The author is indebted to Professor S. Mandelbrojt, who suggested the existence of the present theorem and gave valuable assistance in its development.

THE RICE INSTITUTE

ON ABEL AND LEBESGUE SUMMABILITY

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1. **Introduction.** A series $\sum_1^\infty a_n$ is called Abel summable to the value s if the power series $\sum a_n r^n$ converges for $0<r<1$, and if $\sum a_n r^n \to s$ as $r \uparrow 1$; it is called Lebesgue summable if the sine series

$$\sum_1^\infty a_n \frac{\sin nt}{n} = F(t) \tag{1.1}$$

converges in some interval $0<t<\tau$, and if

$$t^{-1}F(t) \to s \quad \text{as} \quad t \downarrow 0. \tag{1.2}$$

We write in the first case $A\sum a_n = s$, and in the latter case $L\sum a_n = s$ (summability A or L respectively). It is known that convergence does not imply L-summability and conversely L-summability does not imply convergence of $\sum a_n$. Tauberian type problems which arise out of this situation have been discussed.[1] It is also known that either convergence or L-summability imply A-summability. As to the converse (restricting ourselves to real a_n) we have proved the following theorems:

THEOREM 1. [8, pp. 582–583]. *If*

$$\sum_n^{2n} (|a_\nu| - a_\nu) = O(1) \quad \textit{as} \quad n \to \infty, \tag{1.3}$$

and if

$$\sum_1^\infty a_n r^n = O(1) \quad \textit{as} \quad r \uparrow 1, \tag{1.4}$$

Presented to the Society, December 27, 1942; received by the editors December 16, 1942.

[1] See [8], where further references are given; numbers in brackets refer to the bibliography at the end of this paper.

then

$$t^{-1}F(t) = O(1) \quad as \quad t\downarrow 0. \tag{1.5}$$

THEOREM 2. [8, p. 585]. *If* (1.3) *holds and if*

$$\lim_{\lambda\downarrow 1} \liminf_{n\to\infty} \min_{n\leq k\leq \lambda n} \sum_{n}^{k} a_\nu \geqq 0, \tag{1.6}$$

then A-summability implies L-summability.

Note that A-summability and (1.6) (without (1.3), which need not be satisfied) imply convergence (by a theorem of R. Schmidt) and are also necessary for convergence, while the series need not be L-summable.

We remark also that, in the assumption and in the conclusion of Theorem 1, $O(1)$ can be replaced by $o(1)$; for if

$$\sum_{n}^{2n} (|a_\nu| - a_\nu) = o(1) \quad \text{as} \quad n \to \infty, \tag{1.7}$$

then (1.6) holds. Moreover by the previous remark the series $\sum a_n$ converges (to zero).

We shall complete and generalize these results by proving the following theorems:

THEOREM 3. *If* (1.3) *holds then each of the statements* (1.4), (1.5) *and*

$$\sum_{1}^{n} a_\nu = O(1) \quad as \quad n\to\infty \tag{1.8}$$

implies the two others.

THEOREM 4. *If* (1.3) *holds then A-summability implies L-summability, but not necessarily convergence.*

THEOREM 5. *If* (1.3) *holds and if* $\sum a_n$ *converges, then* $\sum a_n \sin nt/nt$ *converges uniformly in* $0<t<\pi$.

This generalizes Theorem 6′ of my paper [8].

2. **Proof of Theorem 3.** We prove the following lemma.

Lemma 1. *If* (1.3) *and* (1.4) *hold, then*

$$s_n = \sum_{1}^{n} a_\nu = O(1), \qquad \sum_{n}^{2n} |a_\nu| = O(1), \qquad \sum_{1}^{n} \nu |a_\nu| = O(n), \tag{2.1}$$

$$\sum_{1}^{\infty} \nu^{-1} |a_\nu| < \infty, \qquad \sum_{n}^{\infty} \nu^{-1}|a_\nu| = O(n^{-1}) \quad as \quad n\to\infty. \tag{2.2}$$

