b_0 b_2 \cdots b_{n+1}$, $M_n + M_{n+2} - C - D = a_0 a_1 b_0 b_1 + b_0 a_1 b_1 b_2 > 0$. Hence $M_n + M_{n+2} > C + D \geqq 2M_{n+1}$.

Case 2. Let there be no such sequence of sides. Then there must be a sequence of the form $a_0 \leqq b_0 < b_1 < a_1 \leqq a_{r-1} \leqq b_{s-1} < b_s < a_r$ since there are n intervals $[a_i, a_{i+1})$ and $n+2$ b's in these intervals. We may assume $s \leqq r+1$ since otherwise we could renumber the sides starting from a_{r-1} which we would call a_0. We may even assume that $s = r+1$, for otherwise $b_{r+1} > a_r$ and $b_{n+1} < a_n$ and hence there is a first subscript $t > r$ for which $b_{t+1} < a_t$. Then $b_t \geqq a_{t-1}$ and we may replace t by r and $t+1$ by s so that $s = r+1$. Let $C_{n+1} = a_0 b_1 \cdots b_{s-1} a_r \cdots a_{n-1}$ and $D_{n+1} = b_0 a_1 \cdots a_{r-1} b_s \cdots b_{n+1}$. Then by Lemma 1, $M_n + M_{n+2} \geqq C + D \geqq 2M_{n+1}$.

THEOREM 2. *If M_n is an n-gon of maximum area inscribed in the convex region R, the areas satisfy the inequality* $M_n + M_{n+2} \leqq 2M_{n+1}$.

PROOF. Let $M_n = a_0 a_1 \cdots a_n$, and $M_{n+2} = b_0 b_1 \cdots b_{n+1}$.

Case 1. Let there be a sequence of sides $a_0 \leqq b_0 < b_1 < b_2 < a_1$. In this case, if $C_{n+1} = a_0 b_1 a_1 \cdots a_{n-1}$ and $D_{n+1} = b_0 b_2 \cdots b_{n-1}$, $C + D - M_n - M_{n+2} = a_0 b_1 a_1 - b_0 b_1 b_2$ which is greater than 0 as can be shown using Lemma 2 and elementary geometry. Hence $M_n + M_{n+2} < C + D \leqq 2M_{n+1}$.

Case 2. Let there be no such sequence of sides. The proof is similar to that of Case 2 of Theorem 1 except that Lemma 3 is used instead of Lemma 1.

THEOREM 3. *If R is a convex region symmetric about a point* 0, *among the 2n-gons of minimum area circumscribed about R there is one which is symmetric about* 0.

PROOF. Let $M_{2n} = a_0 a_1 \cdots a_{2n-1}$ be a minimum circumscribed $2n$-gon and let $M' = b_0 b_1 \cdots b_{2n-1}$ where b_i is the image in 0 of $a_{i \pm n}$. If M_{2n} is not symmetric there is a side $a_i \neq b_i$. We may assume $a_0 \neq b_0$. By reordering the sides if necessary we may assume $a_0 < b_0$ and hence $b_n < a_n$. Let t be the largest number, $0 \leqq t < n$, such that $a_t \leqq b_t$. Then $b_{t+1} < a_{t+1}$ and if we renumber starting from a_t, which we shall call a_0, we have $a_0 \leqq b_0 < b_1 < a_1$. Since a_0 and a_1 are successive sides of M_{2n}, $a_1 < b_n$. Hence $a_0 \leqq b_0 < b_1 < a_1 < b_n \leqq a_n < a_{n+1} < b_{n+1}$. Let $C_{2n} = a_0 b_1 b_2 \cdots b_n a_{n+1} \cdots a_{2n-1}$ and $D_{2n} = b_0 a_1 a_2 \cdots a_n b_{n+1} \cdots b_{2n-1}$. Then, by Lemma 1, $M + M' \geqq C + D$. But C and D are both symmetric and M and M' are equal. Therefore either C or D is a symmetric circumscribed $2n$-gon of area not greater than M_{2n}.

THEOREM 4. *If R is a convex region symmetric about a point* 0, *among the 2n-gons of maximum area inscribed in R there is one which is symmetric about* 0.

PROOF. The proof is similar to that of Theorem 3 except that Lemma 3 is used instead of Lemma 1.

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