ON THE CONVERGENCE OF CERTAIN PARTIAL SUMS OF A TAYLOR SERIES WITH GAPS

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We consider the function $f(z)$ determined by the power series

$$f(z) = \sum_{1}^{\infty} c_n z^{\lambda_n} \tag{1}$$

and its direct analytic continuation. For simplicity, it is supposed that $\limsup |c_n|^{1/\lambda_n} = 1$.

We write

$$S_n(z) = \sum_{1}^{n} c_p z^{\lambda_p},$$

$$M(r) = \max_{|z|=r} |f(z)| \qquad (0 < r < 1),$$

$$M(r) = 1 \qquad (r \leqq 0),$$

$$\theta_n = \lambda_{n+1}/\lambda_n - 1.$$

Ostrowski has proved[1] that if $\{\theta_{n_i}\}$ is a sequence extracted from the sequence $\{\theta_n\}$ such that $\liminf \theta_{n_i} > 0$, then every regular point of $f(z)$ on the circle $|z| = 1$ is the center of a circle in which the sequence $\{S_{n_i}(z)\}$ converges uniformly to $f(z)$. Restricting ourselves to the question of convergence at the regular points themselves, we shall prove the following theorem:

If

$$\limsup_{i\to\infty} \frac{\log (M(1 - \theta_{n_i}^2)/\theta_{n_i})}{\lambda_{n_i}\theta_{n_i}^2} < \infty, \tag{2}$$

then $\lim S_{n_i}(z) = f(z)$ *at all regular points of* (1) *on the circle* $|z| = 1$.

For the proof, we shall assume that $\lim \theta_{n_i} = 0$; afterwards, we shall remove this restriction, with the aid of Ostrowski's theorem.

Let z_1 be a regular point for (1) on the circle $|z| = 1$, and let z_0 be a point on the segment joining z_1 to the origin. We write $|z_1 - z_0| = a$, and for every positive integer i we define the three circles

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[1] A. Ostrowski, *Über eine Eigenschaft gewisser Potenzreihen mit unendlich vielen verschwindenden Koefficienten*. Preuss. Akad. Wiss. Sitzungsber. vol. 34 (1921) pp. 557–565. Essentially the same proof is to be found in P. Montel's *Leçons sur les familles normales de fonctions analytiques et leurs applications*, pp. 204–207.

$$\Gamma_i': \ |z - z_0| = \rho_i' \qquad (0 < \rho_i' < a),$$
$$\Gamma(a): \ |z - z_0| = a, \qquad \Gamma_i'': \ |z - z_0| = \rho_i'' \qquad (\rho_i'' > a),$$

where ρ_i'' shall be chosen so that the function $f(z)$ is holomorphic in the closed region bounded by Γ_i''. By M_i', $M_i(a)$, and M_i'' we denote the maximum value of $|f(z) - S_{n_i}(z)|$ on Γ_i', $\Gamma(a)$, and Γ_i'', respectively. We write

$$R_i' = 1 + \rho_i' - a, \qquad R_i'' = 1 + \rho_i'' - a,$$

and we choose r_i so that $R_i' < r_i < 1$.

By Cauchy's formula we have, on the circle $|z| = R_i'$,

$$|f(z) - S_{n_i}(z)| \leqq \sum_{p=\lambda_{n_i+1}}^{\infty} \frac{M(r_i)}{r_i^p} R_i'^p = \frac{M(r_i) r_i}{r_i - R_i'} \left(\frac{R_i'}{r_i}\right)^{\lambda_{n_i}(1+\theta_{n_i})},$$

and by the principle of the maximum

$$\log M_i' \leqq \log \frac{M(r_i) r_i}{r_i - R_i'} + \lambda_{n_i}(1 + \theta_{n_i}) \log \frac{R_i'}{r_i}.$$

On the circle $|z| = R_i''$, and therefore also on Γ_i'', we have

$$|S_{n_i}(z)| \leqq M(r_i) \sum_0^{\lambda_{ni}} \left(\frac{R_i''}{r_i}\right)^p < \frac{M(r_i) R_i''}{R_i'' - r_i} \left(\frac{R_i''}{r_i}\right)^{\lambda_{ni}}.$$