The statement $s_n = O(1)$ is an immediate corollary of a previous result [**6**, Lemma 2]. Combining it with (1.3) we get

$$\sum_n^{2n} | a_\nu | = \sum_n^{2n} (| a_\nu | - a_\nu) + s_{2n} - s_{n-1} = O(1) \quad \text{as} \quad n \to \infty.$$

Furthermore, where $\sum_\alpha^\beta$ means summation over the range $\alpha < \nu \leqq \beta$,

$$\sum_1^n \nu | a_\nu | = \sum_{k=0}^n \sum_{n/2^{k+1}}^{n/2^k} \nu | a_\nu | \leqq \sum_{k=0}^n \left(\frac{n}{2^k} \sum_{n/2^{k+1}}^{n/2^k} | a_\nu | \right)$$
$$= O\left(n \sum_0^\infty 2^{-k} \right) = O(n).$$

(2.1) is now proved. We have thus $\sum_n^{2n} | a_\nu | < c$, a positive constant, and $\sum_n^{2n} \nu^{-1} | a_\nu | < c/n$, hence

$$\sum_1^n \nu^{-1} | a_\nu | \leqq \sum_{k=1}^n \sum_{2^{k-1}}^{2^k} \nu^{-1} | a_\nu | < c \sum_{k=1}^\infty 2^{1-k} = 2c.$$

This proves the first part of (2.2). Finally

$$\sum_n^\infty \nu^{-1} | a_\nu | \leqq \sum_{k=1}^\infty \sum_{n \cdot 2^{k-1}}^{n \cdot 2^k} \nu^{-1} | a_\nu | < \frac{c}{n} \sum_1^\infty 2^{1-k} = \frac{2c}{n},$$

which proves the lemma.

We now prove Theorem 3. If (1.3) holds, then (1.8) implies (1.5) by Theorem 5 of my paper [**8**], and (1.4) follows from the remark to the same theorem. By the same remark (1.4) implies (1.8), hence also (1.5). Finally, assuming (1.5), to prove (1.8) we write

$$t^{-1}F(t) - s_n = \sum_1^n a_\nu \left(\frac{\sin \nu t}{\nu t} - 1 \right) + \sum_{n+1}^\infty a_\nu \frac{\sin \nu t}{\nu t} \equiv S_1 + S_2.$$

From $0 < 1 - \sin \nu t / \nu t < \nu^2 t^2$ we get

$$| S_1 | < t^2 \sum_1^n \nu^2 | a_\nu | < n t^2 \sum_1^n \nu | a_\nu | = t^2 O(n^2);$$

furthermore, by Lemma 1,

$$| S_2 | < t^{-1} \sum_n^\infty \nu^{-1} | a_\nu | = O(n^{-1} t^{-1}).$$

On putting now $t = n^{-1}$ we get

$$nF(n^{-1}) - s_n = O(1) \quad \text{as} \quad n \to \infty;$$

this proves (1.8) and a fortiori (1.4), which completes the proof of Theorem 3.

3. **Proof of Theorem 4.** We first prove the following lemmas.

LEMMA 2. *Let*

$$\Delta_n = \sin nt/nt - \sin (n+1)t/(n+1)t,$$
$$\Delta_n^2 = \Delta(\Delta_n) = \sin nt/nt - 2 \sin (n+1)t/(n+1)t + \sin (n+2)t/(n+2)t;$$

then

$$0 < \Delta_n^2 < t^2 \quad \textit{for} \quad (n+2)t < \pi/2, \tag{3.1}$$

$$|\Delta_n| < 2/n \quad \textit{for} \quad nt > 1. \tag{3.2}$$

Applying the mean value theorem to Δ^2 we get easily (see [**8**, Lemma 4])

$$0 < \Delta_n^2 < t^2 \quad \text{for} \quad (n+2)t < \pi/2.$$

Furthermore

$$\Delta_n = \frac{\sin (n+1)t}{n(n+1)t} - 2\,\frac{\sin (t/2) \cos ((2n+1)t/2)}{nt},$$

which yields

$$|\Delta_n| < 1/n(n+1)t + 1/n < 2/n \qquad \text{for } nt > 1.$$

LEMMA 3. *If $\sum a_n$ is Abel summable and if* (1.3) *holds, then $\sum a_n$ is Cesàro summable of any order $\alpha > 0$.*

By Lemma 1, $s_n = O(1)$; this and A-summability imply $(C, 1)$ summability, as was proved first by Littlewood in 1910. For a short proof (with a more general assumption) cf. [**5**]. That Abel summability and $s_n = O(1)$ imply (C, α) summability for any $\alpha > 0$ has been proved by Andersen [**1**, p. 80]. We shall apply only the case $\alpha = 1$.

