Applied Mathematics

277. Garrett Birkhoff: *The reversibility paradox and camber.*

A comparison is made between the predictions of the (reversible) Kutta-Joukowsky lift theory and the experimental findings of Eiffel, for circular airfoils. It is found that the predicted angle of the chord to the airfoil with the airstream in the position of zero lift ("first axis") varies from two to seven times the observed angle. Whereas with camber 1/7, the observed increase in lift/effective angle of attack is about 83 per cent; the predicted increase 4 per cent. A possible irreversible explanation of this is pointed out. (Received August 14, 1943.)

278. H. B. Curry: *The method of steepest descent for nonlinear minimization problems.*

A practical method for calculating approximately a stationary value of a function $G(x_1, x_2, \cdots, x_n)$ is desirable in connection with certain nonlinear least square problems (abstract 49-11-286). Such a method may be exhibited as follows. Let x_i^0 be an initial approximation. The direction in which $g(t)=G(x_i^0+t\xi_i)$ decreases most rapidly is obtained by putting $\xi_i=-\partial G/\partial x_i$. By trial t_0 can be determined so that $g(t_0)<g(0)$. Then the point $x_i^0+t_0\xi_i$ can be used as a new approximation and the process repeated. It is shown here that if t_0 is chosen so that $g'(t_0)=0$, $g'(t)<0$ for $0\leqq t<t_0$, then the successive approximations converge to a point where $\partial G/\partial x_i=0$ provided that G takes a smaller value at x_i^0 than at any point on the boundary of that region S within which differentiability and other usual conditions hold. Although the process is well known in analysis, it does not appear to have been noticed recently in this connection. It is applicable when the initial approximation x_i^0 is too rough for the standard least square procedure. The problem includes that of solving n simultaneous nonlinear equations in n unknowns which was handled by Cauchy. (Received October 1, 1943.)

279. D. W. Dudley and Hillel Poritsky: *The geometry of cutting and hobbing of worms and gears.*

The exact shape of the teeth of a worm W or a helical gear G which is produced by a milling cutter C or a hob H of a given tooth profile is investigated. This problem and its converse are of great interest in the manufacture of gears and worms. Assuming numerous cutter teeth one may replace the cutter C by a surface of revolution S. In its motion relative to W, S occupies a one-parameter family of positions whose envelope is W. The curve of contact of S_0, any position of S, with the envelope represents C, the curve of deepest cutting. C may be determined from the condition that along it the velocity of the motion of S relative to W is tangent to S. In the converse problem where W is given and S is sought, the motion of W relative to S is utilized in a similar way; this motion can now be simplified to a rotation about the cutter axis. A similar treatment applies to the hob-gear problem except that here a two-parameter family of motions is involved and two kinematic conditions are now required to determine the deepest cutting. (Received August 12, 1943.)

280. Bernard Friedman: *A method of approximating the complex roots of polynomial equations.*

By using successive divisions, an iterative process is set up to approximate the quadratic factors of a polynomial. In this way, the complex roots of largest and smallest absolute value can be determined. Conditions for convergence and for the rapidity of convergence are also investigated. (Received August 31, 1943.)

281. R. E. Gaskell: *On moment balancing in structural dynamics.*

The moment balancing process (H. Cross, Transactions of the American Society of Civil Engineers vol. 96 p. 1) is developed for the dynamics of structures with the help of the dynamic form of the four-moment equation (W. Prager, Ingenieur-Archiv, vol. 1, p. 527). This combination gives a method of finding stresses in plane, rigid-frame structures, each of whose bars is of uniform cross section and whose joints are not subject to translation, when an oscillating load is applied. The method converges at least for oscillations whose frequency is less than the lowest natural frequency of the structure. (Received August 5, 1943.)

282. G. H. Handelman: *On a principle of M. A. Sadowsky.* Preliminary report.

Sadowsky (Journal of Applied Mechanics vol. 10 (1943) pp. A-65) has applied a heuristic principle of maximum plastic resistance to several types of combined states of stress. The principle has been proved by Prager for a beam under combined tension and torsion. This paper is concerned with the additional case of a symmetric beam subject to bending and torsion. The basic differential equation, derived from the stress-strain relations and the yield condition, is shown to be the Euler-Lagrange equation for the following variational problem: to find that stress distribution for a given applied torque for which the value of the bending couple is stationary. (Received August 9, 1943.)