Since $f(z)$ is holomorphic in the closed region bounded by Γ_i'', and since at least one of the expressions $R_i''/(R_i'' - r_i)$, $(R_i''/r_i)^{\lambda_{n_i}}$ tends to ∞ as $i \to \infty$, we may write, for any positive η and for i sufficiently large,

$$\log M_i'' < \log \frac{M(r_i)(1 + \eta) R_i''}{R_i'' - r_i} + \lambda_{n_i} \log \frac{R_i''}{r_i}.$$

Applying Hadamard's three-circle theorem to the function $S_{n_i}(z) - f(z)$ on the circles Γ_i', $\Gamma(a)$, Γ_i'', we have now

$$
\begin{aligned}
\log \frac{\rho_i''}{\rho_i'} \log M_i(a) &\leqq \log\left(\frac{\rho_i''}{a}\right) \log M_i' + \log\left(\frac{a}{\rho_i'}\right) \log M_i'' \\
&< \log\left(\frac{\rho_i''}{a}\right) \left\{\log \frac{M(r_i) r_i}{r_i - R_i'} + \lambda_{n_i}(1 + \theta_{n_i}) \log \frac{R_i'}{r_i}\right\} \qquad (3) \\
&\quad + \log\left(\frac{a}{\rho_i'}\right) \left\{\log \frac{M(r_i)(1 + \eta) R_i''}{R_i'' - r_i} + \lambda_{n_i} \log \frac{R_i''}{r_i}\right\}.
\end{aligned}
$$

In the proof of his theorem Ostrowski now takes the point z_0 near to the point z_1 and chooses for Γ_i', $\Gamma(a)$, and Γ_i'' three fixed circles with radii sufficiently near to $|z_1 - z_0|$ (with $\Gamma(a)$ including the point z_1 instead of passing through it). In our case it is necessary to take z_0 near to the origin and to let ρ_i' and ρ_i'' tend to a as i becomes large. We choose $\rho_i' = (1-b_i)a$, $\rho_i'' = (1+b_i)a$, $r_i = 1-k_i$, where $0<b_i<1$ and $0<k_i<ab_i$; substituting these values in (3), expanding terms such as $\log(1+ab_i)$ in power series, and dividing both sides of the inequality by b_i, we get

$$
\begin{aligned}
&2(1 + b_i^2/3 + \cdots) \log M_i(a) < (1 - b_i/2 + b_i^2/3 - \cdots) \\
&\qquad \cdot \left\{ \log \frac{M(1-k_i)(1-k_i)}{(1-k_i/ab_i)ab_i} + \lambda_{n_i}(1+\theta_{n_i}) \log \frac{1-ab_i}{1-k_i} \right\} \\
&\quad + (1 + b_i/2 + b_i^2/3 + \cdots) \\
&\qquad \cdot \left\{ \log \frac{M(1-k_i)(1-k_i)(1+\eta)}{(1+k_i/ab_i)ab_i} + \lambda_{n_i} \log \frac{1+ab_i}{1-k_i} \right\} \\
&\quad < 2\eta + 3k_i + 3k_i/ab_i - 3\log a \\
&\quad + 2(1 + b_i^2/3 + \cdots) \log \frac{M(1-k_i)}{b_i} \\
&\quad + \lambda_{n_i}\{\theta_{n_i}[1 - b_i/2 + b_i^2/3 - \cdots] \\
&\qquad \cdot [-ab_i - a^2b_i^2/2 - \cdots + k_i + k_i^2/2 + \cdots] \\
&\quad + [1 - b_i/2 + b_i^2/3 - \cdots][-ab_i - a^2b_i^2/2 - \cdots] \\
&\quad + [1 + b_i/2 + b_i^2/3 + \cdots][ab_i - a^2b_i^2/2 + \cdots] \\
&\quad + 2[1 + b_i^2/3 + \cdots][k_i + k_i^2/2 + \cdots]\} \\
&\quad < 2\eta + 3k_i + 3k_i/ab_i - 3\log a + 3\log \frac{M(1-k_i)}{b_i} \\
&\quad - (\lambda_{n_i}/2)\{ab_i[\theta_{n_i} - b_i(1-a)] - 3k_i\}
\end{aligned} \tag{4}
$$

provided θ_{n_i} and b_i are sufficiently small and the sum of the terms in the braces of the last term is positive.