Let now $\sum_1^n s_\nu = s_n'$, then $n^{-1}s_n'$ tends to a limit s; we can assume without loss of generality that $s = 0$ (otherwise replace a_1 by $a_1 - s$). To a given positive $\epsilon < 1/2$ we now choose $n_0(\epsilon)$ so that

$$|s_n'| < \epsilon^3 n \quad \text{for} \quad n > n_0(\epsilon) > 3. \tag{3.3}$$

By (2.2) $\sum \nu^{-1} a_\nu \sin \nu t$ converges absolutely; we write

$$t^{-1}F(t) = \sum_1^\infty a_\nu \frac{\sin \nu t}{\nu t} = \sum_1^n + \sum_{n+1}^\infty \equiv T_1 + T_2.$$

We restrict ourselves to $0<t<n_0^{-1}$, and choose $n=1+[\epsilon^{-1}t^{-1}]>\epsilon^{-1}t^{-1}>\epsilon^{-1}n_0>2n_0$; Abel's summation by parts yields

$$T_1 = s_n \frac{\sin nt}{nt} + s'_{n-1}\Delta_{n-1} + \sum_1^{n-2} s'_\nu \Delta^2_\nu.$$

Now $nt>\epsilon^{-1}$. Hence

(3.4) $$|s_n \sin nt/nt| < |s_n|/nt < \epsilon|s_n| = \epsilon O(1) \quad \text{as} \quad t\downarrow 0,$$

and, from (3.2) and (3.3),

(3.5) $$|s'_{n-1}\Delta_{n-1}| < 2\epsilon^3;$$

furthermore

(3.6) $$|T_2| < t^{-1}\sum_n^\infty \nu^{-1}|a_\nu| = O(n^{-1}t^{-1}) = O(\epsilon) \quad \text{as} \quad t\downarrow 0.$$

Finally, write

$$\sum_1^{n-2} s'_\nu\Delta^2_\nu = \left(\sum_1^{k-1} + \sum_k^{n-2}\right) s'_\nu\Delta^2_\nu, \qquad 2 \leqq k \leqq n-2,$$

and choose

$$k = 1 + [t^{-1}] > t^{-1} > n_0(\epsilon) > 3.$$

By (3.1), as $(k+1)t<(2+t^{-1})t<3/2<\pi/2$,

(3.7) $$\left|\sum_1^{k-1} s'_\nu\Delta^2_\nu\right| < t^2 \sum_1^k |s'_\nu| = o(t^2k^2) = o(1).$$

It remains to estimate $\sum_k^{n-2} s'_\nu\Delta^2_\nu$. We decompose this sum according to the changes of sign of the factors Δ^2_ν, and write

$$\sum_k^{n-2} s'_\nu\Delta^2_\nu = \sum_1 + \sum_2 + \cdots + \sum_\rho.$$

To estimate ρ we note that there are not more changes of sign in the sequence Δ^2_ν than there are zeros $x_1, x_2, \cdots$ of $D_2(x^{-1}\sin x)$ in the interval $0<x<(n-1)t$. A simple calculation yields for x_ν the estimate

$$x_\nu = (\nu+1)\pi - \psi_\nu, \quad 0 < \psi_\nu < \pi/3, \quad \nu = 1, 2, 3, \cdots;$$

hence,

$$\rho\pi < x_\rho < (n-1)t < \epsilon^{-1}.$$

But each $\sum$ is in absolute value less than $4\epsilon^3nk^{-1}$ (from (3.2) and (3.3)), and

$$\epsilon^3 n k^{-1} < \epsilon^3 nt < 2\epsilon^2;$$

thus

$$(3.8)\qquad \left| \sum_{k}^{n-2} s_\nu' \Delta_\nu^2 \right| < 2\rho\epsilon^2 < \epsilon.$$

Collecting the estimates (3.4) to (3.8) we find

$$\left| t^{-1}F(t) \right| < \epsilon O(1) + o(1) \quad \text{as} \quad t \downarrow 0;$$

ϵ being arbitrarily small the positive part of Theorem 4 follows. For the negative part we refer to the examples in §5.

