CONVERGENCE REGIONS FOR THE GENERAL CONTINUED FRACTION

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The purpose of this note is to prove convergence region theorems for continued fractions

$$(1)\qquad \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots,$$

where the a_n and b_n are complex numbers. As an application we obtain a new theorem for the associated continued fraction

$$(2)\qquad \frac{a_1}{b_1 + z} + \frac{a_2}{b_2 + z} + \cdots.$$

This continued fraction plays a rôle in the moment problem and is also otherwise of interest as a means of representing analytic functions. For details see Perron [**4**, chapters 8 and 9].[1]

Recently a continued fraction very similar to (2) has been investigated by Hellinger and Wall [**1**]. They chose the name "J-fraction." A J-fraction is a continued fraction of the form

$$(2)'\qquad \frac{1}{b_1 + z} - \frac{c_1^2}{b_2 + z} - \frac{c_2^2}{b_3 + z} - \cdots.$$

Our result for J-fractions is stated in Theorem C.

The results of this note are closely related to an as yet unpublished work of Wall and Wetzel on "positive definite J-fractions." In particular Theorem C seems to be contained in a theorem of theirs.

In what follows we shall denote by $H(b, \gamma)$ the half-plane (including the boundary) defined by the relation $z \in H(b, \gamma)$ if $\Re(ze^{-i\gamma}) \geqq b$. For the open half-plane we shall use the notation $H_0(b, \gamma)$. It is clear from the context that b is a real number. Further for $a>0$, $P(a, \gamma)$ shall be the parabolic region (including the boundary) bounded by the curve

$$\rho \leqq \frac{a^2/2}{1 - \cos(\theta - 2\gamma)}.$$

For $a=0$, $P(a, \gamma)$ is to be the totality of points $r \cdot e^{i\gamma}$, $r \geqq 0$.

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[1] Numbers in brackets refer to the references listed at the end of the paper.

We are now in a position to state our theorems.

THEOREM A. *The continued fraction* (1) *converges if all the* a_n *lie in a bounded part of the set* $P(a, \gamma)$, $a \geqq 0$, *and if all* b_n *lie in the half-plane* $H(a+\epsilon, \gamma)$, *where* ϵ *is an arbitrary small positive number.*

Even though its statement is somewhat involved it seems worth while to state the following theorem too, as it is a generalization of the parabola theorems[2] for the continued fractions with $b_n = 1$.

THEOREM A′. *In the continued fraction* (1) *let all* b_n *lie in the angular opening*

$$\left| \arg e^{-i\gamma}(b_n - 1) \right| < \pi/2 - \epsilon,$$

where $-\pi/2 < \gamma < \pi/2$ *and* ϵ *is an arbitrary small positive number. Further let all the* a_n *lie in the parabolic region* $P(\cos \gamma, \gamma)$. *Then the continued fraction* (1) *converges if and only if* $\sum |b_n d_n| = \infty$, *where* $d_1 = 1/a_1$, $d_n = 1/d_{n-1}a_n$.

It is well known that the last condition of the theorem is satisfied if $\liminf a_n < \infty$. We note that neither of these two theorems is contained in the other. Both theorems could be used to obtain new "multiple" convergence regions for continued fractions $1+K(a_n/1)$. For both of these theorems there are corresponding theorems for associated continued fractions and J-fractions. We only state the theorems corresponding to Theorem A.

THEOREM B. *If real numbers* $a \geqq 0$, $M > 0$, b *and* γ *exist such that for all* $n \geqq 1$, $a_n \in P(a, \gamma)$, $|a_n| < M$ *and* $b_n \in H(b, \gamma)$ *then the associated continued fraction* (2) *converges uniformly and hence to a holomorphic function of the complex variable* z *in every closed region contained in the half-plane* $H_0(a-b, \gamma)$.

It is easily seen that if $a_n \in P(a, 0)$ then $c_n = a_n^{1/2}$ satisfies the relation $|\Im(c_n)| \leqq a/2$ and conversely. Hence the following theorem for J-fractions is an immediate consequence of Theorem B with $\gamma = 0$.

THEOREM C. *If real numbers* $a \geqq 0$, $M > 0$ *and* b *exist such that for all* $n \geqq 1$, $|\Im(c_n)| \leqq a/2$, $|c_n| < M$ *and* $\Im(b_n) \geqq b$ *then the* J*-fraction* (2)′ *converges uniformly and hence to a holomorphic function of the complex variable* z *in every closed region contained in the half-plane* $\Im(z) > a - b$.

The theorems are based on the following lemma.

LEMMA. *All the approximants of the continued fraction* (1) *lie in the half-plane* $H(-b/2, \gamma)$ *if for all* $n \geqq 1$, $a_n \in P(b, \gamma)$ *and* $b_n \in H(b, \gamma)$, *where* $b \geqq 0$.