283. Carl Holtom: *Permanent configurations in the n-body problem.*

Let n masses in a plane be projected with initial velocities and thereafter be allowed to move only under the force of attraction according to the Newtonian law. If the potential function of the system is $U(x_i, y_i)$, then the masses m_i at points (x_i, y_i) form a permanent configuration if there exists a real solution of the equations $\partial U/\partial x_i+m_i\omega^2x_i=0$ and $\partial U/\partial y_i+m_i\omega^2y_i=0$, where ω^2 is the angular velocity of the system. It is shown that for the case of $n-1$ masses there exists at least one real point of liberation at which $m_n=0$ may be placed, and that the solutions vary continuously and remain real as m_n increases to a positive value. For the case $n>4$ a restriction on the masses or mass ratios is sufficient, though it may not be necessary, to insure the existence of a permanent configuration. The equations may be solved for the coordinates as functions of the mass ratios, and several terms in the series solutions are given. (Received August 6, 1943.)

284. L. C. Hutchinson: *Free vibrations in a rectangular rod.* Preliminary report.

In this paper the author studies, directly from the general equations of motion, the possible free vibrations in a rectangular rod of isotropic homogeneous material, and their relationship to the corresponding acoustic and electromagnetic waves. (Received August 6, 1943.)

285. Arthur Korn: *Compressibility with repect to vibrations of high frequency.*

When the velocities and accelerations remain under a certain limit, the compressibility of the air can be quite well handled by means of the well known state equation $p=R\rho T$, in which p is the pressure, ρ the density, T the absolute temperature and R a constant. If one takes account of friction the formula is slightly modified. For high values of velocities and accelerations, especially when one has to do with vibrations of high frequency, new approximations have to be found. The state equation of compressible matter not only contains according to the author's ideas all the secrets of quantum mechanics, but also has important relations to the theory of the electromagnetic field. For vibrations $\bar{v}=\bar{v}_0+\bar{v}_1 \cos \nu t+\bar{v}_2 \sin \nu t$ with frequencies, as they are considered in quantum mechanics, for example, in the theory of the spectra, the state equation may have the form $p=R\rho T+R'\nu\rho(T)^{1/2}$, where R and R' are constants and ν the frequency of the vibration. For high frequencies the pressure will become more and more proportional to the frequency. When the frequency still becomes higher and higher, one may make the assumption that for such vibrations approximate incompressibility takes place. (Received August 12, 1943.)

286. Kenneth Levenberg: *A method for the solution of certain non-linear problems in least squares.*

The standard method of least squares for problems involving a set of residual functions $f_i(\alpha, \beta, \gamma, \cdots)$, $i=1, 2, \cdots, n$, which are nonlinear in the parameters $\alpha, \beta, \gamma, \cdots$, consists of reducing the $f_i(\alpha, \beta, \gamma, \cdots)$ to linear form by first order Taylor approximations $F_i(\alpha, \beta, \gamma, \cdots)$, taken about an initial trial solution for the parameters $\alpha_0, \beta_0, \gamma_0, \cdots$, and minimizing $S=\sum_1^n F_i^2(\alpha, \beta, \gamma, \cdots)$ by solving the corresponding linear normal equations (Whittaker and Robinson, Calculus of Observations, 1937, p. 214). If $\alpha_1, \beta_1, \gamma_1, \cdots$ are the solution of these normal equations, it may happen, especially in certain engineering applications, that $\sum_1^n f_i^2(\alpha_1, \beta_1, \gamma_1, \cdots) \geqq \sum_1^n f_i^2(\alpha_0, \beta_0, \gamma_0, \cdots)$ in which case the process has failed to improve the initial solution. This paper offers a method of solving the problem in this case. It is shown that the initial solution can be improved by the minimization of $wS+Q$, where Q is a positive definite quadratic form in $\alpha-\alpha_0, \beta-\beta_0, \gamma-\gamma_0, \cdots$, and w is a sufficiently small positive quantity for which an approximate expression, which may be improved by trial, is given. For $Q=(\alpha-\alpha_0)^2+(\beta-\beta_0)^2+(\gamma-\gamma_0)^2+\cdots$, the method consists of revising the normal equations by the addition of a constant $1/w$ to the coefficients of the principal diagonal. (Received October 1, 1943.)