4. **Proof of Theorem 5.** We write, for $\lambda > 1$,

$$\sum_{n+1}^{\infty} a_\nu \frac{\sin \nu t}{\nu t} = \sum_{n+1}^{\lambda n} + \sum_{\nu > \lambda n} = R_1 + R_2,$$

say; then by (2.2)

$$\left| R_2 \right| < t^{-1} \sum_{\nu > \lambda n} \nu^{-1} \left| a_\nu \right| = \frac{1}{\lambda n t} O(1).$$

Abel's summation by parts yields

$$\sum_{1}^{n} a_\nu \frac{\sin \nu t}{\nu t} = s_n \frac{\sin nt}{nt} + \sum_{1}^{n-1} s_\nu \Delta_\nu,$$

whence

$$\sum_{n+1}^{n+k} a_\nu \frac{\sin \nu t}{\nu t} = s_{n+k} \frac{\sin (n+k)t}{(n+k)t} - s_n \frac{\sin nt}{nt} + \sum_{n}^{n+k-1} s_\nu \Delta_\nu.$$

We may assume that the limit of s_n is zero; given $\epsilon > 0$, we choose $n_0(\epsilon)$ so that $|s_n| < \epsilon^3$ for $n > n_0$; then

$$\left| s_{n+k} \frac{\sin (n+k)t}{(n+k)t} - s_n \frac{\sin nt}{nt} \right| < 2\epsilon^3 \quad \text{for} \quad n > n_0(\epsilon).$$

We define k by $n+k = [\lambda n]$, thus $k = [\lambda n] - n \leqq (\lambda - 1)n$. We subdivide the range $n \leqq \nu < \lambda n$ into consecutive parts in each of which Δ_ν has constant sign; denote the number of subdivisions by σ. Denoting the positive zeros of $u^{-1} \sin u$ by $u_1 < u_2 < \cdots$, we find easily $u_\nu = \nu\pi + \alpha_\nu$, where $0 < \alpha_\nu < \pi/2$; the number of zeros in the interval $nt < u < \lambda nt$ is therefore less than $2\lambda nt/\pi$, and

$$\sigma \leqq \lambda nt + 2.$$

In each section $|\sum s_\nu \Delta_\nu| < 2\epsilon^3$, hence

$$\left|\sum_n^{n+k-1} s_\nu \Delta_\nu\right| < 2\epsilon^3(2 + \lambda nt),$$

and

$$|R_1| < 2\epsilon^3(3 + \lambda nt).$$

We now choose $\lambda = 1/\epsilon^2 nt$, for whatever $n > n_0(\epsilon)$ and any $0 < t < \pi$, if $\epsilon^2 nt < 1$, and put $\lambda = 1$ (that is $R_1 \equiv 0$) otherwise. In the latter case $|\sum_{n+1}^\infty a_\nu \sin(\nu t)/\nu t| < (nt)^{-1}O(1) < \epsilon^2 O(1)$, while in the first case

$$\left|\sum_{n+1}^\infty a_\nu \frac{\sin \nu t}{\nu t}\right| < \epsilon^2 O(1) + 2\epsilon^3\left(3 + \frac{1}{\epsilon^2}\right) < \epsilon O(1)$$

for $n > n_0(\epsilon)$ and $0 < t < \pi$. This proves our theorem.