Now suppose that

$$\limsup_{i\to\infty} \frac{\log(M(1-\theta_{n_i}^2)/\theta_{n_i})}{\lambda_{n_i}\theta_{n_i}^2} < H \qquad (1 < H < \infty).$$

Choose $b_i = b\theta_{n_i}$, $k_i = b_i\theta_{n_i}/b = \theta_{n_i}^2$, where $b > 14H$. The last member of (4) becomes

$$-(\lambda_n \theta_{n_i}^2 b/2)\{a[1 - b(1-a)] - 3/b\}.$$

If a is chosen sufficiently near to 1, $b\{a[1-b(1-a)]-3/b\}>10H$, and for sufficiently large values of i (4) becomes

$$(2+\epsilon)\log M_i(a) < K(a, b) + 3\log M(1-\theta_{n_i}^2)/\theta_{n_i} - 5\lambda_{n_i}\theta_{n_i}^2 H,$$

where $K(a, b)$ depends on a and b only. But $\lim_{i\to\infty}\theta_{n_i}=0$ implies, together with the validity of (2), that $\lim_{i\to\infty}\lambda_{n_i}\theta_{n_i}^2=\infty$. For sufficiently large i we have

$$(2+\epsilon)\log M_i(a) < -\lambda_{n_i}\theta_{n_i}^2 H,$$

that is,

$$\lim_{i\to\infty}\log M_i(a) = -\infty,$$

and, in particular,

$$\lim_{i\to\infty} S_{n_i}(z_1) = f(z_1).$$

Now let $\{\theta_{n_i}\}$ be any sequence of values θ_{n_i} for which (2) is satisfied, and let z_1 be a regular point for (1). From every subsequence of $\{\theta_{n_i}\}$ we can extract a further subsequence $\{\theta_{m_j}\}$ such that either $\lim\theta_{m_j}=0$ or $\liminf\theta_{m_j}>0$. In the first case, $\lim S_{m_j}(z_1)=f(z_1)$ by what we have just proved; in the second case, by Ostrowski's theorem. From every subsequence of $\{n_i\}$ we can therefore extract a further subsequence $\{m_j\}$ such that $\lim S_{m_j}(z_1)=f(z_1)$. It follows that the sequence $\{S_{n_i}(z_1)\}$ itself tends to $f(z_1)$, and our theorem is proved.

Condition (2) may be replaced by one that is somewhat less general, but can be expressed more immediately in terms of the behavior of $M(r)$:

If $\liminf\theta_{n_i}=\theta$, we define, for $h>\theta$,

$$\lambda(h) = \underset{\theta_{n_i}\leqq h}{\text{g.l.b.}}\ \lambda_{n_i};$$

for $0\leqq h\leqq\theta$, we write $\lambda(h)=\infty$. It follows from our theorem that $\lim S_{n_i}(z_1)=f(z_1)$ whenever z_1 is a regular point for (1) on the circle $|z|=1$, provided

$$\limsup_{h\to 0+}\frac{\log(M(1-h^2)/h)}{h^2\lambda(h)}<\infty. \tag{5}$$

If we write $r=1-h^2$, (5) becomes

$$\limsup_{r\to 1-0}\frac{\log[M(r)/(1-r)^{1/2}]}{(1-r)\lambda(1-r)^{1/2}}<\infty.$$

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ON ABEL AND LEBESGUE SUMMABILITY

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1. **Introduction.** A series $\sum_1^\infty a_n$ is called Abel summable to the value s if the power series $\sum a_n r^n$ converges for $0<r<1$, and if $\sum a_n r^n \to s$ as $r \uparrow 1$; it is called Lebesgue summable if the sine series

$$\sum_1^\infty a_n \frac{\sin nt}{n} = F(t) \tag{1.1}$$

converges in some interval $0<t<\tau$, and if

$$t^{-1}F(t) \to s \quad \text{as} \quad t \downarrow 0. \tag{1.2}$$

We write in the first case $A\sum a_n = s$, and in the latter case $L\sum a_n = s$ (summability A or L respectively). It is known that convergence does not imply L-summability and conversely L-summability does not imply convergence of $\sum a_n$. Tauberian type problems which arise out of this situation have been discussed.[1] It is also known that either convergence or L-summability imply A-summability. As to the converse (restricting ourselves to real a_n) we have proved the following theorems:

THEOREM 1. [8, pp. 582–583]. *If*

$$\sum_n^{2n} (|a_\nu| - a_\nu) = O(1) \quad \textit{as} \quad n \to \infty, \tag{1.3}$$

and if

$$\sum_1^\infty a_n r^n = O(1) \quad \textit{as} \quad r \uparrow 1, \tag{1.4}$$

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[1] See [8], where further references are given; numbers in brackets refer to the bibliography at the end of this paper.