ness of his work, if he had dwelled longer on the elementary material of his first chapter. The reader who tackles his book without previous knowledge of Laguerre geometry will find the going hard at the beginning. Once he has overcome the initial difficulties, however, he will be richly rewarded by the great number of beautiful results which fall into his lap, and by the mastery of a method which will allow him to find many more results by his own effort.

The book has a complete bibliography and an excellent index.

D. J. STRUIK

The theory and applications of harmonic integrals. By W. V. D. Hodge. Cambridge, University Press; New York, Macmillan, 1941. 9+281 pp. $4.50.

This is one of those books which everyone who specializes in a particular branch of group theory, of the theory of algebraic surfaces, of the theory of Riemann surfaces, of topology or of the tensor analysis should consult. It shows how all these different fields are connected, and not connected in some superficial way or in the form of an analogy, but in an essential manner, so that interesting and profound theorems in one field cannot be understood without a thorough knowledge of other fields. In reading this book one is reminded of books like Klein's "Ikosaeder," which is also a blend of several important fields. The task of the reviewer of such a book is hard, because he has seldom the enviable mastery of the different branches of mathematics which the author possesses. At the same time he must praise the author for the beautiful exposition of so many and different fields.

There are chapters on Riemannian manifolds, on integrals and their periods, on harmonic integrals, on their applications to algebraic varieties and on their applications to the theory of continuous groups. The first chapter, on Riemannian manifolds, is divided into a part on tensor calculus and into a part on the topology of such manifolds. All these topics are prepared in a very careful way, with precautions which will satisfy strict demands of rigor. There are references at the end of each chapter.

To indicate the contents of this book by simply copying the chapter headings is to do a great injustice to the author and his work. There is a leading thought in the choice of subject matter, and that is the study of harmonic integrals. Harmonic integrals, however, are not the capricious invention of an imaginative scholar. They appear quite naturally in the generalization of the problems set by

Riemann's theory of integration on an algebraic surface, and in particular if we ask for the existence of m-fold integrals on an algebraic manifold of m dimensions which are everywhere finite and of zero period. The investigation is facilitated by the Cartan notation in which differentials alternate and in which the exterior derivative of a "p-form" $A_{i_1\cdots i_p}dx^{i_1}\cdots dx^{i_p}=p!.A$ is defined in such a way that Stokes' theorem takes the form $\int A=\int A_x$, the second integral taken over a chain C_p and the first one over its boundary. This again requires not only some understanding of tensor algebra, but also of the topology of manifolds, which the author provides in his early chapters. There are two theorems of DeRham (J. Math. Pures Appl. (9) vol. 10 (1931) p. 115) which lead up to the heart of the problem and which are demonstrated in this book. The first of these theorems states (p. 88) that if Γ_p^i $(i=1, \cdots, R_p)$ is any base for the p-cycles of a manifold M, then there exists a regular closed ϕ-form on M for which the periods $\nu^i=\int\phi$ (on Γ_p^i) are R_p arbitrary real numbers. The second theorem of DeRham states (p. 100) that if ϕ is a closed form on M whose integral has all its periods equal to zero, then ϕ is a null form. Closed forms are forms with exterior derivatives $A_x=0$, and null forms are the exterior derivatives of other forms, hence forms with zero periods.

The author derives DeRham's second theorem from his main existence theorem for harmonic integrals. This requires the introduction of a Riemannian metric. A function which satisfies the Laplacian equation in this metric is called harmonic. A p-form P is harmonic when it is regular and closed, as well as its dual, the $(n-p)$-form P^*; $p!P$ and $p!P^*$ correspond to the harmonic tensors $P_{i_1\cdots i_p}$ and $g^{1/2}P^{k_1\cdots k_p}{}_{k_1\cdots k_p k_{p+1}\cdots k_n}$. A harmonic integral is the integral of a harmonic form. The gradient of a harmonic function ϕ_i is a simple example of a harmonic tensor. The real and imaginary paths of an integral of a complex variable are harmonic.

One of the main objects of the third chapter is the proof that on a Riemannian manifold M such harmonic integrals actually exist. The existence theorem states (p. 119) that there exists a harmonic integral having arbitrarily assigned periods on R_p independent p-cycles of M. The proof requires considerable labor and demands the knowledge of the theory of integral equations. Once established, the main existence theorem is used for the study of algebraic varieties. It is interesting that an ancient theorem of G. Mannoury (Nieuw Archief voor Wiskunde vol. 4 (1898) p. 112)—the deeper meaning of which had always puzzled the reviewer—is used to show how Riemannian manifolds can be constructed corresponding to an algebraic variety V_m

without singularities in a projective space S_r. In this representation the S_r is given by a real analytical locus of $2r$ dimensions. A large number of applications are presented together with a classification of harmonic integrals. This again connects with Lefschetz' classification of p-cycles. Period matrices of so-called "effective" integrals are studied in detail, "effective" integrals being integrals of p-forms for which a certain related $(p-2)$-form is a null form. Among the many results are several due to Severi and others which are new. In the application to continuous groups the representation of closed semi-simple groups by group manifolds is used. The main purpose of this chapter is to show how harmonic integrals provide a convenient method of discussing the invariant integrals of Cartan's theory.

A flaw in the proof of the theorem establishing the existence of harmonic integrals with preassigned periods has been corrected by H. Weyl in the Annals of Mathematics vol. 44 (1943) pp. 1–6.

The nineteen pages devoted to the exposition of the tensor calculus can be recommended to those who like a short and precise introduction into this theory.

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