Note that convergence of $\sum a_n$ is a necessary condition for the uniform convergence of $\sum a_n \sin(nt)/nt$. For if, for any $\epsilon > 0$,

$$\left|\sum_{n+1}^{n+k} a_\nu \frac{\sin \nu t}{\nu t}\right| < \epsilon \quad \text{for} \quad n > n_0(\epsilon), \quad k = 1, 2, 3, \cdots, 0 < t < \pi,$$

then, letting $t \downarrow 0$ we get $|\sum_{n+1}^{n+k} a_\nu| \leqq \epsilon$. Moreover we have uniform convergence in the closed interval.

It is shown easily that the assumption (1.3) is equivalent to either of the following conditions: There exists a constant $\lambda > 1$ such that

$$\sum_n^{\lambda n} (|a_\nu| - a_\nu) = O(1); \tag{4.1}$$

$$\sum_1^n \nu(|a_\nu| - a_\nu) = O(n), \quad \text{as} \quad n \to \infty. \tag{4.2}$$

For a more general statement see [7, p. 129].

A consequence of our results is the following theorem:

THEOREM 6. *If*

$$\lim_{\lambda \downarrow 1} \limsup_{n \to \infty} \sum_n^{\lambda n} (|a_\nu| - a_\nu) = 0, \tag{4.3}$$

then A-summability of $\sum a_n$ implies uniform convergence of the series $\sum a_n \sin(nt)/nt$ in $0 < t < \pi$.

Clearly (4.3) implies (4.1), whence (1.3). Now, by Theorem 4, $\sum a_n$ is L-summable; furthermore by Theorem 4 of our paper [8] L-summability and (4.3) imply convergence of $\sum a_n$. Theorem 6 now follows from Theorem 5.

5. **Negative results.** We quote the following lemma.

LEMMA 4. *Let* $n \geqq 1$ *and*

$$P_n(z) = \frac{1}{n} + \frac{z}{n-1} + \cdots + \frac{z^{n-1}}{1} - \frac{z^n}{1} - \cdots - \frac{z^{2n-1}}{n};$$

then, when $|z| \leqq 1$,

$$|P_n(z)| < 6.$$

For the proof see Fejér [2, pp. 36–37].

Consider the polynomial series $\sum_1^\infty n^{-2} z^{\lambda_n} P_{k_n}(z)$, where $\lambda_1 = 1$, $k_1 = 3$, $2\lambda_n = 2^{n^2}$, $2k_n = \lambda_{n+1} - \lambda_n$, $n \geqq 2$. In view of the above lemma the series converges uniformly in $|z| \leqq 1$, so that the function

$$F(z) = \sum_1^\infty n^{-2} z^{\lambda_n} P_{k_n}(z)$$

is regular in $|z| < 1$ and continuous in $|z| \leqq 1$. The degree of the nth term is $2k_n + \lambda_n - 1 < \lambda_{n+1}$, hence writing out the polynomials explicitly we get a power series, convergent for $|z| < 1$,

$$F(z) = \sum a_n z^n. \tag{5.1}$$

For $|z| = 1$ we get a Fourier power series of a *continuous* function $F(e^{it})$. The structure of P_n and the inequality $(n+1)^{-2} \log k_n < \log 2$ easily yield

$$\sum_n^{2n} |a_\nu| = O(1) \quad as \quad n \to \infty.$$

But $\sum a_n$ diverges, as there are sections $\sum a_\nu = n^{-2} \sum_1^{k_n} 1/\nu$ which do not tend to zero. On the other hand the series (5.1) is evidently L-summable at every point on $|z| = 1$.

Next we define a series $\sum a_n$ by putting $s_n = 1$ for $n = 2^k$, $k = 0, 1, 2, \cdots$, and $s_n = 0$ otherwise. Now $n^{-1} \sum_1^n s_\nu \to 0$, moreover $\sum_n^{2n} |a_\nu| \leqq 3$, hence the series is summable L. But $\sum a_n$ diverges, in fact $\limsup |a_n| = 1$, and $\sum a_n \cos nt$ is not a Fourier series.

Another example of this kind is due to Neder [4].

In contrast Menchoff [3] tried to prove that A-summability and (1.3) imply convergence of $\sum a_n$; the error lies in his Lemma 4 which is false. It is based on a false interpretation of an argument used by Landau.

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