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SOLUTION OF THE "PROBLÈME DES MÉNAGES"

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The *problème des ménages* asks for the number of ways of seating n husbands and n wives at a circular table, men alternating with women, so that no husband sits next to his wife. Despite the considerable literature devoted to this problem (cf. the appended bibliography), the following simple solution seems to have been missed.

It is convenient first to solve two preliminary problems, perhaps of some interest in themselves.

LEMMA 1. *The number of ways of selecting k objects, no two consecutive, from n objects arrayed in a row is ${}_{n-k+1}C_k$.*

Let $f(n, k)$ be the desired number. We split the selections into two subsets: those which include the last of the n objects and those which do not. The former are $f(n-2, k-1)$ in number (since further selection of the second last object is forbidden); the latter are $f(n-1, k)$ in number. Hence

$$f(n, k) = f(n - 1, k) + f(n - 2, k - 1),$$

and, combining this with $f(n, 1) = n$, we readily prove by induction that $f(n, k) = {}_{n-k+1}C_k$.

LEMMA 2. *The number of ways of selecting k objects, no two consecutive, from n objects arrayed in a circle is ${}_{n-k}C_k n/(n-k)$.*

This differs from the preceding problem only in the imposition of the further restriction that no selection is to include both the first and last objects; and the number of such selections which are otherwise acceptable is $f(n-4, k-2)$. Hence the desired result is $f(n, k) - f(n-4, k-2) = {}_{n-k}C_k n/(n-k)$.

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We now restate the *problème des ménages* in the usual fashion by observing that the answer is $2n!u_n$, where u_n is the number of permutations of $1, \cdots, n$ which do not satisfy any of the following $2n$ conditions: 1 is 1st or 2nd, 2 is 2nd or 3rd, $\cdots$, n is nth or 1st. Now let us select a subset of k conditions from the above $2n$ and inquire how many permutations of $1, \cdots, n$ there are which satisfy all k; the answer is $(n-k)!$ or 0 according as the k conditions are compatible or not. If we further denote by v_k the number of ways of selecting k compatible conditions from the $2n$, we have, by the familiar argument of inclusion and exclusion, $u_n = \sum (-1)^k v_k (n-k)!$. It remains to evaluate v_k, for which purpose we note that the $2n$ conditions, when arrayed in a circle, have the property that only consecutive ones are not compatible. It follows from Lemma 2 that $v_k = {}_{2n-k}C_k 2n/(2n-k)$, and hence

$$u_n = n! - \frac{2n}{2n-1}\,{}_{2n-1}C_1(n-1)! + \frac{2n}{2n-2}\,{}_{2n-2}C_2(n-2)! - \cdots .$$

From this result it follows without difficulty that $u_n/n! \to e^{-2}$ as $n \to \infty$.

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