287. L. L. Merrill: *The mathematical determination of cooling rates during arc welding.*

This is an extension of work done by Dr. Daniel Rosenthal who determined the temperature distribution in very thick or very thin plates during arc welding under the simplifying assumptions that the heat source is a point source and the plate metal in the neighborhood of the source undergoes no change. The present investigation considers plates of various finite thickness and, by introducing an "input factor," takes into account the facts that the heat source is not a point source and that a pool of molten metal trails the moving electrode. To accomplish this two temperature-time records for each plate thickness must be determined experimentally at a point at the edge of the molten metal, one with the plate initially at room temperature and the other with the plate at some convenient pre-heat temperature. It is then possible to predict the cooling rate at any temperature during the cooling, for any initial plate temperature, for any travel speed, and for any heat input within the range of interest. Results of the investigation make it possible to determine welding conditions necessary to produce any desired cooling rate at any temperature level during cooling and thus to make a weld with desired characteristics. (Received August 4, 1943.)

288. A. D. Michal: *An analogue of the Maupertuis-Jacobi "least" action principle for dynamical systems of infinite degrees of freedom.*

More than a century ago, Jacobi gave his famous least action principle for the dynamical trajectories of dynamical systems with finite degrees of freedom. Although Hamilton's principle has been used widely in the derivation of the equations of motion for infinite degree of freedom problems in physics and engineering, analogues of the Jacobi least action principle do not seem to be in the literature. In this paper, analogues of the Jacobi least action principle are given for various infinite degree of freedom problems. The functionals that are made stationary are more general than those considered in the usual calculus of variations. (Received August 6, 1943.)

289. A. D. Michal: *Physical models of some curved differential-geometric metric spaces of infinite dimensions.* I. *Wave motion as a study in geodesics.*

In this paper, the second order partial differential equations of wave motion are shown to define dynamical states that can be considered as generating geodesics in some of the author's general "Riemannian" spaces with linear topological coordinates (Bull. Amer. Math. Soc. vol. 45 (1939) pp. 529–563, especially pp. 551–559). The author's earlier geometries (1927–1931) in infinitely dimensional spaces are not suitable for these particular physical models partly because some of the functionals have a higher order continuity than zero. (Received August 6, 1943.)

290. Isaac Opatowski: *Cantiliever beam of circular cross section and constant strength under the action of its own weight.*

The determination of the cross section consists in the solution of a nonlinear integral equation, which gives the profile of the beam in the shape of two parabolas tangent at their common vertex. A calculation of deflections, based on the usual approximation concerning the curvature, gives an infinite deflection at the free end. The use of an exact expression of the curvature gives an imaginary deflection curve, at least within a certain range containing the free end. These impossible conclusions are due to the use of the theory of prismatic beams for a beam of variable cross section. An extension is made to the case in which, besides the beam's own weight, a known force F acts at its free end. This leads to a hyperelliptic integral which gives a known formula when the beam's own weight is negligible with respect to F (see, for example, Hütte, *Des Ingenieurs Taschenbuch*, 25th edition, Berlin, 1925, vol. 1, p. 627). (Received August 6, 1943.)

291. M. O. Peach: *Simplified technique for constructing orthonormal functions.*

To orthonormalize the functions $f_1, f_2, \cdots$ over a given region R evaluate the quantities $d_{p,q}=\int_R f_p \bar{f}_q dR$, $0<p$, $q\leqq n$, where n orthonormal functions are desired.

A computer, whose mathematical attainments need extend no further than a knowledge of the algebraic rules of sign, may complete the calculations. The quantities $d_{p,q}$ are formed into a matrix, the unit matrix is adjoined, and a process P is applied to the combined matrix which reduces its order by one. Repeated application of P yields directly the coefficients of all n orthogonal functions. A root extraction is required to obtain the normalizing factor for each function. The process P consists of the repeated evaluation of certain second-order determinants. Proof depends upon certain properties of P and is obtained largely from the elementary properties of determinants. The method is particularly adapted to the use of modern computing machines. (Received August 9, 1943.)

292. A. E. Ross: *A note on forced oscillations of a conducting sphere.*

J. A. Stratton and L. J. Chu (Journal of Applied Physics vol. 12 (1941) pp. 236–240) studied the problem of forced oscillations in a conducting sphere. Taking the equatorial line as the effective region of an applied oscillatory field E', they obtained a series expansion of the current I_0 crossing the equatorial plane and used it to compute and plot the curves for the radiation conductance, and susceptance, and input admittance of the first few modes of the sphere, and also for the total radiation conductance. It was noted by Stratton and Chu that the above series expansions were not convergent. In the present paper, at the suggestion of Professor Brillouin, the author takes an equatorial strip of finite fixed width $\Delta\theta$ as the effective region of the applied field E', shows that the resulting expansion for the current I_θ through the surface $\theta=$const. is convergent, and employs it to plot the graphs of the above mentioned quantities. It is of interest to observe that to different values of $\Delta\theta$ correspond different selections of dominant modes in the expansions of the input impedance Y_i. (Received August 6, 1943.)