[2] See Leighton and Thron [2] and Paydon and Wall [3].

The proof of the lemma is by induction. To perform the induction we require the following two relations: for an arbitrary choice of $a_n \in P(b, \gamma)$, $b_n \in H(b, \gamma)$ and $v \in H(-b/2, \gamma)$ the two relations

$$a_n/b_n \in H(-b/2, \gamma) \tag{3}$$

and

$$\frac{a_n}{b_n + v} \in H(-b/2, \gamma) \tag{4}$$

must be satisfied. We note that (3) is a consequence of (4) as $v = 0 \in H(-b/2, \gamma)$. Further as b_n and v vary independently over their respective regions their sum $b+v$ varies over the half-plane $H(b/2, \gamma)$. It is then easily seen that relation (4) is satisfied if $P(b, \gamma)$ is the part common to all the regions $c \cdot H(-b/2, \gamma)$, $(z \in c \cdot H(-b/2, \gamma)$ if $z = c \cdot v$, where $v \in H(-b/2, \gamma))$ where c varies over $H(b/2, \gamma)$. For the proof of this fact we refer the reader to [2, §2], where a very similar fact is proved in detail.

We now proceed to the proof of Theorem B. Under the conditions of that theorem $b_n + z \in H_0(a, \gamma)$ for all $n \geqq 1$. According to the lemma all the approximants of the continued fraction (2) then lie in the half-plane $H(-a/2, \gamma)$. All approximants are finite. This is seen as follows: In order that an approximant be infinite it is necessary that

$$-(b_1 + z) = \frac{a_2}{b_2 + z +} \quad \cdots \quad \frac{a_n}{+ b_n + z} \cdot$$

This is impossible as the regions $-H_0(a, \gamma)$ and $H(a/2, \gamma)$ have no point in common.

The approximants of (2) are rational functions of z; for z in $H_0(a-b, \gamma)$ they are finite and do not take on certain values (more than two), hence by Montel's Theorem the sequence of approximants forms a normal family of holomorphic functions for z in $H_0(a-b, \gamma)$.

As the $|a_n|$ are bounded there exists an M such that for $|z| > M$, $z \in H_0(a-b, \gamma)$,

$$|b_n + z| > |a_n| + 1$$

for all n. For these values of z the continued fraction converges by Pringsheim's Theorem [4, p. 254]. It then follows from the generalized Stieltjes-Vitali Theorem that the continued fraction converges uniformly in every closed region contained in the half-plane $H_0(a-b, \gamma)$. This completes the proof of Theorem B.

Theorem A follows from Theorem B if we set $a = b$ and let $z = \epsilon e^{i\gamma}$.

For the proof of Theorem A′ we consider the continued fraction

$$\frac{a_1}{1 + d_1 e^{i\theta_1 z}} + \frac{a_2}{1 + d_2 e^{i\theta_2 z}} + \cdots, \tag{5}$$

where $a_n \in P(\cos\gamma, \gamma)$, $d_n \geqq 0$ and $|\theta_n - \gamma| < \pi/2 - \epsilon$ for all $n \geqq 1$.

Under these conditions the approximants of (5) form a normal family of holomorphic functions for z in the region D defined by

$$-\delta < \Re(z) < 1 + \delta,$$

where δ is positive and depends on γ and ϵ. The proof of this fact is similar to the proof used in the previous case. For $\Re(z) = 0$ the continued fraction converges. In this case the partial denominators are real and greater than or equal to 1 and hence (5) can be transformed into a continued fraction of the form

$$\frac{g_1}{1} + \frac{g_2}{1} + \cdots,$$

where all $g_n \in P(\cos\gamma, \gamma)$. The convergence of this continued fraction follows from the parabola theorems (we are assuming that all conditions of Theorem A′ are satisfied). The Stieltjes-Vitali Theorem then insures the convergence of (5) for all z in D. If we recall that $\sum |c_n| = \infty$ is a necessary condition for convergence we have Theorem A′ by setting $z = 1$ in (5).

References

1. E. D. Hellinger and H. S. Wall, *Contributions to the analytic theory of continued fractions*, Ann. of Math. vol. 44 (1943) pp. 103–127.

2. W. Leighton and W. J. Thron, *Continued fractions with complex elements*, Duke Math. J. vol. 9 (1942) pp. 763–772.

3. J. F. Paydon and H. S. Wall, *The continued fraction as a sequence of linear transformations*, Duke Math. J. vol. 9 (1942) pp. 360–372.

4. O. Perron, *Die Lehre von den Kettenbrüchen*, Leipzig, 1929.

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