293. J. A. Shohat: *Parseval formula in its application to Van der Pol's and generalized equations.*

First, an error is indicated in the paper by the author: *A new analytical method* · · · , Journal of Applied Physics vol. 14 (1943) pp. 40–48. It is further shown, by a simple application of the Parseval formula for Fourier series, that the main result of Van der Pol—approximating the solution by a fundamental oscillation—is mathematically correct, and the error of such an approximation is estimated. Its amplitude is derived anew, with the same numerical result as previously obtained. The present method is shown to apply to the more general equation $d^2u/dt^2-\epsilon F(u)du/dt+u=0$. (Received September 15, 1943.)

294. C. A. Truesdell: *The differential equations of the membrane theory of shells of revolution.*

Previously the "membrane equations" have been derived from a figure. Here they are shown to be consequences of the general equations of elasticity. First a generalization of spherical polar coordinates is introduced in which one family of coordinate surfaces is that obtained by revolving curves parallel to some fixed curve about an axis. Then the assumption is made that there is no force normal to the shell across any cut, and no force component in any direction across the surface of the shell or across any parallel surface. There result equations similar to those usually given for the membrane theory and not more difficult to solve. If the thickness of the shell is small compared with its radii of curvature, the usual equations can be deduced, in which the stress does not vary along a normal. The significance of the assumptions is discussed. Finally it is shown that solutions of the equations of the membrane theory lead to a state of strain which does not satisfy the conditions of compatibility, so that displacements calculated in the membrane theory violate geometry. (Received August 6, 1943.)

295. D. V. Widder: *Positive temperatures on an infinite rod.*

In this paper it is proved that any solution $u(x, t)$ of the heat equation $\partial^2u/\partial x^2 = \partial u/\partial t$ which is non-negative for positive t and which vanishes for $t=0$ is identically zero. By use of this result it is shown that any solution which is non-negative for $t>0$ has the Poisson-Stieltjes representation $u(x, t)=\int_{-\infty}^{\infty}k(x-y, t)d\alpha(y)$. Here $\alpha(y)$ is nondecreasing and $k(x, t)$ is the familiar source solution $(4\pi t)^{-1/2}\exp(-x^2/4t)$. As a consequence any such solution is analytic in x and in t. (Received August 24, 1943.)

Geometry

296. R. C. Buck: *Partition of space.*

By an application of elementary topology, it is shown that n hyperplanes, with general intersection, partition Euclidean r-space into $M_r(p, n)$ p-dimensional regions, where $M_r(p, n)=\sum_{r-p}^{r} C_{n,k}C_{k,r-p}$, of which $C_{r,p}C_{n,r+1}\,(r+1)/(n+p-r)$ are bounded. The problem is also solved for projective r-space, yielding $\sum_0^{[(r-p)/2]} C_{r-2k,r-p}C_{n,r-2k}$ as the number of p-dimensional regions. This completely solves the well known "cheese slicing" problem. (Received September 11, 1943.)

297. John DeCicco: *Dynamical and curvature trajectories in space.*

Kasner has studied the geometry of dynamical trajectories in the plane and in space in the Princeton Colloquium (Amer. Math. Soc. Colloquium Publications, vol. 3). This paper considers the problem of determining all quintuply-infinite systems of curves in space which are at once dynamical and curvature trajectories. In the plane, Kasner has shown that the appropriate families are the trajectories of all central or parallel fields of force. It is shown that the systems of ∞^5 curves which are simultaneously dynamical and curvature trajectories are the dynamical trajectories of the following three distinct types of fields of force: (I) Those whose lines of force all lie in a pencil of planes. (II) Those whose lines of force are orthogonal to a family of ∞^2 circular helices, all of which possess the same axis and the same period. (III) Those of the central or parallel type. Each of these types is projectively invariant. (Received August 11, 1943.)

298. Edward Kasner and John DeCicco: *Union-preserving transformation of space.*

Sophus Lie showed that the only lineal-element transformations of the contact type are the extended point transformations. This result is extended by studying transformations from differential curve-elements of order n: $(x, y, z, y', z', \cdots, y^{(n)}, z^{(n)})$, where n is 2 or more, into lineal-elements (X, Y, Z, Y', Z'). The entire class of the union-preserving transformations is determined. Any general union-preserving transformation from curve-elements of order n into lineal-elements is completely determined by a new directrix equation $\Omega(X, Y, Z, x, y, z, y', z', \cdots, y^{(n-2)}, z^{(n-2)})=0$. The only available union-preserving transformations (in the whole domain of